Two results on the interaction energy

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The interaction energy

- For a nonnegative density $\rho \in L^1_+(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, consider its interaction energy, given by

$$\mathcal{E}_W[\rho] := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \rho(x)\rho(y)W(x-y)dxdy = \int_{\mathbb{R}^n} \rho(\rho * W)dx,$$

  where $W \in C^1(\mathbb{R}^n \setminus \{0\}) \cap L^1_{loc}(\mathbb{R}^n)$ is a **radially decreasing** interaction potential.

- Examples of W:

  Newtonian potential: $\mathcal{N} = \begin{cases} c_n|x|^{2-n} & n > 2, \\ -\frac{1}{2\pi}\log|x| & n = 2, \end{cases}$

  which belongs to the broader class of Riesz potentials:

$$W_k = \begin{cases} -\frac{|x|^k}{k} & \text{for } k \neq 0, \\ -\log|x| & \text{for } k = 0. \end{cases}$$

Riesz rearrangement inequality

- Riesz's rearrangement inequality:

$$f, g, h \geq 0 \Longrightarrow \int f(g * h)dx \leq \int f^*(g^* * h^*)dx,$$

  here f^* is the radially decreasing rearrangement of f.
- A direct consequence (using that W is radially decreasing):

$$\mathcal{E}_W[\rho] \leq \mathcal{E}_W[\rho^*].$$

- If W is strictly radially decreasing, Lieb '77 showed that "$=$" is achieved iff $\rho = \rho^*$ up to a translation.

Questions

Are there any stability estimate of the form

$$\mathcal{E}_W[\rho^*] - \mathcal{E}_W[\rho] \geq d(\rho, \rho^*) \geq 0?$$

Here $d(\rho, \rho^*)$ measures the "asymmetry" of ρ, and should $= 0$ iff $\rho = T_a\rho^*$ under some translation T_a.

Stability estimates for Riesz potentials: current results

- One natural way to measure the asymmetry of ρ:

$$\delta(\rho) := \inf_{a\in\mathbb{R}^n} \frac{\|T_a\rho - \rho^*\|_{L^1(\mathbb{R}^n)}}{2\|\rho\|_{L^1(\mathbb{R}^n)}}.$$

- For Newtonian potential $\mathcal{N}$ in $\mathbb{R}^3$, Burchard–Chambers '15 obtained:

$$\mathcal{E}_{\mathcal{N}}[1_D^*] - \mathcal{E}_{\mathcal{N}}[1_D] \geq c|D|^{5/3}\,\delta(1_D)^2.$$

- For Riesz potential W_k, Fusco–Pratelli '19, Burchard–Chambers '20:

$$\mathcal{E}_{W_k}[1_D^*] - \mathcal{E}_{W_k}[1_D] \geq c(n,k)|D|^{2+\frac{k}{n}}\,\delta(1_D)^2 \quad \text{for } k \in (-n+1, 0).$$

Proofs based on delicate geometrical + mass transportation arguments; works for $\rho = 1_D$, cannot be easily extended to general densities.

Stability estimates for Riesz potentials: current results

- For Riesz potential $\mathcal{W}_k$, Frank–Lieb '19 proved the following for $0 \le \rho \le 1$:

$$\mathcal{E}_{W_k}[1_{E^*}] - \mathcal{E}_{W_k}[\rho] \ge c(n,k) \|\rho\|_{L^1}^{2+\frac{k}{n}} \tilde{\delta}(\rho, 1_{E^*})^2 \quad \text{for } k \in (-n, \infty),$$

where E^* is a ball centered at origin with $|E^*| = \int \rho dx$.

- Note that it does **not** imply a stability estimate for $\mathcal{E}_{W_k}[\rho^*] - \mathcal{E}_{W_k}[\rho]$.
- Proof is built on a deep result by Christ '17: if B is a ball centered at the origin satisfying $\delta \le \frac{|B|^{1/n}}{2\|\rho\|_1^{1/n}} \le 1 - \delta$,

$$\mathcal{E}_{1_B}[1_{E^*}] - \mathcal{E}_{1_B}[\rho] \ge c(n,\delta) \, \|\rho\|_1^2 \, \tilde{\delta}(\rho, 1_{E^*})^2.$$

Questions and a conjecture

Questions

1. Can we obtain sharp stability estimates for $\mathcal{E}_W[\rho^*] - \mathcal{E}_W[\rho]$ for general densities ρ that are **not** characteristic functions?

2. Other than $\delta(\rho)$, are there any other natural ways to measure the asymmetry of ρ?

Special case: For $W = \mathcal{N}$, $\mathcal{E}_\mathcal{N}$ is *positive definite*: for all $f \in L^1 \cap L^\infty$ (can be sign-changing),

$$\mathcal{E}_\mathcal{N}[f] = \int f(-\Delta)^{-1} f dx = \|f\|^2_{\dot{H}^{-1}} \geq 0.$$

Perhaps we can use $\mathcal{E}_\mathcal{N}$ itself to measure the asymmetry of ρ?

Conjecture (Yan Guo)

Is it true that for all $\rho \in L^1_+(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$,

$$\mathcal{E}_\mathcal{N}[\rho^*] - \mathcal{E}_\mathcal{N}[\rho] \overset{?}{\geq} c(n) \inf_a \mathcal{E}_\mathcal{N}[T_a\rho - \rho^*]?$$

Here no normalization is required, as both sides scale in the same way.

Our result: Sharp stability estimate for general densities

Assumptions on potential. Assume $W \in C^1(\mathbb{R}^n \setminus \{0\}) \cap L^1_{loc}(\mathbb{R}^n)$ is radially decreasing, and $W(x) = w(r)$ satisfies

- $w'(r) < 0$ for all $r > 0$;
- There is $c > 0$ such that $w'(r) \le -cr$ for $r \in (0,1)$.

Example. The Riesz potentials W_k satisfies the assumptions for $k \in (-n, 2]$, but not for $k > 2$.

Theorem (Yan–Y. '20)

Let $\rho \in L^1_+(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, with $\operatorname{supp} \rho^* \subset B(0, R_*)$. Then for $n \ge 2$ we have:

$$\mathcal{E}_W[\rho^*] - \mathcal{E}_W[\rho] \ge c(n, W, R_*) \|\rho\|_{L^1}^{2+\frac{2}{n}} \|\rho\|_{L^\infty}^{-\frac{2}{n}} \delta(\rho)^2.$$

In particular, for $W = W_k$ with $k \in (-n, 2]$, $c(n, W, R_*) = c(n) R_*^{k-2}$.

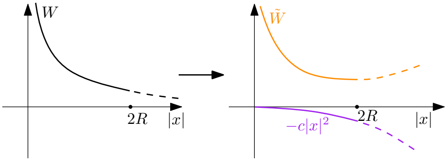
Note: For $W = W_k$, the power R_*^{k-2} in c is sharp. And when $\rho = 1_D$, it recovers the sharp estimates for characteristic functions.

Idea of proof: "take out" a small quadratic potential

- We assumed $\operatorname{supp} \rho^* \subset B(0, R_*)$, but $\operatorname{supp} \rho$ can be unbounded.
- A rather standard step: reduce the proof to the case when $\operatorname{supp} \rho \subset B(0, R)$ with $R = 20R_*$.
- Key idea: "take out" a small quadratic potential from W, and decompose it as

$$W(x) = -c_{R,W}|x|^2 + \tilde{W}(x),$$

where $c_{R,W} > 0$, and $\tilde{W}$ is radially decreasing in $B(0, 2R)$.



- This leads to

$$\mathcal{E}_W[\rho^*] - \mathcal{E}_W[\rho] = -c_{R,W}\left(\mathcal{E}_{|x|^2}[\rho^*] - \mathcal{E}_{|x|^2}[\rho]\right) + \left(\mathcal{E}_{\tilde{W}}[\rho^*] - \mathcal{E}_{\tilde{W}}[\rho]\right).$$

Idea of proof: magic of quadratic potentials

- Magic of interaction potential $|x|^2$: $\mathcal{E}_{|x|^2}$ is closely related to the second moment $M_2[\rho] := \int \rho |x|^2 dx$:

$$\mathcal{E}_{|x|^2}[\rho] = 2\|\rho\|_{L^1} M_2[T_{x_0}\rho],$$

where x_0 is the center of mass of ρ (so $T_{x_0}\rho$ has center of mass 0).

- This directly yields

$$\begin{aligned}\mathcal{E}_W[\rho^*] - \mathcal{E}_W[\rho] &\geq c_{R,W}\left(\mathcal{E}_{|x|^2}[\rho] - \mathcal{E}_{|x|^2}[\rho^*]\right)\\ &\geq 2c_{R,W}\|\rho\|_1 (M_2[T_{x_0}\rho] - M_2[\rho^*])\end{aligned}$$

- Compared to $\mathcal{E}_W$, it is much easier to deal with M_2 since it is linear in ρ: e.g. Lemou '16 proved $M_2[T_{x_0}\rho] - M_2[\rho^*] \gtrsim \delta(\rho)^2$.

Stability with respect to 2-Wasserstein distance

- Question: Other than the L^1 difference between $T_a\rho$ and ρ^*, are there any other natural ways to measure the asymmetry of ρ?
- Our second result is a stability estimate w.r.t 2-Wasserstein distance, which frequently arises in the study of interaction energy.

Theorem (Yan–Y. '20)

Let $\rho \in \mathcal{P}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, with $\operatorname{supp} \rho \subset B(0,R)$, and center of mass x_0. Then

$$\mathcal{E}_W[\rho^*] - \mathcal{E}_W[\rho] \geq c(W,R)\, W_2^2(T_{x_0}\rho, \rho^*).$$

In particular, if $W = W_k$ with $k \in (-n, 2]$, $c(W_k, R) = (2R)^{k-2}$.

- Here the power 2 on W_2 is sharp, so does the power $k-2$ on $c(W_k, R)$.
- Idea of proof: Due to the quadratic potential trick, only need to prove

$$M_2[\rho] - M_2[\rho^*] \geq W_2^2(\rho, \rho^*),$$

which is done by carefully building certain interpolation curve between ρ and ρ^* and track the the evolution of M_2 along the curve.

Guo's conjecture

- Back to Guo's conjecture: is it always true that

$$\mathcal{E}_{\mathcal{N}}[\rho^*] - \mathcal{E}_{\mathcal{N}}[\rho] \overset{?}{\geq} c(n) \inf_a \mathcal{E}_{\mathcal{N}}[T_a\rho - \rho^*]?$$

- A consequence of our Theorem 2:

$$\mathcal{E}_{\mathcal{N}}[\rho^*] - \mathcal{E}_{\mathcal{N}}[\rho] \geq c(n, R)\, W_2^2(T_{x_0}\rho, \rho^*).$$

- Remarkable result by Loeper '06 connecting W_2 distance with H^{-1}: For $\rho_1, \rho_2 \in \mathcal{P}_2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$,

$$\|\rho_1 - \rho_2\|^2_{\dot{H}^{-1}(\mathbb{R}^n)} \leq \max\{\|\rho_1\|_{L^\infty}, \|\rho_2\|_{L^\infty}\} W_2^2(\rho_1, \rho_2).$$

- This directly leads to the following:

Theorem (Yan–Y. '20)

Let $\rho \in L_1^+(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, with $\operatorname{supp}\rho \subset B(0, R)$, and center of mass x_0. Then

$$\mathcal{E}_{\mathcal{N}}[\rho^*] - \mathcal{E}_{\mathcal{N}}[\rho] \geq c(n) \frac{\|\rho\|_{L^1}}{\|\rho\|_{L^\infty} R^n} \mathcal{E}_{\mathcal{N}}[T_{x_0}\rho - \rho^*].$$

Guo's conjecture

Question: Can we get rid of the fraction in the inequality

$$\mathcal{E}_{\mathcal{N}}[\rho^*] - \mathcal{E}_{\mathcal{N}}[\rho] \geq c(n) \frac{\|\rho\|_1}{\|\rho\|_\infty R^n} \inf_a \mathcal{E}_{\mathcal{N}}[T_a\rho - \rho^*]?$$

Answer: Impossible if $n \geq 3$!

For any $n \geq 3$, we can construct an example with $\|\rho\|_1 \sim 1$, $R \sim 1$, $\|\rho\|_\infty = \epsilon^{-(\frac{n}{2}+1)} \gg 1$, such that

$$0 < \mathcal{E}_{\mathcal{N}}[\rho^*] - \mathcal{E}_{\mathcal{N}}[\rho] < C(n)\, \epsilon^{\frac{n}{2}-1} \inf_a \mathcal{E}_{\mathcal{N}}[T_a\rho - \rho^*].$$

Therefore for $n \geq 3$, Guo's conjecture is correct if and only if we allow $c(n)$ to also depend on $\frac{\|\rho\|_1}{\|\rho\|_\infty R^n}$.

Aggregation equation with (degenerate) diffusion

- In the rest of this talk, we consider

$$\rho_t = \underbrace{\Delta\rho^m}_{\text{local repulsion}} + \underbrace{\nabla\cdot(\rho\nabla(W*\rho))}_{\text{nonlocal interaction}} \quad \text{in } \mathbb{R}^d,$$

  where $m \geq 1$, W is radially symmetric, and $W(r)$ is increasing. (So W is an attractive interaction potential).

- The nonlinear diffusion term with $m > 1$ models the anti-overcrowding effect.

  (Boi-Capasso-Morale '00, Topaz-Bertozzi-Lewis '06)

- The global well-posedness v.s. blow-up criteria has been well studied. (e.g. If $W = \mathcal{N}$, then $m > 2 - \frac{2}{d}$ leads to global existence, whereas solution may blow-up if $m < 2 - \frac{2}{d}$.)
  (Blanchet-Carrillo-Laurencot '09, Bedrossian-Rodriguez-Bertozzi '11)

- In the cases that well-posedness is known, long time behavior of solution remains unclear.

Free energy functional

- The associated free energy functional plays an important role:

$$E[\rho] = \underbrace{\frac{1}{m-1}\int \rho^m dx}_{=:S[\rho] \text{ (entropy)}} + \underbrace{\frac{1}{2}\int \rho(\rho * W)dx}_{=:I[\rho] \text{ (interaction energy)}} .$$

(When $m = 1$, the first term becomes $\int \rho \log \rho dx$).

- Formally taking time derivatives along a solution, we have

$$\frac{d}{dt}E[\rho] = -\int \rho \left|\nabla(\frac{m}{m-1}\rho^{m-1} + \rho * W)\right|^2 dx \leq 0.$$

- Formally, the solution is a gradient flow of E in the metric space endowed by the 2-Wasserstein distance. (But rigorously justifying this requires certain convexity of W).
(Villani'03, Ambrosio-Gigli-Savare '08, Craig '17)

Main questions

In order to understand the long-time dynamics, a key step is to identify the stationary solutions.

Questions

1. For a given mass, does there exist a stationary solution?
2. Are they necessarily radially symmetric (up to a translation)?
3. If so, is it unique within the radial class?

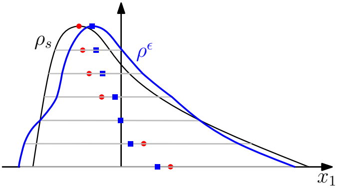
- Existence of stationary solution can be done by a concentration-compactness argument (Lions '84):
  - For power-law kernels $W = |x|^k/k$, there exists a global minimizer when $m > 1 - k/d$.
  - For $m > 2$, there exists a global minimizer for any attractive kernel (Bedrossian '11)
  - For $1 \leq m < 2$, criteria of existence v.s. non-existence are given in Carrillo–Delgadino–Patacchini '18.

Symmetric or not?

- By Riesz rearrangement inequality, a global minimizer of E must be radially decreasing. But must all stationary solutions be radial too?
- Using continuous Steiner symmetrization techniques, we gave a positive answer:

Theorem (Carrillo-Hittmeir-Volzone-Y. '19)

Let W be an attractive potential that is no more singular than Newtonian kernel. Any stationary solution $\rho_s \in L^1_+(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ must be radially decreasing up to a translation.



Unique or not?

Now that all stationary solutions are known to be radially decreasing (up to a translation), a natural question is whether there is uniqueness within this class.

Questions

For attractive kernels, for a given mass, must stationary solutions be unique?

Uniqueness results are only known in the following cases:

- $W = \mathcal{N}$ is the Newtonian potential in $\mathbb{R}^d$, and m is in the diffusion dominated regime. (Lieb–Yau '87)
- $W = \mathcal{N} * h$, where $h \geq 0$ is radially decreasing. (Kim–Yao '12)
- W is an attractive Riesz potential, and m is in the diffusion dominated regime. (Carrillo–Hoffmann–Mainini–Volzone '18, Calvez–Carrillo–Hoffmann '19)
- $m = 2$ and W is a C^2 attractive potential. (Burger–Di Francesco–Franek '13 and Kaib '17)

Uniqueness for $m \geq 2$

Theorem (Delgadino–Yan–Y., '19)

Let $m \geq 2$ and $W \in C^1(\mathbb{R}^d \setminus \{0\})$ be a locally integrable attractive potential. Then for any given mass, there is at most one stationary solution in $L^1_+(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ up to a translation.

Idea of proof (when the gradient flow structure is rigorous):

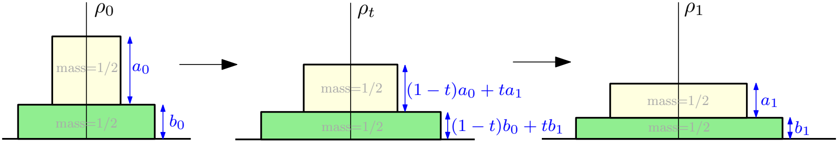
- If ρ_0, ρ_1 are two radial stationary solutions with the same mass, we will construct a curve $\{\rho_t\}_{t=0}^1$ connecting them, such that the energy along this curve is strictly convex for $m \geq 2$.
- Therefore ρ_0 and ρ_1 can't be both critical points!

But how to find such an interpolation curve, if it exists at all?
(Note: linear interpolation or W_2 geodesic do not work!)

The main idea of our proof is the construction of a novel interpolation curve between two radially decreasing ρ_0, ρ_1.

Construction of the interpolation curve

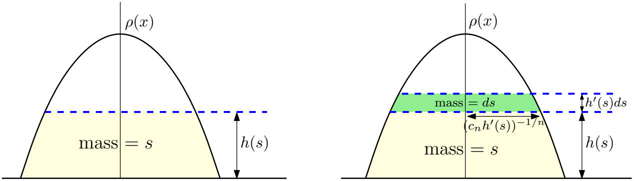
- Suppose ρ_0, ρ_1 are two radially decreasing step functions having N horizontal layers with mass $1/N$ in each layer.
- ρ_t is constructed by deforming each layer so that its height changes linearly, and meanwhile adjust the width so that the mass in each layer remains constant.



- Note that ρ_t is neither the linear interpolation between ρ_0 and ρ_1, nor the geodesic in 2-Wasserstein metric.
- For two radially decreasing function, the interpolation can be seen as a $N \to \infty$ limit of the step-function case.

Construction of the interpolation curve

- For a radially decreasing function ρ with mass 1, define its "height function with respect to mass" $h(s)$ as the left figure:



- $h : [0,1] \to [0, \|\rho\|_\infty]$ is increasing and convex in s. Also, ρ can be uniquely recovered from h (see the right figure):

$$\rho(x) = \int_0^1 1_{B(0,(c_d h'(s))^{-1/d})}(x) h'(s) ds$$

- Let h_0, h_1 be the height function for ρ_0, ρ_1. For $t \in (0,1)$, let

$$h_t(s) = (1-t)h_0(s) + t h_1(s),$$

and let ρ_t be determined by the height function h_t.

Convexity of energy

- For the entropy, an explicit computation gives

$$
\begin{aligned}
S[\rho] &= \int_{\mathbb{R}^d} \frac{1}{m-1} \rho^m dx \\
&= \int_0^{\max \rho} \frac{m}{m-1} h^{m-1} |\{\rho > h\}| dh \\
&= \int_0^1 \frac{m}{m-1} h(s)^{m-1} ds,
\end{aligned}
$$

thus
$$\frac{d^2}{dt^2} S[\rho_t] = m(m-2) \int_0^1 (h_1 - h_0)^2 h_t(s)^{m-3} ds,$$

which is non-negative if and only if $m \geq 2$.

- Key step: the interaction energy $I[\rho] = \int \rho(\rho * W) dx$ is strictly convex along the curve for all attractive potential W. (proof quite technical in multi-dimension)

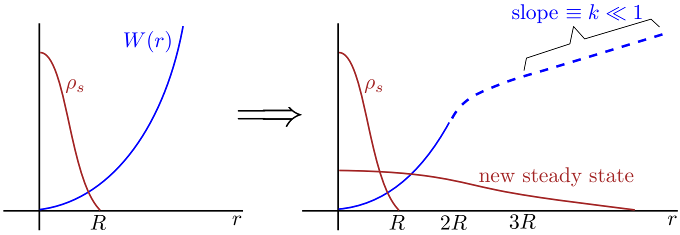
Non-uniqueness for $1 < m < 2$

For all $m < 2$, our uniqueness proof fails. But is there really non-uniqueness in this regime?

Theorem (Delgadino–Yan–Y., '19)

Let $1 < m < 2$. There exists a smooth attractive kernel W which gives an infinite sequence of radially decreasing stationary solutions with the same mass.

- It shows that the uniqueness result for $m \geq 2$ is indeed sharp.



- Idea: If we modify the tail of W and let the slope be $0 < k \ll 1$, it leads to a new stationary solution different from ρ_s.

Thank you for your attention!