Zero sets of Laplace eigenfunctions

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Based on joint works with L.Buhovsky, S.Chanillo, Eu.Malinnikova, N.Nadirashvili, F.Nazarov, M.Sodin

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Overview

1. Three "elementary" questions on spherical harmonics and eigenfunctions on $\mathbb{T}^2$.
2. Geometry of zero sets of Laplace eigenfunctions, Yau's conjecture, Nadirashvili's conjecture.
3. Harmonic functions: Growth vs Zeroes
4. Application: Landis' conjecture on the plane.

Eigenfunctions of the Laplace operator

Let M be a closed Riemannian manifold of dimension n and Δ be the Laplace operator on M. There is a sequence of eigenfunctions:

$$\Delta\varphi = -\lambda\varphi, \quad 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$$

Example 1.

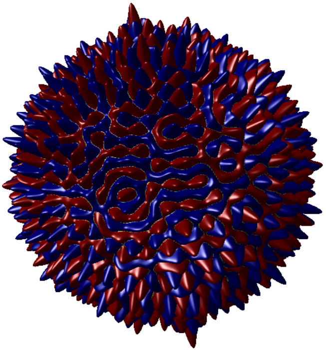
$$\varphi(x, y) = \sin(ax)\sin(by)$$

is an eigenfunction on the torus $\mathbb{T}^2$ with eigenvalue $\lambda = a^2 + b^2$. Linear combinations

$$\sum_{a_k^2+b_k^2=\lambda} c_k \sin(a_k x)\sin(b_k y)$$

.

Spherical harmonics



Value distribution $|\varphi|$ of a spherical harmonic. Red and blue areas represent the sign.

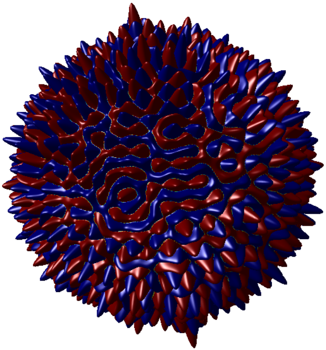
Picture credits:
Matthew de Courcy-Ireland

Example 2. Eigenfunctions on S^2 are restrictions of homogeneous harmonic polynomials in $\mathbb{R}^3$ to S^2. They are called spherical harmonics.

The corresponding eigenvalue is $\lambda = n(n+1)$, where n is the degree of the polynomial. The multiplicity is $2n+1$.

There is a standard basis of each eigenspace consisting of relatively simple polynomials. However, the value distribution of their (random) linear combinations can be complicated.

Three "elementary" questions on eigenfunctions



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Consider any sequence of eigenfunctions φ_λ on S^2 with $\lambda \to \infty$.

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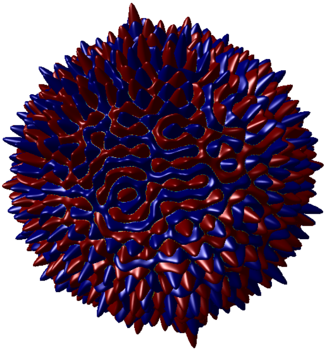

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The number of critical points of φ_λ grows to infinity.

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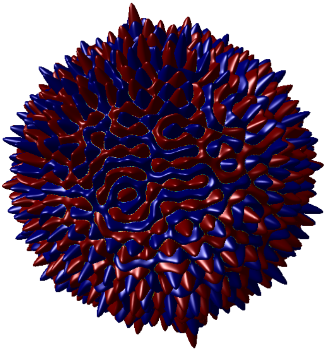

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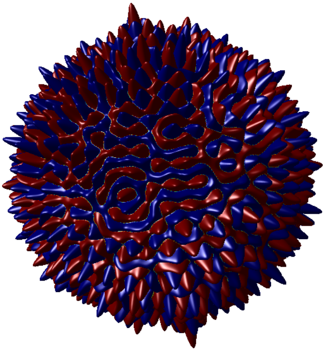

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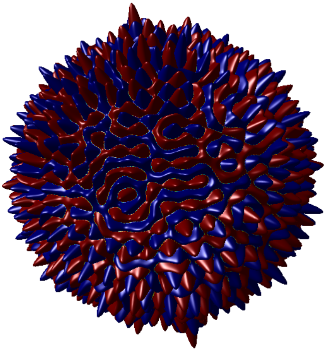

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$$\frac{\text{Area}(\varphi_\lambda > 0)}{\text{Area}(\varphi_\lambda < 0)} \to 1.$$

Thm(Donnelly and Fefferman)

$$c < \frac{\text{Area}(\varphi_\lambda > 0)}{\text{Area}(\varphi_\lambda < 0)} < C.$$

Number of critical points.

Conjecture (Yau) Does the number of the critical points of eigenfunctions φ_λ,

$$C_{\varphi_\lambda} = \{x : \nabla \varphi_\lambda(x) = 0\},$$

tends to infinity as $\lambda \to \infty$?
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Buhovsky & AL & Sodin, 2018: There is a metric on $\mathbb{T}^2$ with an infinite sequence of eigenfunctions ($\lambda \to \infty$) such the number of isolated critical points for each of them is infinite.

Sarnak's Conjecture

- Flat eigenfunctions: Is there a sequence of eigenfunctions φ_λ on S^2 with $\lambda \to \infty$ such that

$$\max_M |\varphi_\lambda| \leq C \|\varphi_\lambda\|_2 ?$$

- Example: On S^1 all eigenfunctions $sin(ax + b)$ are flat.

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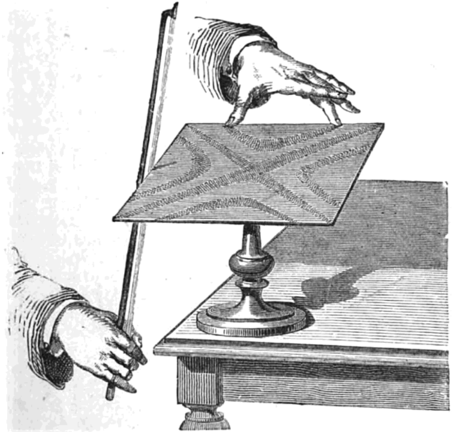
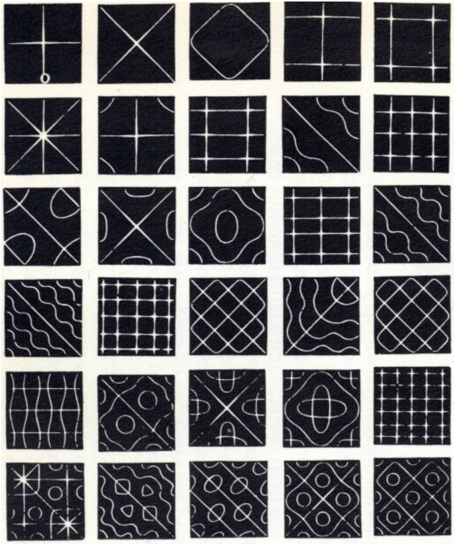
- Example: On S^1 all eigenfunctions $sin(ax + b)$ are flat.
- Ryll & Wojtaszczyk, 1983: There a sequence of flat eigenfunctions on S^{2d+1},
  Bourgain, 1985, 2016: stronger results for S^3 and S^5
- Sarnak's Conjecture: there is no such sequence on S^2.
- No one knows whether there is L^2 basis of spherical harmonics with bounded $L^\infty norm$.

Nodal sets and Chladni's resonance experiments.



Downloaded from William Henry Stone (1879), Elementary Lessons on Sound, Macmillan and Co., London, p. 26, fig. 12;



Chladni patterns published by John Tyndall in 1869.

Nodal geometry

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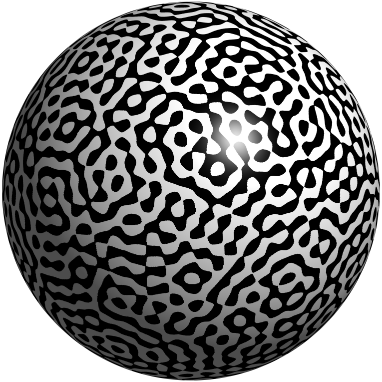
Zeroes of eigenfunctions of the Laplace operator: $\Delta u + \lambda u = 0$.
(a) vibration modes of a plate with half-free boundary conditions, (b) the vibration modes of a membrane, (c) the stationary wave equation, (d) the Helmholtz equation and (e) quantum mechanics.

Nodal domains and Courant theorem



The sign of a spherical harmonic.

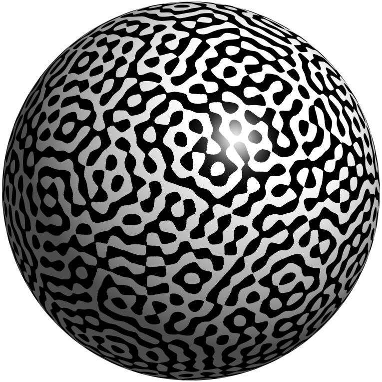
Picture credits: Dmitry Belyaev.

The nodal set separates the manifold M into several connected components, which are called nodal domains.

Thm(Courant, 1923). The k-th eigenfunction of the Laplace operator on any closed manifold has at most k nodal domains.

A. Stern(1924), H. Lewy(1977): there are spherical harmonics of any odd degree with only two nodal domains.

Topology of nodal loops



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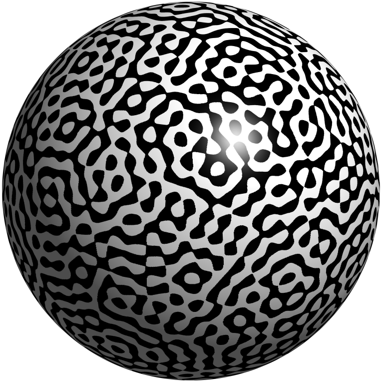
Thm(Eremenko,Nadirashvili,Jacobson). On S^2 every symmetric topological configuration of nodal loops (without intersections) is possible.

Nodal domains and Courant's theorem



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Thm(Courant, 1923). The k-th eigenfunction of the Laplace operator on a closed manifold M has at most k nodal domains.

Thm(Chanillo, AL, Malinnikova, 2019, work in progress)
Local version of Courant's theorem. The number of nodal domains of the k-th eigenfunction, which intersect a geodesic ball B is bounded by

$$k|B|/|M| + Ck^{1-\varepsilon_d}.$$

Spherical harmonic localized near equator

$$u(x, y, z) = \Re(x + iy)^n.$$

$\varphi = u|_{S^2}$ is the k-th eigenfunction on S^2 with

$$k \sim \lambda \sim n^2$$

Nodal domains and Courant's theorem

Thm(Courant, 1923). The k-th eigenfunction of the Laplace operator on a closed manifold M has at most k nodal domains.

Proof is one page long and uses only variational methods (minmax principle) and the fact that eigenfunctions can not vanish on open set.

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Why nodal domains can not be long and narrow?

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- New ingredient: It appears that eigenfunctions can not grow too fast in narrow domains because of some global reasons: the function is defined not only in the nodal domain, but on the whole manifold.

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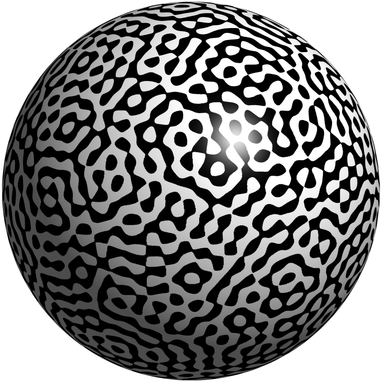
The proof requires to prove sharp BMO bounds Conjecture(Donnelly, Fefferman)/Thm(AL, Malinnikova):

$$\|\log|\varphi_\lambda|\|_{BMO} \leq C\sqrt{\lambda}$$

and to resolve a related question of Landis on three balls inequality for wild sets.

Two conjectures



The sign of a random spherical harmonic.

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Let M be a compact C^∞ -smooth Riemannian manifold M (without boundary) of dimension n.

Fact. For any Laplace eigenfunction φ,
$\Delta\varphi = -\lambda\varphi$,
the nodal set $Z_\varphi = \{x \in M : \varphi(x) = 0\}$ is $C/\sqrt{\lambda}$ dense.

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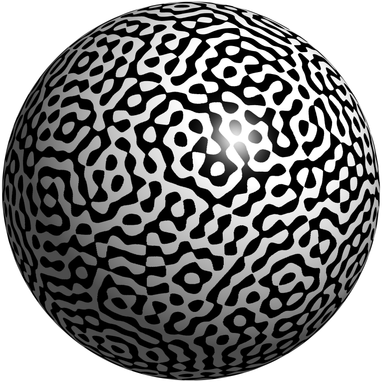

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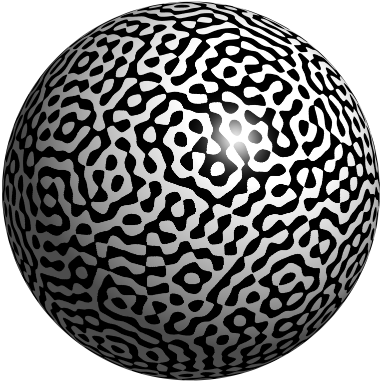

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Quasi-symmetry conjecture

$$c \le \frac{H^n(\varphi > 0)}{H^n(\varphi < 0)} \le C$$

Yau's conjecture: $c\sqrt{\lambda} \leq H^{n-1}(Z_{\varphi_\lambda}) \leq C\sqrt{\lambda}$

Previous bounds

- Brunning 1978, Yau: Lower bound is true for $n = 2$.
- Donnelly & Fefferman 1988: True for real analytic metrics.
- Nadirashvili 1988: $n = 2$, $H^1(Z_{\varphi_\lambda}) \leq C\lambda \log \lambda$
- Donnelly & Fefferman 1990, Dong 1992: $n = 2$, $H^1(Z_{\varphi_\lambda}) \leq C\lambda^{3/4}$
- Hardt & Simon 1989: $n \geq 2$, $H^{n-1}(Z_{\varphi_\lambda}) \leq C\lambda^{C\sqrt{\lambda}}$
- Colding & Minicozzi 2011, Sogge & Zelditch 2011, 2012, Steinerberger 2014: $c\lambda^{\frac{3-n}{4}} \leq H^{n-1}(Z_{\varphi_\lambda})$.

Yau's conjecture: $c\sqrt{\lambda} \le H^{n-1}(Z_{\varphi_\lambda}) \le C\sqrt{\lambda}$

New results

Thm(AL, Eu. Malinnikova, 2016). $n = 2$

$$H^1(Z_{\varphi_\lambda}) \le C\lambda^{3/4-\varepsilon}.$$

Thm(AL, 2016). $n \ge 3$

$$c\sqrt{\lambda} \le H^{n-1}(Z_{\varphi_\lambda}) \le C\lambda^{C_n}.$$

Yau's conjecture: $c\sqrt{\lambda} \leq H^{n-1}(Z_{\varphi_\lambda}) \leq C\sqrt{\lambda}$

Thm(AL, Malinnikova, Nazarov, Nadirashvili, work in progress): Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. Then for the eigenfunctions of the Laplace operator in Ω with Dirichlet boundary conditions

$$\Delta\varphi = -\lambda\varphi, \ \ \varphi|_{\partial\Omega} = 0$$

we have

$$H^{n-1}(Z_{\varphi_\lambda}) \leq C\sqrt{\lambda}.$$

Nadirashvili's conjecture

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- Thm(2016). Yes.
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$$\text{Area}(\{u = 0\} \cap B_1(0)) \geq c > 0,$$

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  where c is a universal constant.
- Rescaled version in $\mathbb{R}^n$:
  If $u(0) = 0$, then

$$H^{n-1}(\{u = 0\} \cap B_R(0)) \geq c_n R^{n-1}.$$

From Laplace eigenfunctions to harmonic functions

$$\Delta\varphi + \lambda\varphi = 0 \quad \text{vs} \quad \Delta u = 0.$$

Let φ satisfy $\Delta\varphi + \lambda\varphi = 0$ in $\mathbb{R}^n$.

Old trick: define a harmonic function u in $\mathbb{R}^{n+1}$ by

$$u(x,t) = \varphi(x)\exp(\sqrt{\lambda}t),$$

$$Z_u = Z_\varphi \times \mathbb{R}.$$

The same lifting trick works for eigenfunctions on manifolds.

From Nadirashvili's conjecture to Yau's conjecture

- Let φ satisfy $\Delta\varphi + \lambda\varphi = 0$ in $\mathbb{R}^n$.
  Why $H^{n-1}(Z_\varphi \cap \{|x| < 1\}) \geq c\sqrt{\lambda}$ for $\lambda > \lambda_0$?

- We will use another fact: Z_φ is $\frac{C}{\sqrt{\lambda}}$ dense in $\mathbb{R}^n$.

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- One can find $\sim \lambda^{n/2}$ disjoint balls $B(x_i, \frac{1}{\sqrt{\lambda}})$ in B_1 such that $\varphi(x_i) = 0$.

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- One can find $\sim \lambda^{n/2}$ disjoint balls $B(x_i, \frac{1}{\sqrt{\lambda}})$ in B_1 such that $\varphi(x_i) = 0$.
- Using Nadirashvili's conjecture on the scale $1/\sqrt{\lambda}$ and the lifting trick we have

$$H^{n-1}(Z_\varphi \cap B_{1/\sqrt{\lambda}}(x_i)) \geq c\left(\frac{1}{\sqrt{\lambda}}\right)^{n-1}.$$

  Thus $H^{n-1}(Z_\varphi \cap \{|x| < 1\}) \geq c\sqrt{\lambda}$.

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  Thus $H^{n-1}(Z_\varphi \cap \{|x| < 1\}) \geq c\sqrt{\lambda}$.
- The proof of Nadirashvili's conjecture is beyond the scope of this lecture.

Style of the proofs.

The works of Donnelly and Fefferman brought many ideas to nodal geometry. In particular they explained how to use complex and harmonic analysis to study nodal sets and proved Yau's and quasi-symmetry conjectures in the case of real-analytic Riemannian metrics. One of their ideas: geometry of nodal sets is controlled growth properties of functions.

The proof of Nadirashvili's conjecture (3D) is a multiscale induction argument. Complex analysis tools are not working for Nadirashvili's conjecture (at least we don't know how).

Tools in the proof Nadirashvili's conjecture: monotonicity formulas and unique continuation for elliptic PDE.

Growth of Laplace eigenfunctions on compact manifolds

$$\Delta \varphi + \lambda \varphi = 0$$

Donnelly-Fefferman growth estimate for Laplace eigenfunctions on compact Riemannian manifolds:

For any geodesic ball $B_r(x) \subset M$

$$\log \frac{\max_{B_{2r}(x)} |\varphi|}{\max_{B_r(x)} |\varphi|} \leq C\sqrt{\lambda}.$$

$2r$ is assumed to be smaller than the injectivity radius of M.

Harmonic counterpart of Yau's conjecture

Yau's conjecture: $H^{n-1}(Z_{\varphi_\lambda}) \leq C\sqrt{\lambda}$.
Lifting trick: $u(x,t) = \varphi(x)\exp(\sqrt{\lambda}t)$
satisfies an elliptic PDE of the second order in the divergence form

$$div(A\nabla u) = 0.$$

Doubling index:

$$N(B_r) = \log \frac{\max_{B_{2r}} |u|}{\max_{B_r} |u|}$$

Harmonic counterpart of Yau's conjecture:

$$H^{n-1}(Z_u \cap B_1) \leq CN(B_1).$$

Recent result (2016):

$$H^{n-1}(Z_u \cap B_1) \leq CN(B_1)^{C_n}.$$

Zeroes and growth of harmonic functions on the plane

For entire functions one can estimate the number of zeroes from above in terms of growth. But there is a plenty of holomorphic functions that have no zeroes.

Let u be a harmonic function (real valued) in $\mathbb{R}^2$.
Doubling index:

$$N(B_r) = \log \frac{\max_{B_{2r}} |u|}{\max_{B_r} |u|}$$

Thm(Gelfond, Robertson, Nadirashvili)

$$cN(B_{1/4}) - C \leq H^1(Z_u \cap B_1) \leq CN(B_2) + C$$

Length of nodal lines and doubling index

Let $n = 2$. So M is a surface and nodal sets are unions of curves. Consider an eigenfunction $\varphi : \Delta\varphi + \lambda\varphi = 0$.

Fact. On the scale $1/\sqrt{\lambda}$ eigenfunctions behave like harmonic functions.

Estimate of length of nodal lines (Donnelly-Fefferman, Nadirashvili, Nazarov-Polterovich-Sodin, Roy-Fortin).

$$cN(B_{\frac{1}{4\sqrt{\lambda}}}(x)) - C \leq \sqrt{\lambda} \cdot H^1(Z_\varphi \cap B_{\frac{1}{\sqrt{\lambda}}}(x)) \leq CN(B_{\frac{1}{\sqrt{\lambda}}}(x)) + C$$

Distribution of doubling index

Let $n = 2$. So M is a surface and nodal sets are unions of curves. Let M be covered by $\sim \lambda$ geodesic balls B_i of radius $1/\sqrt{\lambda}$ so that each point of M is covered at most 10 times.

Conjecture(Nazarov-Polterovich-Sodin). There is a numerical constant C (independent of λ and of the covering) such that

$$\frac{\sum N(B_i)}{\# B_i} \leq C.$$

Weak form. At least half of B_i have a bounded doubling index.

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Comment. In the case when the metric is real analytic Donnelly and Fefferman proved the weak conjecture on the distribution of doubling indices and used it show that quasisymmetry holds.

Comment. The weak conjecture implies the quasisymmetry conjecture:

$$c < \frac{Area(\varphi > 0)}{Area(\varphi < 0)} < C.$$

Comment. The strong NPS conjecture is equivalent to the Yau conjecture in dimension 2.

Upper bounds in Yau's conjecture, $n \geq 3$

Yau's conjecture: $H^{n-1}(Z_{\varphi_\lambda}) \leq C\sqrt{\lambda}$.
Lifting trick: $u(x,t) = \varphi(x)\exp(\sqrt{\lambda}t)$
satisfies an elliptic PDE of second order in divergence form

$$div(A\nabla u) = 0.$$

Doubling index:

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Harmonic counterpart of Yau's conjecture:

$$H^{n-1}(Z_u \cap B_1) \leq CN(B_1).$$

Lemma on distribution of doubling indices.

Consider a harmonic function u in $\mathbb{R}^n$ and let Q be a unit cube.

$$N = N_u(Q) = \log \frac{\max_{2Q} |u|}{\max_{Q} |u|}.$$

Let's partition Q into K^n equal cubes q_i of size $1/K$.

Lemma on distribution of doubling index.
If K and N are sufficiently large, then there are at least $K^n - \frac{1}{2}K^{n-1}$ good cubes q_i such that $N(q_i) \leq N/2$.

A version of the lemma above is used in the multiscale argument to prove polynomial upper bounds in Yau's conjecture and the lower bound.

Toolbox: Monotonicity of the doubling index for harmonic functions

$$N_u(rB) \leq (1+\varepsilon)N_u(B) + C(\varepsilon)$$

for any $r \in (0,1)$ and any harmonic function u in $\mathbb{R}^n$.

Toolbox: Monotonicity of the doubling index for harmonic functions

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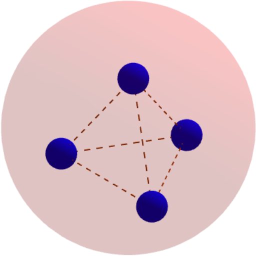
for any $r \in (0,1)$ and any harmonic function u in $\mathbb{R}^n$.
Monotonicity of the frequency function.

$$H_u(x,r) = |\partial B_r|^{-1} \int_{\partial B_r(x)} |u|^2, \quad F_u(x,r) = \frac{rH'(r)}{H(r)}.$$

$F_u(x,r)$ is monotone in r.
For more general elliptic equations Garofalo and Lin showed that $F_u(x,r)e^{Cr}$ is a non-decreasing function

Simplex lemma



Simplex Lemma (informal formulation):
Let u be a harmonic function in $\mathbb{R}^3$ such that for each blue ball $N(B_i) \geq A > 1000$, $n = 1, 2, 3, 4$.

Then the doubling index of the giant red ball, which contains small blue balls, is larger than A:

$$N(B) > A(1 + c), c > 0.$$

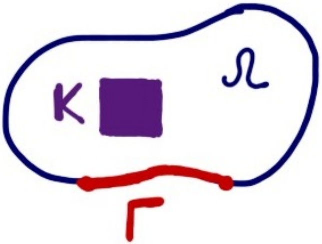
Toolbox: three balls theorem

Let u be a harmonic function. If $\max_B |u| \leq 1$ and $\max_{\frac{1}{4}B} |u| \leq \varepsilon$, then

$$\max_{\frac{1}{2}B} |u| \leq C\varepsilon^{\alpha}.$$

for some $\alpha \in (0,1)$ and C that do not depend on u.

Toolbox: quantitative Cauchy uniqueness.



$$\mathrm{div}(A\nabla u) = 0, \quad A \text{ is elliptic}$$

and with Lipschitz coefficients.

If $\Gamma \subset \partial\Omega$ is relatively open and $K \subset \Omega$ is a compact set, then

$$\max_K |\nabla u| \leq C \sup_\Gamma |\nabla u|^\beta \sup_\Omega |\nabla u|^{1-\beta}$$

Second question from Nadirashvili's plan

Cauchy uniqueness problem.
Let u be a harmonic function in a unit ball $B \subset \mathbb{R}^3$. Assume that $u \in C^\infty(\overline{B})$ and $\nabla u = 0$ on a set $S \subset \partial B$ with positive area. Does it imply that $\nabla u \equiv 0$?

Comment. If S is a relatively open subset the answer is yes. It is also true in dimension two for any set of positive length. In $\mathbb{R}^3$ if C^∞ class of functions is replaced by $C^{1,\varepsilon}$ the answer is no (Bourgain, Wolff). Attempts to construct C^2 counterexamples were not successful.

Application of zero sets and quasiconformal mappings: Landis conjecture

Let u be a solution to $\Delta u + Vu = 0$ in $\mathbb{R}^2$,
where V is a bounded potential: $|V| < 1$.
Landis' conjecture: if $|u(x)| \leq exp(-|x|^{1+\varepsilon})$, then $u \equiv 0$.

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Example: The function $exp(-|x|)$ decays exponentially and outside of the unit ball $|\Delta \exp(-|x|)| \leq C \exp(-|x|)$. One can construct a solution in the whole $\mathbb{R}^2$, which decays exponentially.

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Example: The function $exp(-|x|)$ decays exponentially and outside of the unit ball $|\Delta \exp(-|x|)| \leq C \exp(-|x|)$. One can construct a solution in the whole $\mathbb{R}^2$, which decays exponentially.
Meshkov: Landis conjecture is false for complex-valued potentials. There is a non-zero complex solution u: $|\Delta u| \leq |u|$ such that $|u(x)| \leq \exp(-c|x|^{4/3})$.

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Thm(AL, Malinnikova, Nadirashvili, Nazarov, work in progress)
Landis' conjecture is true for real potentials.

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The proof is using zero sets and quasiconformal mappings.

Application of zero sets and quasiconformal mappings: Landis conjecture

Landis conjecture is a problem about solutions to $\Delta u + Vu = 0$ on the plane.
Quasiconformal mappings and nodal sets help to reduce the problem to a simpler one about a harmonic function $h : \Delta h = 0$ on the plane with holes.

Toy problem. Let $\{z_i\}$ be a set of points in $\mathbb{R}^2$ with $|z_i - z_j| > 10$.

$$\Omega = \mathbb{R}^2 \setminus \cup B_1(z_i)$$

Let h be a harmonic function in Ω with

unusual boundary conditions:
h does not change sign in each of the annuli $B_2(z_i) \setminus B_1(z_i)$.

Show that $|h(z)|$ cannot be too small near infinity:

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One can reduce the quantitative version of Landis conjecture to the quantitative version of the toy problem using quasiconformal mappings and nodal sets.

Question

Can one find a way in higher dimensions to simplify PDEs? Quasiconformal mappings allow to find a smart change of variables in 2D, which transforms the solution of

$$div(A\nabla u) = 0$$

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The change of variables depends on the solution itself, but has good quantitative estimates that depend on A only.

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Quasiconformal mappings allow to find a smart change of variables in 2D, which transforms the solution of

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The change of variables depends on the solution itself, but has good quantitative estimates that depend on A only.
In higher dimensions there is no hope to simplify the equation to the equation with constant coefficients.

Question. Can one find a change of variables for one fixed solution to $div(A\nabla u) = 0$ in $\mathbb{R}^3$ such that the new equation has a symmetry (is not depending on one of the coordinates)?

Non-standard logic: There is one fixed function and we study all Riemannian metrics such that the function is harmonic with respect to the metric. So it is the equation for the metric. There are many metrics, which solve it and we want to find the one, which is simple.

The change of variables/metric should depend on the solution and cannot serve for all solutions at the same time.

Question

Thm(AL, Malinnikova, Nadirashvili, Nazarov, work in progress) If M is a closed Riemannian surface an u is a real-valued function on M with $|\Delta u| \leq \lambda |u|$, then the vanishing order of u at any point is smaller than $C\lambda^{1/2+\varepsilon}$

Question If M is a closed Riemannian surface an u is a real-valued function on M with $|\Delta u| \leq \lambda |u|$ is it true that

$$H^1(Z_u) \leq C\lambda^{1/2+\varepsilon}?$$