Stable Big Bang Formation in General Relativity: The Complete Sub-Critical Regime

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Cauchy Problem for Einstein's equations

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- Some results I will describe hold when $\phi = 0$
- Data on $\Sigma_1 = \mathbb{T}^D$ are tensors $(\mathring{g}, \mathring{k}, \mathring{\phi}_0, \mathring{\phi}_1)$ verifying the Gauss and Codazzi constraints
- Our data will be Sobolev-close to Kasner data
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"Dynamic stability of the Big Bang"

Some sources of inspiration

- Hawking–Penrose “singularity” theorems.
- Explicit solutions, especially FLRW and Kasner.
- Heuristics from the physics literature.
- Numerical work on singularities.
- Rigorous results in symmetry and analytic class.
- Dafermos–Luk.

"Generalized" Kasner solutions

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$$\sum_{I=1}^{D} q_I = 1, \qquad \sum_{I=1}^{D} (q_I)^2 = 1 - B^2$$

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where $C > 0$ (unless one q_I equals 1 and the rest vanish)

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"Big Bang" singularity at $t = 0$

Hawking's incompleteness theorem

Theorem (Hawking)

Assume

- $(\mathcal{M}, \mathbf{g}, \phi)$ *is the maximal globally hyperbolic development of data* $(\mathring{g}, \mathring{k}, \mathring{\phi}_0, \mathring{\phi}_1)$ *on* $\Sigma_1 \simeq \mathbb{T}^D$
- $\mathrm{tr}\mathring{k} < -C < 0$

Then no past-directed timelike geodesic emanating from Σ_1 *is longer than* $C' < \infty$.

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- Why are the timelike geodesics incomplete?
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New result with Rodnianski and Fournodavlos: Kasner Big Bang is dynamically stable assuming a sub-criticality condition:

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Key takeways:

- In GR, distinct kinds of incompleteness occurs in different solution regimes

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Key takeaways:

- In GR, distinct kinds of incompleteness occurs in different solution regimes
- In principle, other stable pathologies could dynamically develop in other (not-yet-understood)

Inspiration from physics

Belinskiĭ–Khalatnikov–Lifshitz considered tensorfields:

$$\mathbf{g}_{BKL} = -dt \otimes dt + \sum_{I=1}^{D} t^{2q_I(x)} dx^I \otimes dx^I, \ \phi_{BKL} = B(x) \ln t,$$

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“Monotonic” regimes

Works by BK, Barrow, Demaret–Henneaux–Spindel, Andersson–Rendall, Damour–Henneaux–Rendall–Weaver suggest that a $D-$dimensional Kasner Big Bang might be dynamically stable under the sub-criticality condition:

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- Significance: Heuristics suggest that time derivative terms will dominate: “Asymptotically Velocity Term Dominated”
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The singularity industry: A sampler

- **Numerical works**: e.g. Berger, Garfinkle, Isenberg, Lim, Moncrief, Weaver, $\cdots$
- **Symmetry**: e.g. Alexakis–Fournodavlos, Chruściel–Isenberg–Moncrief, Ellis, Isenberg–Kichenassamy, Isenberg–Moncrief, Liebscher, Ringström, Wainwright, $\cdots$
- **Linear**: e.g. Alho–Franzen–Fournodavlos, Ringström
- **Construction of singular solutions**: e.g. Ames, Andersson, Anguige, Beyer, Choquet-Bruhat, Damour, Demaret, Fournodavlos, Henneaux, Isenberg, LeFloch, Luk, Kichenassamy, Rendall, Spindel, Ståhl, Todd, Weaver, $\cdots$
- **Oscillatory investigations**: e.g. BKL, Damour, van Elst, Heinzle, Hsu, Lecian, Liebscher, Misner, Nicolai, Uggla, Reiterer, Ringström, Tchapnda, Trubowitz, $\cdots$

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Theorem (JS, G. Fournodavlos, and I. Rodnianski (to appear))

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holds, then near its Big Bang, $\mathbf{g}_{KAS} := -dt \otimes dt + \sum_{I=1}^{D} t^{2q_I} dx^I \otimes dx^I$, $\phi_{KAS} = B \ln t$ *is a dynamically stable solution to the Einstein-scalar field system under Sobolev-class perturbations of the data on* $\{t = 1\}$.

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Moreover, when $D = 3$ *and* $B = 0$*, under polarized* $U(1)$*-symmetric perturbations (i.e.,* $g_{13} = g_{23} \equiv 0$ *and no* x^3*-dependence),* ***all*** *Kasner Big Bangs are dynamically stable.*

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- Previously with Rodnianski, we had treated i) $D = 3$ with $q_1 = q_2 = q_3 = 1/3$, i.e., stability for FLRW, and ii) $D \geq 38$ with $\max_{I=1,\cdots,D} |q_I| < 1/6$ and $\phi = 0$.

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Crushing singularities

The singularities in our main results are crushing:

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This shows that in the chosen gauge, the solution cannot be continued weakly.

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Key new ingredient:

Fermi-Walker-propagated Σ_t-tangent orthonormal spatial frame $\{e_I\}_{I=1,\cdots,D}$; with $e_I = e_I^c \partial_c$:

$$e_0 e_I^i = k_{IC} e_C^i$$

Proof philosophy

Recast Einstein's equations as an elliptic-hyperbolic PDE system for scalar frame-component functions

The unknowns are:

- The lapse n
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Einstein-vacuum equations in our gauge

Evolution equations

$$\begin{aligned}
\partial_t k_{IJ} &= -\frac{n}{t} k_{IJ} - e_I e_J n + n e_C \gamma_{IJC} - n e_I \gamma_{CJC} \\
&\quad + \gamma_{IJC} e_C n - n \gamma_{DIC} \gamma_{CJD} - n \gamma_{DDC} \gamma_{IJC}, \\
\partial_t \gamma_{IJB} &= n e_B k_{IJ} - n e_J k_{BI} \\
&\quad - n k_{IC} \gamma_{BJC} + n k_{IC} \gamma_{JBC} + n k_{IC} \gamma_{CJB} \\
&\quad - n k_{CJ} \gamma_{BIC} + n k_{BC} \gamma_{JIC} \\
&\quad + (e_B n) k_{IJ} - (e_J n) k_{BI}
\end{aligned}$$

Elliptic lapse PDE

$$\begin{aligned}
e_C e_C (n-1) - t^{-2}(n-1) = \gamma_{CCD} e_D (n-1) + 2 n e_C \gamma_{DDC} \\
- n \left\{ \gamma_{CDE} \gamma_{EDC} + \gamma_{CCD} \gamma_{EED} \right\}
\end{aligned}$$

Constraint equations

$$\begin{aligned}
k_{CD} k_{CD} - t^{-2} &= 2 e_C \gamma_{DDC} - \gamma_{CDE} \gamma_{EDC} - \gamma_{CCD} \gamma_{EED}, \\
e_C k_{CI} &= \gamma_{CCD} k_{ID} + \gamma_{CID} k_{CD}
\end{aligned}$$

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Asymptotic limits

- Similar argument $\implies \exists\ \kappa_{IJ}^{(\infty)}(x)$ such that $\left| tk_{IJ}(t,x) - \kappa_{IJ}^{(\infty)}(x) \right| \to 0$ as $t \downarrow 0$.
- Eigenvalues of the symmetric matrix $(\kappa_{IJ}(x))_{I,J=1,\cdots,D}$ are functions $\{q_i^{(\infty)}(x)\}_{i=1,\cdots,D}$ on $\mathbb{T}^D$.
- The $\{q_i^{(\infty)}(x)\}_{i=1,\cdots,D}$ are the "asymptotic Kasner exponents" of the perturbed solution.
- The set of "limiting end states" is infinite-dimensional.
- Our proof does not suggest that t-rescaled versions of the component functions $e_I^i(t,x)$ should have finite, non-trivial limits as $t \downarrow 0$.
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Top-order energy estimates

We prove that for $t \in (0, 1]$, we have:

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- Large $A \implies$ very singular top-order energy estimates

Problems to think about

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Thank You!