Roots of Polynomials Under Repeated Differentiation

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UNIVERSITY of WASHINGTON

Outline of the Talk

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2. A Nonlinear PDE

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5. A Nonlinear PDE in the Complex Plane (with Sean O'Rourke)

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where $z_1, \ldots, z_n \sim \mu$ are i.i.d. random variables distributed according to μ.

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Note: The critical points are $w_1, \ldots, w_{n-1} \in \mathbb{C}$. The statement really says that

$$\frac{1}{n-1} \sum_{k=1}^{n-1} \delta_{w_k} \rightharpoonup \mu.$$

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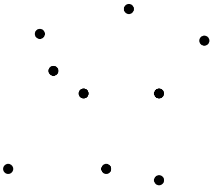

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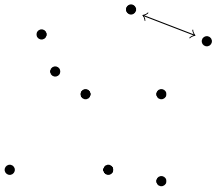

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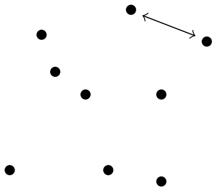

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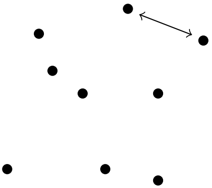
Every root of the derivative satisfies $p_n'(z) = 0$ which means

$$\frac{1}{z - z_\ell} = -\sum_{\substack{k=1 \\ k \neq \ell}}^{n} \frac{1}{z - z_k}.$$



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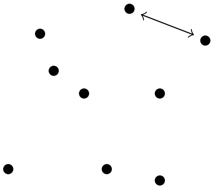

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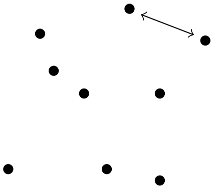

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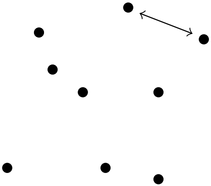

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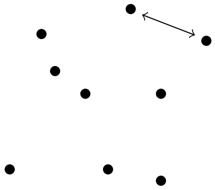

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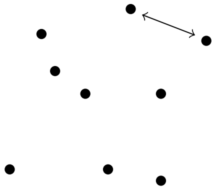
What this argument tells us is roughly the following: if μ is a sufficiently nice measure, then for each (random) root $p_n(z_k) = 0$, we would expect that there is a critical point $p_n'(z) = 0$ that is at most distance $\sim n^{-1}$ nearby.



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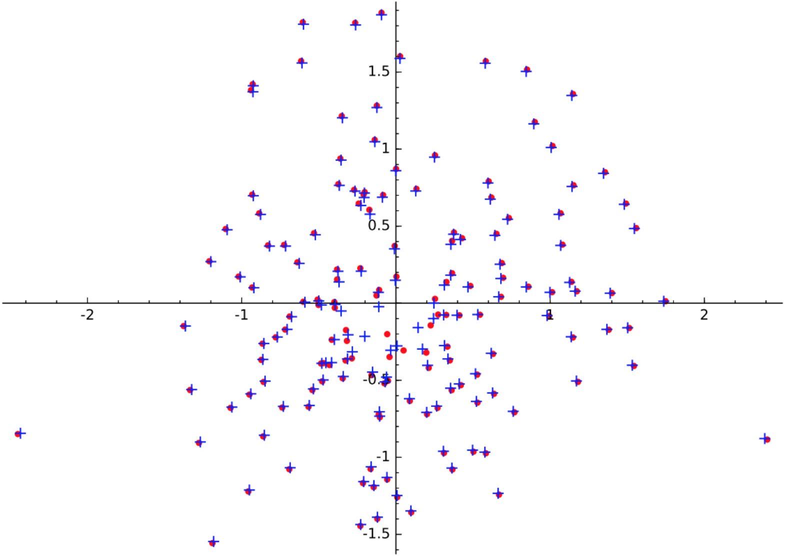
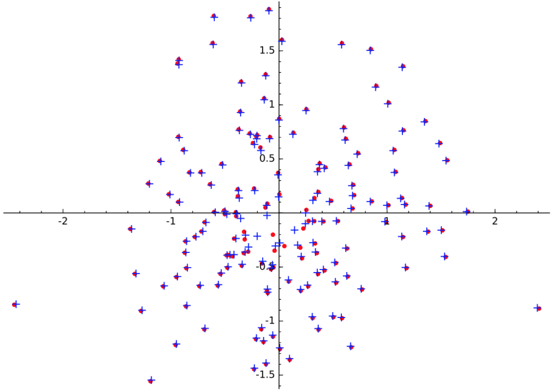


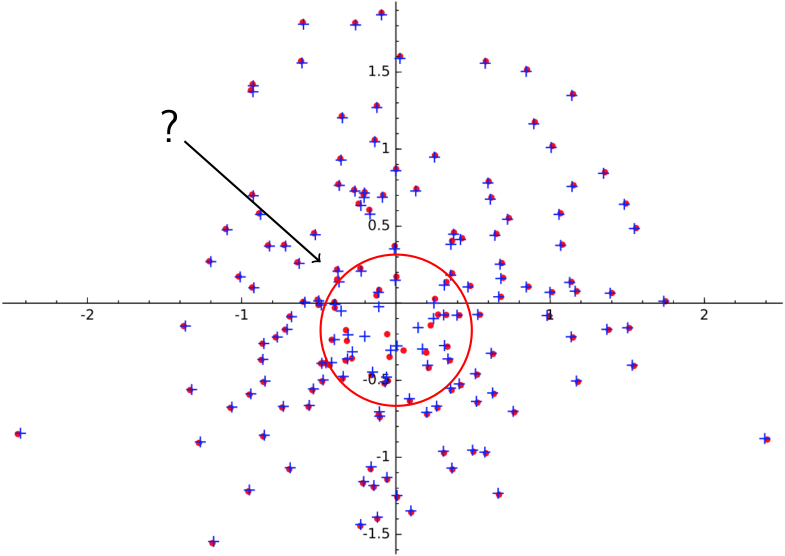
picture from O'Rourke and Williams (2018)



Theorem (O'Rourke and Williams)

Under reasonable assumptions on the measure

$$W_1(\mu_n, \mu'_n) \lesssim \frac{(\log n)^{10}}{n}.$$



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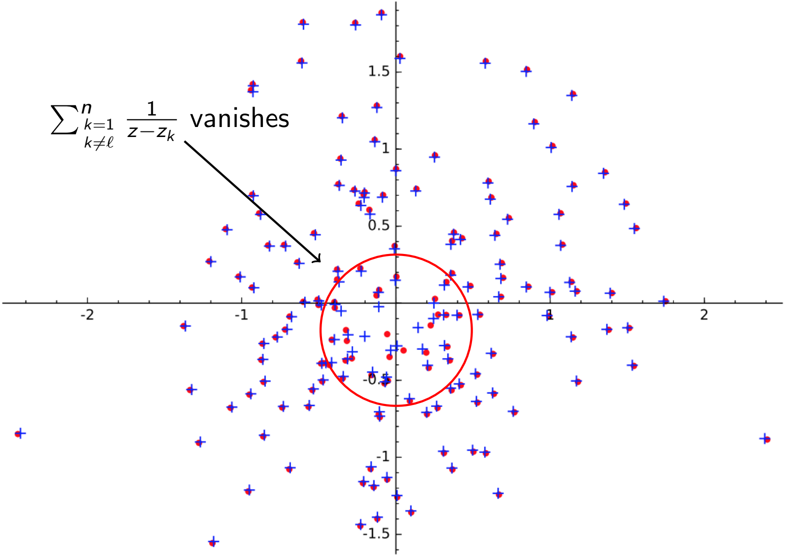

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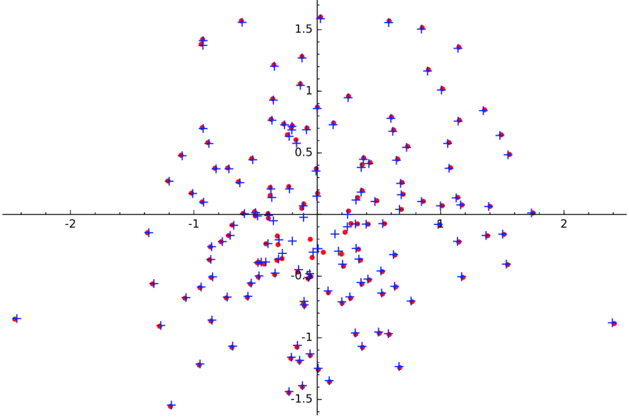
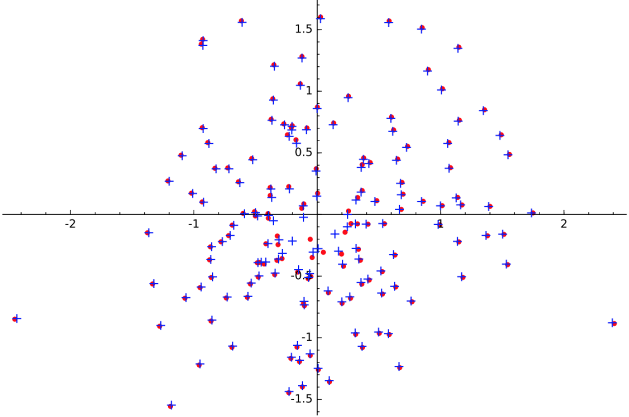
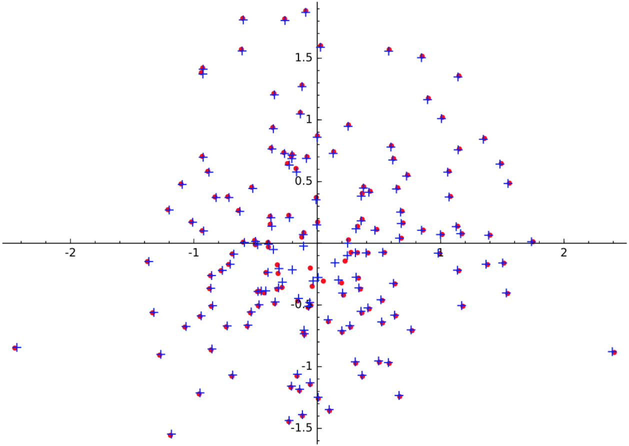


In fact, the bijective relationship has to fail somewhere: there are n roots and $n - 1$ critical points.



In fact, the bijective relationship has to fail somewhere: there are n roots and $n-1$ critical points. The unpaired root is frequently close to the root of

$$V(z) = \sum_{k=1}^{n} \frac{1}{z - z_k}.$$



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This looks almost like a flow of particles captured at nearby times.

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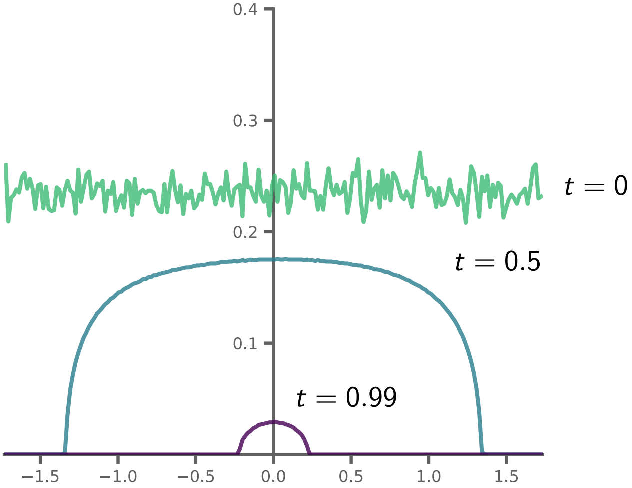

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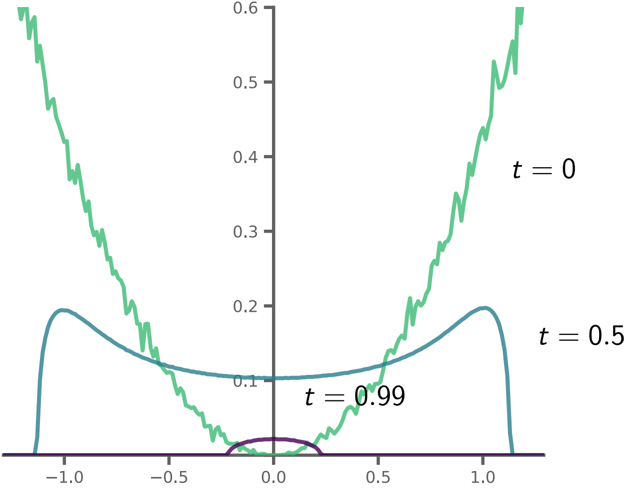

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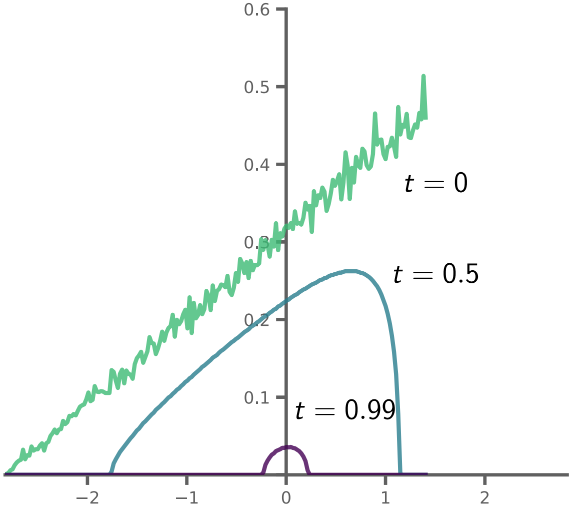

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$$G(p_n) = \min_{i \neq j} |x_i - x_j|,$$
we have (Riesz, Sz-Nagy, Walker, 1920s)
$$G(p_n') \geq G(p_n).$$

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What can one say about $u(t,x)$?

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This means: the distribution shrinks linearly in mass, its mean is preserved and the mass is distributed over area $\sim \sqrt{1-t}$.

An Equation (S. 2018)

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The argument is actually fun and I can give it in full. But before, let's explore this strange equation.

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Presumably there are many others(?)

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Hermite polynomials $H_n : \mathbb{R} \to \mathbb{R}$ satisfy a nice recurrence relation

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This suggests that

$$u(t,x) = \frac{2}{\pi} \sqrt{1 - t - x^2} \cdot \chi_{|x| \leq \sqrt{1-t}} \qquad \text{for } t \leq 1$$

should be a solution of the PDE (and it is).

Hermite Polynomials

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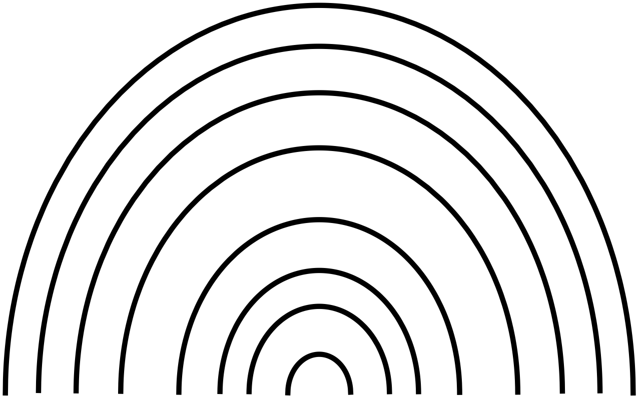

Laguerre Polynomials

(Associated) Laguerre polynomials $H_n : \mathbb{R} \to \mathbb{R}$ satisfy the recurrence relation

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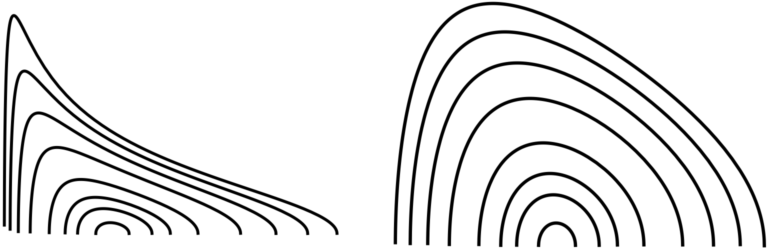

Figure: Marchenko-Pastur solutions $u_c(t,x)$: $c=1$ (left) and $c=15$ (right) shown for $t \in \{0, 0.2, 0.4, 0.6, 0.8, 0.9, 0.95, 0.99\}$.

A Bonus Solution

There are several classical orthogonal polynomials on $[-1, 1]$ (Gegenbauer, Jacobi, ...).

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Theorem (Tricomi?)

Let $f : (-1,1) \to \mathbb{R}_{\geq 0}$. If $Hf \equiv 0$ in $(-1,1)$, then

$$f = \frac{c}{\sqrt{1-x^2}}.$$

$$\frac{\partial u}{\partial t} + \frac{1}{\pi}\frac{\partial}{\partial x} \arctan\left(\frac{Hu}{u}\right) = 0 \qquad \text{on } \operatorname{supp}(u)$$

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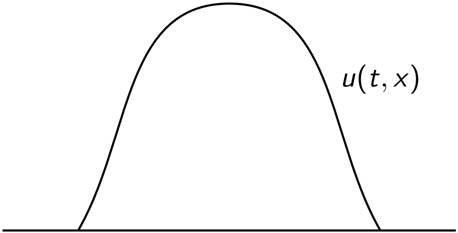

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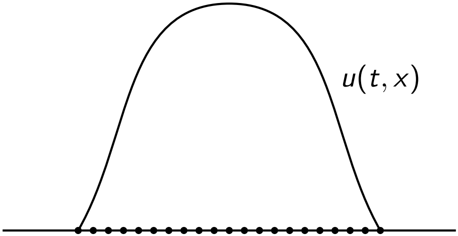

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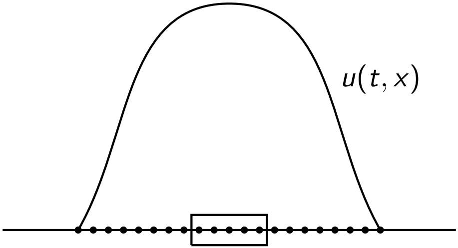

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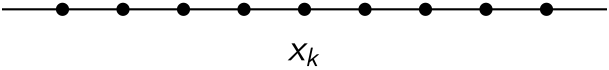

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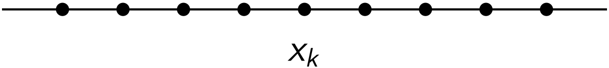

$$\sum_{k=1}^{n} \frac{1}{x - x_k} = \sum_{|x_k - x| \text{ large}} \frac{1}{x - x_k} + \sum_{|x_k - x| \text{ small}} \frac{1}{x - x_k}$$

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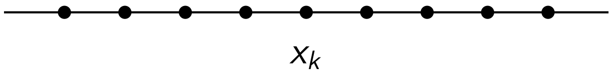

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It thus remains to understand the behavior of the local term.

The local term is

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$$\sum_{|x_k - x| \text{ small}} \frac{1}{x - x_k} \sim \sum_{\ell \in \mathbb{Z}} \frac{1}{x - \left(x_k + \frac{\ell}{u(t,x)n}\right)}.$$

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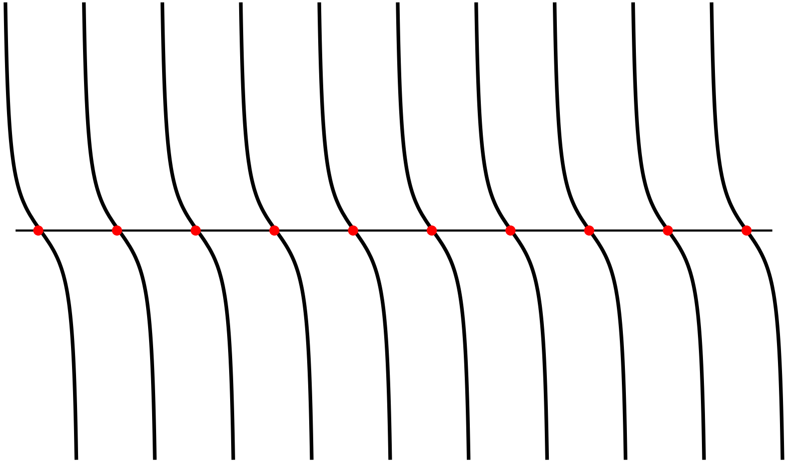
We are in luck: this sum has a closed-form expression due to Euler

$$\pi \cot \pi x = \frac{1}{x} + \sum_{n=1}^{\infty} \left(\frac{1}{x+n} + \frac{1}{x-n} \right) \qquad \text{for } x \in \mathbb{R} \setminus \mathbb{Z}.$$

The Local Field



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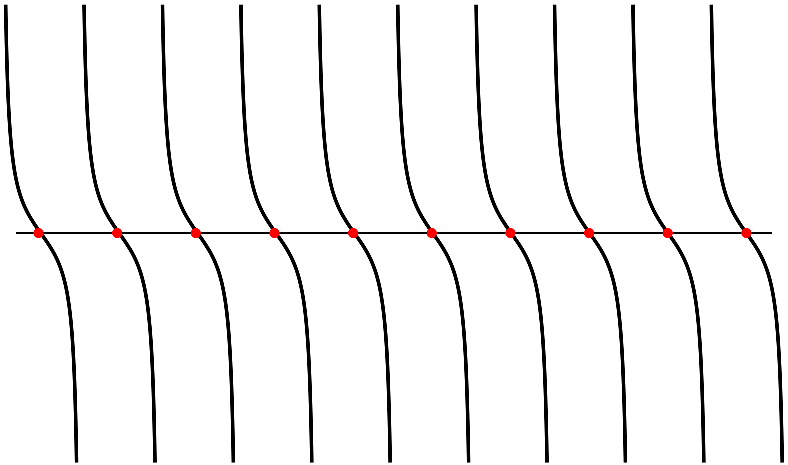
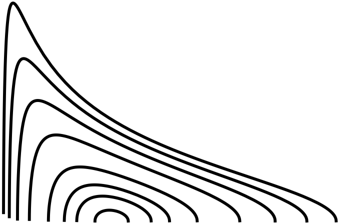


We can then predict the behavior of the roots of the derivative: they are in places where the local (near) field and the global (far) field cancel out. This leads to the desired equation.

A Fast Numerical Algorithm



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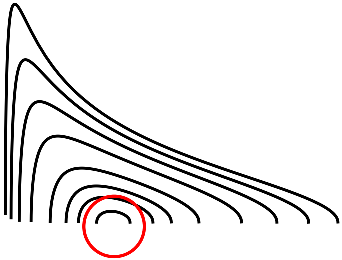

Jeremy Hoskins (U Chicago) used the electrostatic interpretation to produce an algorithm that can compute all derivatives of polynomials up to degree $\sim$ 100.000. **Semicircles**.

Theorem (J. Hoskins and S, 2020)

Let X be a random variable on $\mathbb{R}$ such that all moments are finite and $\mathbb{E}X = 0$ as well as $\mathbb{V}X = 1$.

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Remarks.

1. The roots of the Hermite polynomial have a semicircle density.
2. If $x_1, x_2, \ldots, x_n \sim X$, then

$$\frac{x_1 + \cdots + x_n}{\sqrt{n}} \sim \mathcal{N}(0,1)$$

and the mean of the roots is preserved under differentiation (hence the random shift).

Ideas behind the proof

$$p_n(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

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We need elementary symmetric polynomials

$$e_0(x_1, \ldots, x_n) = 1$$
$$e_1(x_1, \ldots, x_n) = x_1 + \cdots + x_n$$
$$e_2(x_1, \ldots, x_n) = \sum_{i<j} x_i x_j$$
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As it turns out: e_1 determines everything else.

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Lemma

Let $m \in \mathbb{N}$ and let $x_1, \ldots, x_n$ be i.i.d. random variables sampled from a distribution on $\mathbb{R}$ with $\mathbb{E}X = 0$, $\mathbb{E}X^2 = 1$ and $\mathbb{E}|X|^m < \infty$. Then, as $n \to \infty$,

$$\mathbb{E}\left| e_m - \sum_{k=0}^{\lfloor m/2 \rfloor} (-1)^k \frac{1}{k!(m-2k)!2^k} \cdot e_1^{m-2k} n^k \right| \lesssim_X n^{\frac{m-1}{2}}$$

Sep 3, 2020

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Fractional free convolution powers

Dimitri Shlyakhtenko, Terence Tao

The extension $k \mapsto \mu^{\boxplus k}$ of the concept of a free convolution power to the case of non-integer

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(where we use the branch of arctan taking values in $[0,\pi]$) and thus by the change of variables $k = 1/s$ and abbreviating $f := f_{1/s}$,

$$(-s\partial_s + x\partial_x)Hf = \frac{1}{\pi}\partial_x \log((Hf)^2 + f^2)^{1/2} \tag{4.1}$$

and

$$(-s\partial_s + x\partial_x)f = \frac{1}{\pi}\partial_x \arctan\frac{f}{Hf} \tag{4.2}$$

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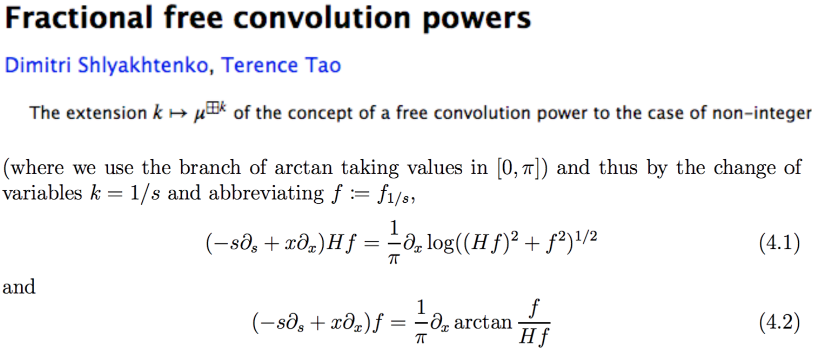

The same PDE in a supposedly different context is presumably not a coincidence.

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Under some reasonable assumptions

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- Fractional Free Convolution preserves **free cumulants**

$$\kappa_1(\mu) = \int_{\mathbb{R}} x d\mu$$

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Infinitely many conserved quantities.

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This would then imply that $u(t, x)$ should be a semicircle for t close to 1.

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[Submitted on 4 Sep 2020]

Universal objects of the infinite beta random matrix theory

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Limit theorems for Bessel and Dunkl processes of large dimensions and free convolutions

Michael Voit, Jeannette H.C. Woerner

which establishes a connection between Bessel processes and free convolution. So I think we are pretty close to having completely rigorous arguments for most things.

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- **What about the complex case?**

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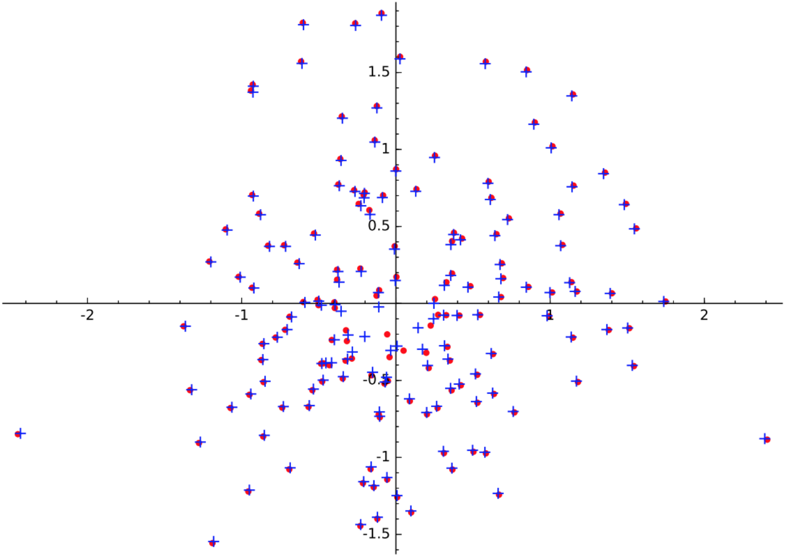

picture from O'Rourke and Williams (2018)

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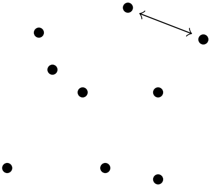

typically $\sim n^{-1/2}$

$$\frac{1}{z - z_\ell} = -\sum_{\substack{k=1 \\ k \neq \ell}}^{n} \frac{1}{z - z_k}.$$

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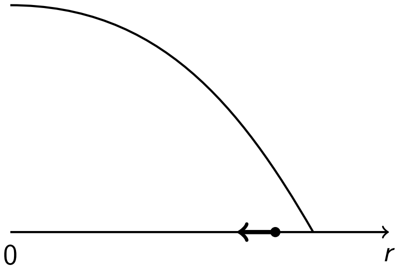

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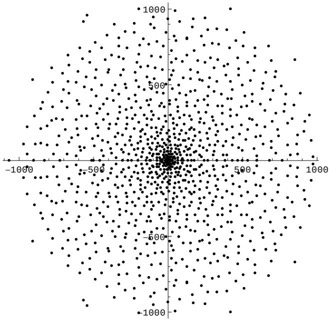
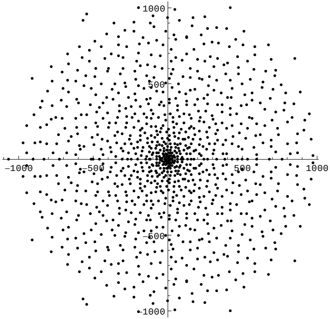
This corresponds to **Random Taylor Polynomials**.

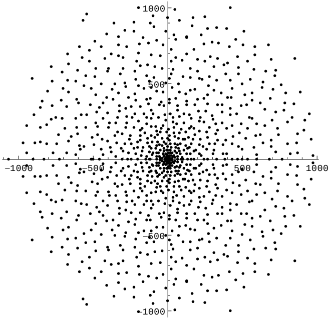




Random Taylor polynomials are defined by

$$p_n = \sum_{k=0}^{n} \gamma_k \frac{z^k}{k!},$$

where $\gamma_k \sim \mathcal{N}(0,1)$.



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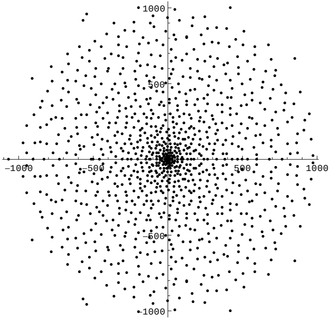

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Theorem (Kabluchko & Zaporozhets)

$$\frac{1}{n}\sum_{k=1}^{n} \delta_{z_k n^{-1}} \to \frac{\chi_{|z|\le 1}}{2\pi|z|} \qquad \text{as } n \to \infty.$$

A final pretty fact: when trying to study L^2-stability of the solution, one runs into the following beautiful inequality.

Lemma

For $f:(0,\infty)\to\mathbb{R}_{\geq 0}$

$$\int_0^\infty \frac{f(x)}{x^2}\left(\int_0^x f(y)dy\right)dx \leq \int_0^\infty \frac{f(x)^2}{x}dx,$$

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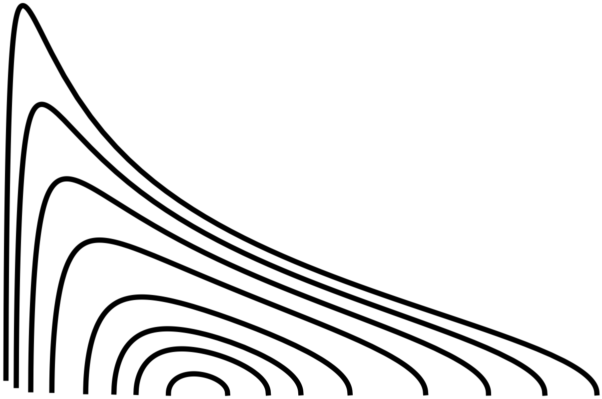
Proof. follows easily from a general Hardy inequality.



Thank you!