Rigidity for measurable sets

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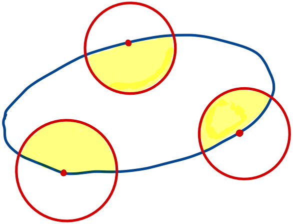
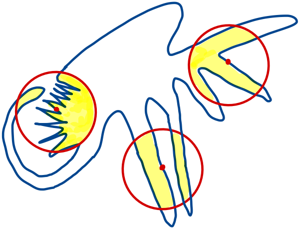
The problem

Characterize sets $\Omega \subset \mathbb{R}^d$, with $|\Omega| < +\infty$, such that, for a given radius $r > 0$,

$$|\Omega \cap B_r(x)| = c > 0 \quad \text{for } x \in \partial\Omega$$





OUTLINE

I. *Historical overview*:
from 1932 to 2018

II. *The main result*:
rigidity for arbitrary measurable sets

III. *A look at the proof*:
rebuild of moving planes methods in a measure theoretic framework

IV. *Further remarks and open questions*

[Cimmino '32] Is it possible to characterize C^0 surfaces Γ in $\mathbb{R}^3$ such that, for any sufficiently small radius $r > 0$,

$$\mathscr{H}^2(\Gamma^+ \cap \partial B_r(x)) = \mathscr{H}^2(\Gamma^- \cap \partial B_r(x)) = 2\pi r^2 \qquad \forall x \in \Gamma?$$

[Nitsche '95]
The only (smooth) surfaces with this property are the plane and the helicoid.

[Meeks-Rosenberg '05]
The plane and the helicoid are the unique simply connected minimal surfaces embedded in $\mathbb{R}^3$.

Reprise in the first 2000s

- The *Matzoh Ball Soup Problem*: Klamkin 1964, Zalcman 1987 solved by Alessandrini in 1990

  $$u_t = \Delta u \text{ in } \Omega \times (0, \infty).$$

  Initially, u = 0. On the boundary u = 1.
  The sphere is the only bounded solid having the property of invariant equipotential surfaces.
  [Magnanini-Sakaguchi '02] only one surface is enough.

- ∂U is isothermal $\Leftrightarrow$ U is B-dense [Magnanini-Prajapat-Sakaguchi '06]
  i.e.

  $$|U \cap (x + rB)| = c(r) \qquad \forall x \in \partial U, \quad \forall r_0 > r > 0.$$

- **Theorem** [Magnanini-Marini '16] U, K convex,
  If U with $|U| \in (0, +\infty)$ is K-dense, then U and K are homothetic ellipsoids.

Relationship with the mean curvature

- Expansion [Hulin-Troyanov '03]

$$|\Omega \cap B_r(x)| = \frac{1}{2}\omega_d r^d - \frac{d-1}{2(d+1)}\omega_{d-1} H_\Omega(x) r^{d+1} + O(r^{d+2}),$$

- Applications in Geometry Processing: integral invariant estimator of the mean curvature.

Theorem [Alexandrov '58, Delgadino-Maggi '19]

- Balls are the unique bounded connected C^2 sets in $\mathbb{R}^d$ having constant mean curvature.
- Finite unions of equal balls are the unique sets of finite Lebesgue measure and finite perimeter in $\mathbb{R}^d$ having constant distributional mean curvature.

Rigidity of measurable sets: r-criticality

By definition, for a FIXED positive radius $r > 0$ we say that Ω is r-critical if,

$$|\Omega \cap B_r(x)| = c \qquad \forall x \in \partial^* \Omega$$

where $\partial^*\Omega$ is the essential boundary of Ω, i.e. the set of points $x \in \mathbb{R}^d$ such that

$$\limsup_{\rho \to 0} \frac{|\Omega \cap B_\rho(x)|}{\rho^d} > 0 \qquad \text{and} \qquad \limsup_{\rho \to 0} \frac{|(\mathbb{R}^d \setminus \Omega) \cap B_\rho(x)|}{\rho^d} > 0 \, .$$

- Remark: $\partial^*\Omega \subseteq \partial\Omega$.

The variational interpretation

- Constant mean curvature sets are stationary for the isoperimetric inequality [De Giorgi '58]

$$\frac{\mathrm{Per}(\Omega)^{\frac{1}{d-1}}}{|\Omega|^{\frac{1}{d}}} \geq \frac{\mathrm{Per}(B)^{\frac{1}{d-1}}}{|B|^{\frac{1}{d}}}.$$

- r-critical sets are stationary for the rearrangement inequality [Riesz '30]

$$\int_\Omega \int_\Omega h(x-y)\,dx\,dy \leq \int_{\Omega^*} \int_{\Omega^*} h(x-y)\,dx\,dy$$

holding for any radially symmetric, decreasing, nonnegative function h, when one chooses $h = \chi_{B_r(0)}$.

Equivalently: $2|\Omega \cap B_r(x)| - \omega_d r^d = |\Omega \cap B_r(x)| - |\Omega^c \cap B_r(x)| \quad \Rightarrow$
r-critical sets are stationary for the nonlocal r-perimeter

$$r - \mathrm{Per}(\Omega) = \int_\Omega \int_{\Omega^c} \chi_{|x-y|<r}\,dx\,dy\,,$$

[Chambolle-Morini-Ponsiglione '15, Mazon-Rossi-Toledo '19] .

The nonlocal side

Fractional perimeter [Caffarelli-Souganidis '08, Caffarelli-Roquejoffre-Savin '10]

$$P_s(\Omega) = \int_\Omega \int_{\Omega^c} \frac{1}{|x-y|^{d+2s}}\, dx\, dy \qquad s \in (0, 1/2)$$

is minimized by balls under volume constraint [Frank-Seiringer '08].

Critical sets have constant fractional mean curvature

$$H_{\Omega,s}(x) := \frac{1}{\omega_{d-2}} PV \int_{\mathbb{R}^d} \frac{\chi_{\Omega^c}(y) - \chi_\Omega(y)}{|y-x|^{d+2s}}\, dy$$

Theorem [Ciraolo-Figalli-Maggi-Novaga, Cabré-Fall-Solá Morales-Weth '18]
If Ω is a bounded open set of class $C^{1,2s}$ and $H_{\Omega,s} = c$ on $\partial\Omega$, then Ω is a ball.

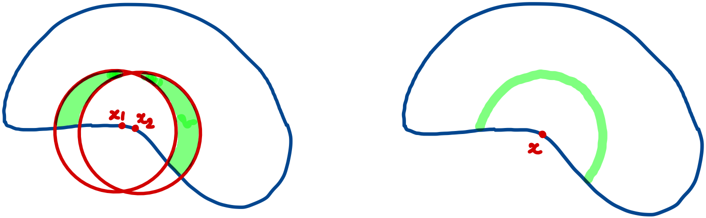
Why our problem is different: the kernel equals χ_{B_r}!

- Bounded kernel allows to deal with measurable sets.
- The flatness (level sets of positive measure): a point can not distinguish how far is from the center of a kernel!
- Compactly supported kernel produces short-range non-local effects.
- Discontinuous kernel enhances the need for some "transmission condition", companion to r-criticality.

The companion condition: *r*-nondegeneracy

By definition, we say that Ω is not r-degenerate if

$$\inf_{x_1,x_2\in\partial^*\Omega} \frac{\left|\Omega\cap(B_r(x_1)\Delta B_r(x_2))\right|}{\|x_1-x_2\|} > 0 .$$



- A sufficient condition is: $\inf_{x\in\mathscr{U}_\varepsilon(\partial^*\Omega)} |D\chi_{B_r(x)}|(\Omega^{(1)}) > 0$.
- Any open connected set is not r-degenerate for any $r <$ diameter.

Theorem [B.-Fragalà '21]
Let Ω be a measurable set with finite Lebesgue measure in $\mathbb{R}^d$, and let $r > 0$. Assume that Ω is r-critical and is not r-degenerate, i.e.

$$|\Omega \cap B_r(x)| = c > 0 \quad \forall x \in \partial^* \Omega, \qquad \inf_{x_1, x_2 \in \partial^* \Omega} \frac{\left|\Omega \cap (B_r(x_1) \Delta B_r(x_2))\right|}{\|x_1 - x_2\|} > 0.$$

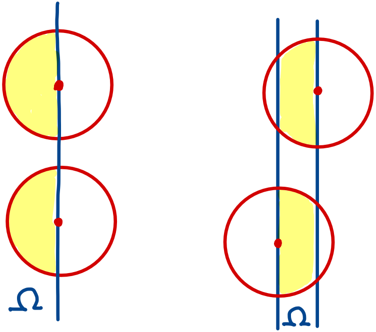
Then Ω is equivalent to a finite union of equal balls of radius $R > \frac{r}{2}$, at mutual distance larger than or equal to r.

- Bubbling is tuned by the initial choice of r.
- The analogue result holds true with ellipsoids in place of balls.

Some sets which escape from rigidity (though being r-critical)

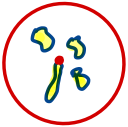
Sets of infinite measure

- *Halfspaces or strips:*

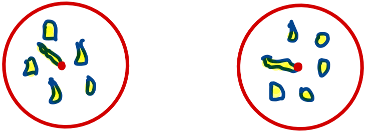


Sets which are r-degenerate

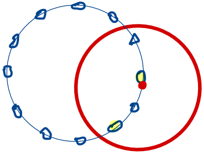
- *Small sets:* any measurable set Ω with $\mathrm{diam}(\Omega) \le r$.



- *Small sets at large distance:* the union of two measurable sets Ω_1, Ω_2 of equal measure, with $\mathrm{diam}(\Omega_i) \le r$ and $\mathrm{dist}(\Omega_1, \Omega_2) \ge r$.



- *Spaced small sets at small distance:* the union of a family of measurable sets of equal measure suitably "equidistributed" along a circle in $\mathbb{R}^2$.



Rigidity for sets enjoying some regularity

Corollary (the case of open sets)

Let Ω be an open set with finite Lebesgue measure in $\mathbb{R}^d$, and let $r > 0$. Assume that

$$|\Omega \cap B_r(x)| = c \qquad \forall x \in \partial\Omega.$$

If Ω is connected and $r < \mathrm{diam}(\Omega)$, then Ω is a ball.

If Ω has multiple components Ω_i and $r < \inf_i\{\mathrm{diam}(\Omega_i)\}$, then Ω is a finite union of equal balls of radius $R > \frac{r}{2}$, at distance larger than or equal to r.

Corollary (the case of sets with finite perimeter)

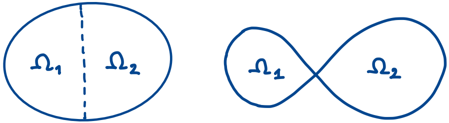
Let Ω be set with finite Lebesgue measure and finite perimeter in $\mathbb{R}^d$, and let $r > 0$. Assume that

$$|\Omega \cap B_r(x)| = c \qquad \text{for } \mathscr{H}^{d-1}\text{-a.e. } x \in \mathscr{F}\Omega.$$

If Ω is indecomposable and $r < \operatorname{diam}(\Omega)$, then Ω is a ball.

If Ω has multiple components Ω_i and $r < \inf_i\{\operatorname{diam}(\Omega_i)\}$, then Ω is a finite union of equal balls of radius $R > \frac{r}{2}$, at distance larger than or equal to r.

- $\mathscr{F}\Omega$ is the reduced boundary, i.e. the collection of points $x \in \operatorname{supp}(D\chi_\Omega)$ such that the generalized normal exists
- Ω is indecomposable if it is not possible to find a partition (Ω_1, Ω_2) of Ω such that $|\Omega_1| > 0$, $|\Omega_2| > 0$ and $\operatorname{Per}(\Omega) = \operatorname{Per}(\Omega_1) + \operatorname{Per}(\Omega_2)$



The method of moving planes

- Alexandrov idea: if the mean curvature of $\partial\Omega$ is constant, Ω has a plane of symmetry in every direction. This can be proved by moving a given plane H_0 to parallel positions H_t and "reflecting as far as possible".

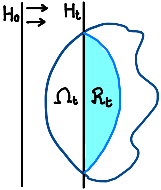
- Applied to overdetermined PDEs [Serrin 71, Gidas-Ni-Nirenberg '79, Caffarelli-Gidas-Spruck 89, Berestycki-Nirenberg '91]

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \\ \frac{\partial u}{\partial n} = constant & \text{on } \partial\Omega \end{cases}$$

then Ω has to be a ball.

How it works

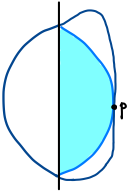
- *Start*: For $t \ll 1$, the reflection $\mathscr{R}_t$ of the cap Ω_t is contained in Ω.

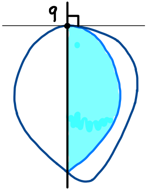


- *Stop*: At the stopping time: 1) interior tangency 2) orthogonality.

  Crucial fact: the boundary is locally the graph of a function u, and

$$H_\Omega = \frac{1}{d-1}\mathrm{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) \quad \Rightarrow \quad \text{a PDE can be exploited!}$$

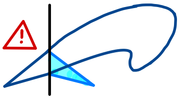




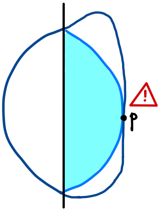
- *Conclusion*: Ω has a plane of symmetry in every direction $\Rightarrow$ it is a ball.

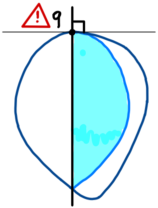
Obstacles in the measurable setting

- *Start*: For $t \ll 1$, $\mathscr{R}_t$ is not contained into Ω

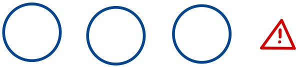


- *Stop*: For nonsmooth sets, tangency and orthogonality make no sense. (No local graphicality and no PDE!)



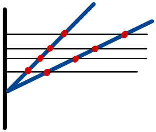


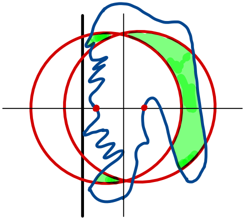
- *Conclusion:* multiple balls have not a plane of symmetry in every direction.



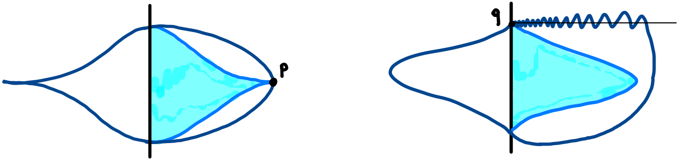
Rebuilding of the main steps

- *Start* : For $t \ll 1$, $\mathscr{R}_t \subset \Omega$ and $\Omega_t \cup \mathscr{R}_t$ is Steiner symmetric about H_t.
  By contradiction, using r-criticality and r-nondegeneracy.

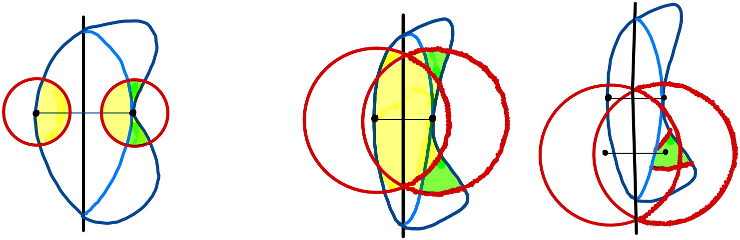




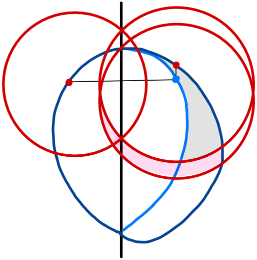
- *Stop*: Replace interior tangency and orthogonality by:
  - Away contact: $p \in \left[\overline{\partial^* \mathscr{R}_t} \cap \overline{\partial^* \Omega}\right] \setminus H_t$
  - Close contact: $q = \lim q_{1,n} = \lim q_{2,n} \in H_t,\ q_{i,n} \in \overline{\partial^* \Omega} \cap \{q + t\nu \,:\, t \in \mathbb{R}\}$



- Away contact implies local symmetry
  (based on symmetry inside r-moons, and on a ping-pong game)



- Close contact is not possible without away contact
  (based on a local analysis which exploits nondegeneracy)



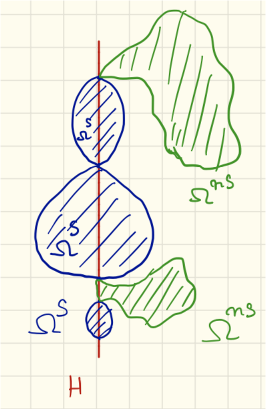
- *Conclusion*

  Under some connectedness hyp: Ω is symmetric about H_t

  $\Rightarrow$ Ω is a ball since it has a plane of symmetry in every direction

  Otherwise: $\Omega = \Omega^s \sqcup \Omega^{ns}$, with Ω^s open (the Steiner symmetric part of Ω)

  $\Rightarrow$ Ω^s is a finite union of balls of equal radii, while Ω^{ns} is negligible.



- What about decreasing radial kernels h other than $\chi_{B_r(0)}$?
- What about measuring area intersection with spheres

$$\forall x \in \partial\Omega, \quad \mathscr{H}^{d-1}(\partial B_r(x) \cap \Omega) = c?$$

- What is our motivation?

Polygonal isoperimetric inequalities...

Thank you for your attention!