Zeros of Fekete polynomials

(joint work with Y. Lamzouri and M. Munsch)

University of Bristol

Outline

- Main theme: “Law of small numbers”

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- Zeros of polynomials: classical results
- Zeros of Fekete polynomials: what we know
- Some new results and ideas of the proof
- Speculations and more results

Question

Given a polynomial $P_N(z) = \sum_{n=0}^{N} a_n z^n$ with $a_n \in \mathbb{C}$, what can we say about the set

$$\{z \in \mathbb{C} | P_N(z) = 0\}?$$

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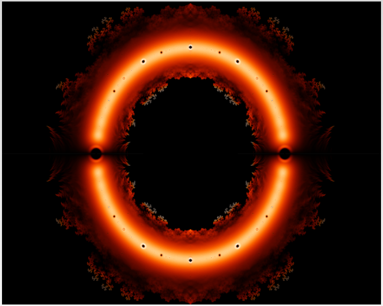

Figure: Roots of polynomials with ± 1 coefficients of degree ≤ 24 (by S. Derbyshire)

Angular distribution of the roots

Given an arc I on the unit circle, let $N(I;P)$ denote the number of zeros $\alpha_j = r_j e^{i\theta_j}$ of P such that $e^{i\theta_j}$ lies on the arc I.

Theorem (Erdős-Turan)

We have the estimate

$$\mathcal{D}(P) := \max_I \left| N(I;P) - \frac{|I|}{2\pi} N \right| \leq \frac{8}{\pi} \sqrt{N h(P)},$$

where

$$h(P) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{|P(e^{i\theta})|}{\sqrt{|a_0|}} d\theta, \qquad \log^+ x = \max(0, \log x).$$

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- Borwein-Erdélyi-Kós: $N_R(P) \ll \sqrt{N}$ (sharp)
- Littlewood-Offord: except for $o(2^{N+1})$ choices of Littlewood polynomials, expected $N_R(P) \ll \log^2 N$
- Kac-Rice: random polynomials with Gaussian coefficients, expected $N_R(P) \sim \frac{2}{\pi} \log N$
- Erdős-Offord: generalization to random Littlewood polynomials, $N_R(P) \sim \frac{2}{\pi} \log N$

Fekete polynomials and L- functions

- $$F_p(t) := \sum_{n=0}^{p-1} \left(\frac{n}{p}\right) t^n.$$
- For D a positive fundamental discriminant, set
$$F_D(t) := \sum_{n=0}^{D-1} \left(\frac{D}{n}\right) t^n.$$
- $$L(s,\chi_D)\Gamma(s) = \int_0^{+\infty} F_D(e^{-x})(1-e^{-|D|x})^{-1} x^{s-1} dx, \qquad \Re(s) > 0$$
where $L(s,\chi_D) = \sum_{n\geq 1} \frac{\chi_D(n)}{n^s}$.

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Fekete: If $F_D(t)$ has no real zeros t with $0 < t < 1$, then $L(s, \left(\frac{D}{\cdot}\right))$ has no real zeros $s > 0$ (*Siegel zero*).

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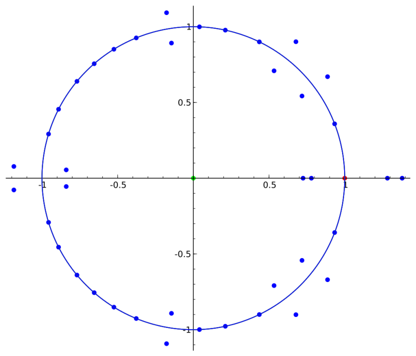
Zeros of Fekete polynomials



Roots of the Fekete polynomial F_{43}.

Complex zeros of Fekete polynomials

Let $\xi_p = e^{2\pi i/p}$ and $H_p(z) = z^{-p/2}F_p(z)$, then for $1 \leq k \leq p-1$:

- $$H_p(\xi_p) = 2\sum_{a=1}^{(p-1)/2} \left(\frac{a}{p}\right)\cos((2a-p)\pi t)$$

- $$H_p(\xi_p^k) = (-1)^k\left(\frac{k}{p}\right)H_p(\xi_p)$$

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- $$N_R(F_p(e^{i\theta})) \geq |\{k \leq p-2| \left(\frac{k}{p}\right) = \left(\frac{k+1}{p}\right)\}| = \frac{p-3}{2}$$

- Up to $p < 500$, those are all zeros! One finds more for $p = 661$.

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Theorem (Conrey-Granville-Poonen-Soundararajan, 1999)

There exists a constant $1/2 < \kappa_0 < 1$ such that, as $p \to +\infty$

$$\#\{z : |z| = 1 \text{ and } F_p(z) = 0\} \sim \kappa_0 p.$$

- $0.500667 < k_0 < 0.500883$

Problem [Littlewood]: Given finite $\mathcal{A} \in \mathbb{Z}$, let

$$f_{\mathcal{A}}(z) = \sum_{k \in A} \cos(kz).$$

What is $\min_{|\mathcal{A}|=n} N_R(f_{\mathcal{A}})$?

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Borwein-Erdélyi-Fergusson-Lockhart, Juckevicius-Sahasrabudhe:

$$(\log\log\log n)^{1-\epsilon} \ll \min_{|\mathcal{A}|=n} N_R(f_{\mathcal{A}}) \ll n^{2/3}(\log n)^{1/3}$$

Real zeros of Fekete polynomials

Conjecture (Baker-Montgomery, Conrey-Granville-Poonen-Soundararajan)

"It seems likely that for almost all $D \in \mathbb{N}$, the corresponding $F_D(t)$ has

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Theorem (Baker-Montgomery, 92)

For almost all discriminants, the number of zeros of $F_D(t)$ in $(0, 1)$ goes to $+\infty$ with D.

Lower bound

Let $\mathcal{F}(x)$ be the set of fundamental discriminants $|D| \leq x$

Theorem (K-L-M, 20+)

For almost all fundamental discriminants $D \in \mathcal{F}(x)$, $F_D(x)$ *has*

$$\gg \frac{\log\log D}{\log\log\log\log D}$$

zeros in $(0, 1)$.

What about upper bounds?

Reminder (Erdélyi, survey): the number of real zeros of F_D is $\ll \sqrt{D}$.

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- *For* $\gg N^{2/3}$ *fundamental discriminants* $D \leq N$, $F_D(z)$ *has* $O(D^{1/4})$ *real zeros.*

Lower bound: ideas of the proof

$$L_D(s)\Gamma(s)\left(\frac{L_D'}{L_D}(s)+\frac{\Gamma'}{\Gamma}(s)\right)=\int_0^\infty F_D(e^{-x})(1-e^{-Dx})^{-1}x^{s-1}(\log x)dx.$$

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Karlin: $\mathcal{S}(f,-\infty,+\infty)\geq\mathcal{S}(\mathcal{L},0,+\infty)$, where $\mathcal{L}(s)$ is the Laplace transform of f.

Plan of the proof

- Statistical study of the functions $(L_D'/L_D)(s)$ for s close to $1/2$.
- Approximation by a short sum over primes for almost all discriminants.
- Define a random model for this sum using random variables indexed by primes.
- Convergence to a suitable normal variable.
- Good choice of points s_i making random variables at these points "independent" and creating change of signs.
- Discrepancy between random model and the approximation.

Approximation of $-L'/L(s, \chi_D)$: First part

Let $1/2 + (\log\log x)^2/\log x \leq \sigma \leq 1$, and put $A = 12/(\sigma - 1/2)$ and $y = (\log |D|)^A$. Then for all fundamental discriminants $|D| \leq x$ except for a set $\mathcal{E}(x)$ with cardinality

$$|\mathcal{E}(x)| \ll x^{1-(\sigma-1/2)/5}(\log x)^{72},$$

we have

$$-\frac{L'}{L}(s, \chi_D) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n} \chi_D(n) e^{-n/y} + O\left(\frac{1}{\log |D|}\right),$$

where $\Lambda(D) = \log p$ is $D = p^k$ and 0 otherwise.
Proof uses standard zero density estimate, contour shifting.

Approximation of $-L'/L(s,\chi_D)$: Short sum over primes

Proposition (K-L-M, 20+)

Let $s = 1/2 + 1/g(x)$ *where* $(\log\log x)^2 \le g(x) \le \sqrt{\log x}/(\log\log x)^2$. *Put*

$$u(s) = \exp\left(\frac{g(x)}{\log\log g(x)}\right), \text{ and } v(s) = \exp\big(g(x)\log\log g(x)\big).$$

Then for almost all fundamental discriminants $|D| \le x$ *we have*

$$\left|\frac{L'}{L}(s,\chi_D) + \sum_{u(s)\le p\le v(s)} \frac{\chi_D(p)\log p}{p^s}\right| \le 2g(x).$$

High moments estimates via the large sieve inequality.

Random model for the short sum over primes

Consider a sequence of independent random variables $\{X(p)\}_p$, , indexed by the primes, and taking values in $\{0, -1, 1\}$ such that

$$\mathbb{P}(\mathbb{X}(p) = -1) = \mathbb{P}(\mathbb{X}(p) = 1) = \frac{p}{2(p+1)}, \mathbb{P}(\mathbb{X}(p) = 0) = \frac{1}{p+1}.$$

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The random variable

$$\mathbb{X}(s) = \sum_{u(s)<p<v(s)} \frac{\mathbb{X}(p) \log p}{p^s}$$

converges to a centered normal distribution with variance

$$\sum_{u(s)<p<v(s)} \frac{(\log p)^2 p}{p^{2s}(p+1)} = \frac{(1+o(1))}{(2s-1)^2}.$$

Choice of the points

Take $R := \left\lfloor \frac{\delta \log_2 x}{3 \log_4 x} \right\rfloor$, for $R/5 \leq r \leq R$ and let

$$s_r := \frac{1}{2} + \frac{1}{(\log r)^{3r}},$$

$(\log x)^{-\delta} < s_r - 1/2 < (\log x)^{-\delta/5}$ for all $R/5 \leq r \leq R$.

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We also define u_r and v_r as before. This gives a sequence of points s_r such that $2 \leq u_r < v_r < u_{r+1} < v_{r+1} \leq x$ for all $R/5 \leq r \leq R$.

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Goal: Show that there are many sign changes at these points.

"Large values" in both sides

We consider for $\mathrm{s} = (s_{R/5}, \ldots, s_R)$ the probabilistic random vector

$$L_R(\mathrm{s}, \mathbb{X}) = \left(\sum_{u_r < p < v_r} \frac{\mathbb{X}(p) \log p}{p^{s_r}} \right)_{R/5 \leq r \leq R}.$$

$$\begin{aligned} \mathbb{P}\left(\mathbb{X}(s_r) > 4(\log r)^{3r}\right) &= \mathbb{P}\left(\mathbb{X}(s_r) < -4(\log r)^{3r}\right) \\ &= (1 + o(1)) \frac{1}{\sqrt{2\pi}} \int_a^\infty e^{-x^2/2} dx > 0. \end{aligned}$$

Moreover, distinct points s_r produce non overlapping intervals of primes that do not see each other ="Independence".

Discrepancy estimate

Define the vector of Dirichlet polynomials:

$$L_R(s, \chi_D) = \left(\sum_{u_r < p < v_r} \frac{\chi_D(p) \log p}{p^{s_r}} \right)_{R/5 \le r \le R}.$$

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$$D(\mathrm{s}) = \sup_{\mathcal{R}} \left| \frac{1}{|\mathcal{F}(x)|} \left| \left\{ D \in \mathcal{F}(x) : L_R(\mathrm{s}, \chi_D) \in \mathcal{R} \right\} \right| - \mathbb{P}\big(L_R(\mathrm{s}, \mathbb{X}) \in \mathcal{R}\big) \right|$$

where the supremum is taken over all rectangular boxes (possibly unbounded) $\mathcal{R} \subset \mathbb{R}^R$ with sides parallel to coordinates axes.

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Using techniques (Fourier analytic) of Lamzouri-Lester-Radziwiłł :

Proposition

We have

$$D(\mathrm{s}) \ll \frac{1}{(\log x)^{1/5}}.$$

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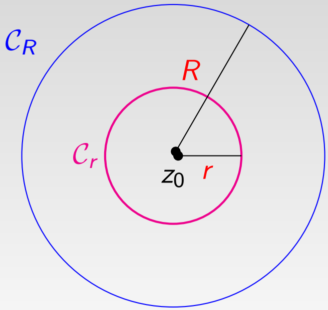
There are a lot of sign changes for $L'_D(s_r)/L_D(s_r)$.

Jensen's formula and concentric circles

$$\#\{\text{zeros of P inside } \mathcal{C}_r\} \leq \left(\log \frac{\max_{|z-z_0|=R} |P(z)|}{|P(z_0)|} \right) / \log(R/r).$$



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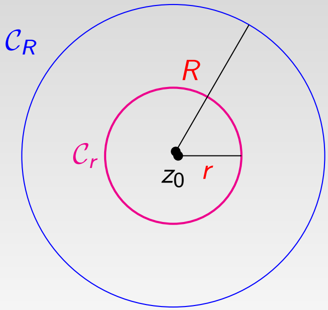


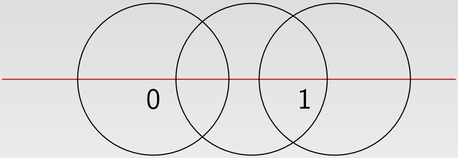
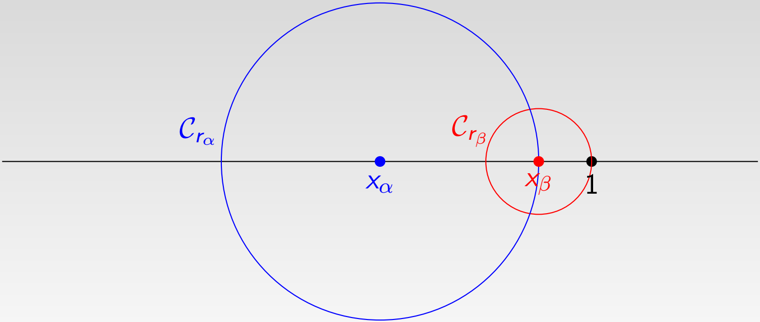


Figure: **Covering the line**

Covering with two circles

Let: $x_\alpha = \exp(-1/x^\alpha)$ and $x_\beta = \exp(-1/x^\beta)$ for $0 < \alpha < 1/2 < \beta < 1$.

<u>Idea</u>: For many discriminants D, F_D is not too small simultaneously at x_α and x_β.



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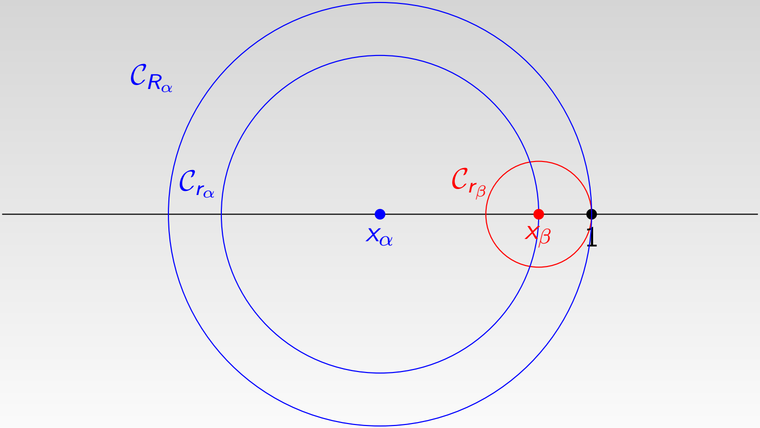

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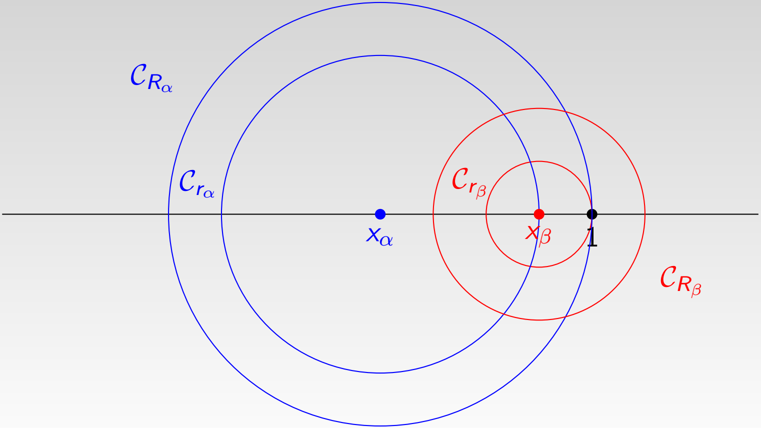

Simultaneous "size" and mixed moments

Consider $\mathcal{L}(x)$ the set of discrimants such that

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We show that $\mathcal{L}(x)$ is relatively "large" by bounding the mixed moments:

$$S_1(\alpha_1, \alpha_2, \ldots, \alpha_k) = \sum_{D \in \mathcal{F}(x)} F_D(x_{\alpha_1}) F_D(x_{\alpha_2}) \ldots F_D(x_{\alpha_k}),$$

and

$$S_2(\alpha_1, \alpha_2, \ldots, \alpha_k) = \sum_{D \in \mathcal{F}(x)} (F_D(x_{\alpha_1}) F_D(x_{\alpha_2}) \ldots F_D(x_{\alpha_k}))^2.$$

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Proposition

Under the conditions $1/2 < \alpha + \beta < 1$ the following inequality holds $S_1(\alpha, \beta) \gg x^{1+\alpha/2+\beta/2} \log x$ and $S_2(\alpha, \beta) \ll x^{1/2+2\alpha+2\beta+\epsilon}$.

In practice we use more circles and higher mixed moments.

Fekete polynomials with no real zeros

More general Heuristic [Sarnak]: there exist infinitely many D such that F_D does not vanish on $(0, 1)$.

Sarnak in his letter to Bachmat: GL_n analogues of Fekete polynomials and asks about the existence of such polynomials without zeros on subintervals of $(0, 1)$.

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$$F_D(t) := \sum_{n=0}^{D-1} \left(\frac{D}{n}\right) t^n = (1-t) \sum_{k \leq D-1} S_k(D) t^k,$$

where $S_k(D) = \sum_{n \leq k} \left(\frac{D}{n}\right)$.

Polynomials with no real zeros in some interval

Proposition (K-L-M, 20+)

There exists at least $x^{1-o(1)}$ *fundamental discriminants* $0 < D \leq x$ *such that* $F_D(z)$ *has no zeros in the interval*

$$\left[0, 1 - \frac{1}{(\log x)^{\sqrt{e}+\varepsilon}}\right].$$

Borwein-Erdélyi-Kós: Any Littlewood polynomial has at most $O(\log x)$ zeros in $(0, 1 - 1/\log x)$.