Tiling the integers with translates of one tile: the Coven-Meyerowitz tiling conditions for three prime factors

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May 2021

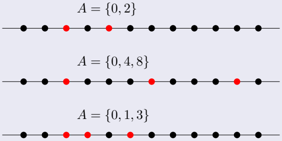
Tiling the integers: an introduction

Tiling the integers with a finite set

Let $A \subset \mathbb{Z}$ be a finite set. We say that A *tiles* $\mathbb{Z}$ *by translations* if $\mathbb{Z}$ can be covered by a union of disjoint translates of A. (There is an infinite set $T \subset \mathbb{Z}$ such that every $x \in \mathbb{Z}$ can be uniquely represented as $x = a + t$, with $a \in A$, $t \in T$.)

Tiling the integers with a finite set

Let $A \subset \mathbb{Z}$ be a finite set. We say that A *tiles* $\mathbb{Z}$ *by translations* if $\mathbb{Z}$ can be covered by a union of disjoint translates of A. (There is an infinite set $T \subset \mathbb{Z}$ such that every $x \in \mathbb{Z}$ can be uniquely represented as $x = a + t$, with $a \in A$, $t \in T$.)



$A = \{0, 2\}$ and $A = \{0, 4, 8\}$ tile $\mathbb{Z}$; $A = \{0, 1, 3\}$ does not.

How to determine whether a given A tiles the integers?

Periodicity

Newman (1977): all tilings of $\mathbb{Z}$ by a finite set A are periodic, with period M bounded by $2^{\max(A)-\min(A)}$.

- In principle, this makes the tiling property computationally decidable. However, the bound on M can be very large even if the cardinality of A is small.

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- In principle, this makes the tiling property computationally decidable. However, the bound on M can be very large even if the cardinality of A is small.
- But it does reduce the problem to the study of tilings of finite cyclic groups $\mathbb{Z}_M = \{0, 1, \ldots, M-1\}$, with addition mod M. Notation:

$$A \oplus B = \mathbb{Z}_M.$$

Further reductions

- Tijdeman (1993): if A tiles the integers, then it also tiles a finite cyclic group $\mathbb{Z}_M$, where M has the same prime factors as $|A|$.

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- Sands (1979): let $A, B \subset \mathbb{Z}_M$. Then $A \oplus B = \mathbb{Z}_M$ if and only if $|A|\,|B| = M$ and

$$\mathrm{Div}(A) \cap \mathrm{Div}(B) = \{M\},$$

where $\mathrm{Div}(A) = \{(a - a', M) : \ a, a' \in A\}$.

Geometric representation of tilings

Suppose that $A \oplus B = \mathbb{Z}_M$, with $M = \prod_{i=1}^{K} p_i^{n_i}$, p_i distinct primes, $n_i \geq 1$. By the Sun-Tsze Remainder Theorem (aka the "Chinese Remainder Theorem"), we have

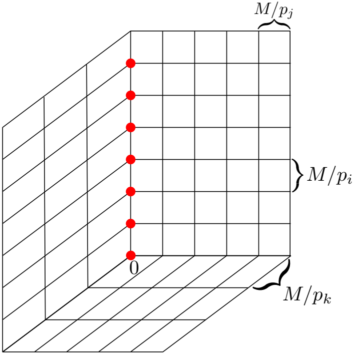
$$\mathbb{Z}_M = \mathbb{Z}_{p_1^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{p_K{}^{n_K}},$$

which we can represent geometrically as a K-dimensional lattice. Then $A \oplus B$ can be interpreted as a tiling of that lattice.

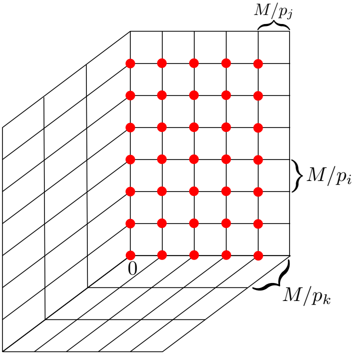
(However, this is more than just a multidimensional tiling. It will be important that the side lengths in different dimensions are powers of distinct primes.)

Geometric representation of sets

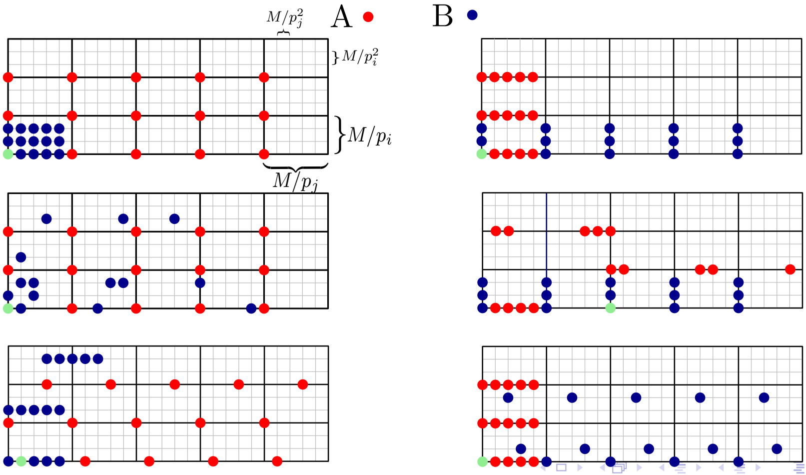
$A = \{0, M/p_i, 2M/p_i, \ldots, (p_i - 1)M/p_i\}$



$A = \{x \in \mathbb{Z}_M : M/p_ip_j | x\}$



Examples of tilings



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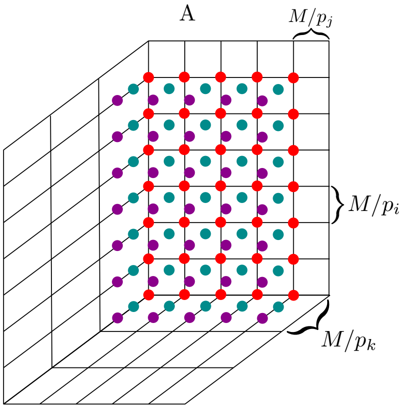

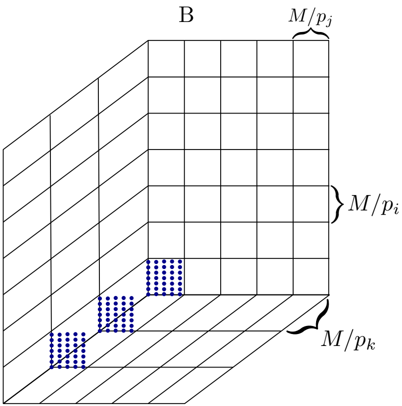

The Coven-Meyerowitz tiling conditions

Polynomial formulation

By translational invariance, we may assume that $A, B \subset \{0, 1, \dots\}$ and that $0 \in A \cap B$. The *characteristic polynomials* (aka *mask polynomials*) of A and B are

$$A(X) = \sum_{a \in A} X^a, \; B(X) = \sum_{b \in B} X^b.$$

Then $A \oplus B = \mathbb{Z}_M$ is equivalent to

$$A(X)B(X) = 1 + X + \cdots + X^{M-1} \mod (X^M - 1).$$

Now use factorization of polynomials.

Cyclotomic polynomials

The s-th *cyclotomic polynomial* is the unique monic, irreducible polynomial $\Phi_s(X)$ whose roots are the primitive s-th roots of unity. Alternatively, Φ_s can be defined inductively via

$$X^n - 1 = \prod_{s|n} \Phi_s(X).$$

Then the tiling condition $A(X)B(X) = 1 + X + \cdots + X^{M-1} \mod (X^M - 1)$ is equivalent to

$$|A||B| = M \text{ and } \Phi_s(X) \mid A(X)B(X) \text{ for all } s|M,\ s \neq 1.$$

Since Φ_s are irreducible, each $\Phi_s(X)$ with $s|M$, $s \neq 1$, must divide at least one of $A(X)$ and $B(X)$.

Tiling equivalences

To summarize, the following are equivalent:

- $A \oplus B = \mathbb{Z}_M$
- $|A||B| = M$ and $\mathrm{Div}(A) \cap \mathrm{Div}(B) = \{M\}$
- $A(X) \cdot B(X) = 1 + X + \ldots + X^{M-1} \mod X^M - 1$
- $A(1) \cdot B(1) = M$ and each $\Phi_s(X)$ with $s|M$, $s \neq 1$, must divide at least one of $A(X)$ and $B(X)$

The Coven-Meyerowitz Theorem (1998)

Let $S_A = \{p^\alpha : \ \Phi_{p^\alpha}(X) | A(X)\}$. Consider the following conditions.

(T1) $A(1) = \prod_{s \in S_A} \Phi_s(1)$,

(T2) if $s_1, \ldots, s_k \in S_A$ *are powers of different primes, then* $\Phi_{s_1 \ldots s_k}(X)$ *divides* $A(X)$.

Then:

- if A satisfies (T1), (T2), then A tiles $\mathbb{Z}$;
- if A tiles $\mathbb{Z}$ then (T1) holds;
- if A tiles $\mathbb{Z}$ and $|A|$ has *at most two prime factors*, then (T2) holds.

Cyclotomic polynomials and distribution

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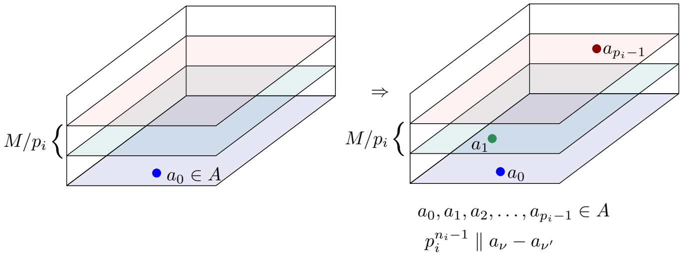
- $\Phi_{p_i}|A \Leftrightarrow A$ is equidistributed mod p_i,
- $\Phi_{p_i^{n_i}}|A \Leftrightarrow A$ is equidistributed mod $p_i^{n_i}$ within residue classes mod $p_i^{n_i-1}$.



Alternative formulation of T2

Assume $A \oplus B = \mathbb{Z}_M$, with $M = \prod_i p_i^{n_i}$. Let $M_i = M/p_i^{n_i}$. Define the *standard tiling set* $A^\flat \subset \mathbb{Z}_M$ via

$$A^\flat(X) = \prod_i \prod_{\alpha : \Phi_{p_i^\alpha} | A} \Phi_{p_i}\left(X^{M_i p_i^{\alpha-1}}\right).$$

This has the same prime power cyclotomic divisors as A, but also has a "lattice structure". Then B satisfies T2 if and only if

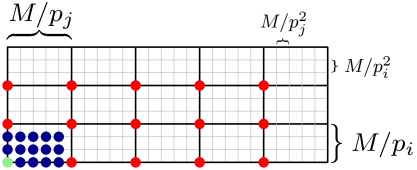
$$A^\flat \oplus B = \mathbb{Z}_M$$

is also a tiling. (C-M used this set to prove T2 $\Rightarrow$ tiling.)

Standard T2 sets

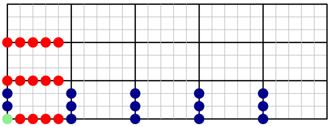
What does a standard set looks like?

$$A = A^\flat, B = B^\flat$$



$\Phi_{p_i^2}\Phi_{p_j^2} | A$

$\Phi_{p_i}\Phi_{p_j} | B$



$\Phi_{p_i^2}\Phi_{p_j} | A$

$\Phi_{p_i}\Phi_{p_j^2} | B$

A (red dots) B (blue dots)

Connection to Fuglede's spectral set conjecture

Conjecture (Fuglede, 1974): A set $\Omega \subset \mathbb{R}^n$ tiles $\mathbb{R}^n$ by translations if and only if $L^2(\Omega)$ admits an orthogonal basis of exponential functions.

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- Łaba (2001): T2 implies spectrality (finite groups, unions of finitely many unit intervals in $\mathbb{R}$).

Main result

Main result (Łaba-Londner 2021)

Theorem. Suppose that $A \oplus B = \mathbb{Z}_M$, with $M = \prod_{i=1}^{3} p_i^2$. (This is the simplest case that cannot be reduced to two prime factors.) Assume that $p_i \neq 2$ for all i. Then A and B satisfy T2.

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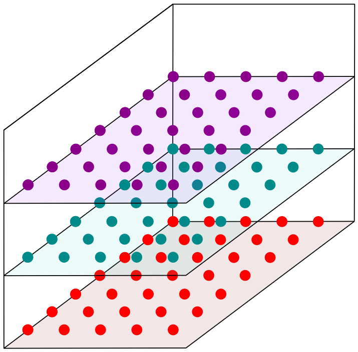
Additionally:

- The proof essentially provides a classification of all tilings of period $M = \prod_{i=1}^{3} p_i^2$. (It does not get much more complicated than Szabó-type examples.)
- We are also close to finishing the even case.
- Methods and some intermediate results extend to more general M.

Why are 3 prime factors more difficult?

Example due to Szabó (1985)



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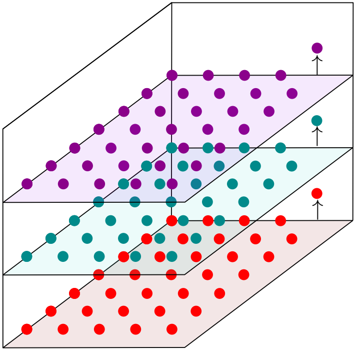

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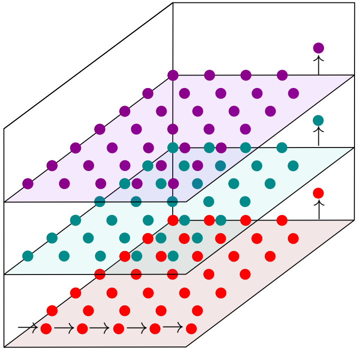

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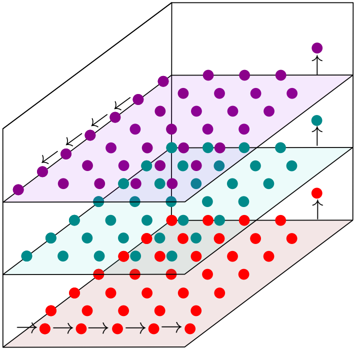


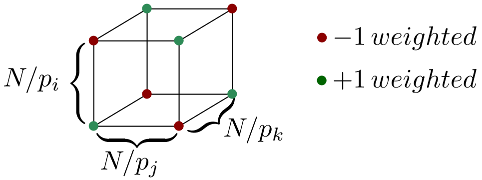
Main ingredients of the proof

Cuboids

Let $N = p_1^{\alpha_1} \dots p_K^{\alpha_K}$, where $p_1, \dots, p_K$ are distinct primes. An N-cuboid in $\mathbb{Z}_N$ is a weighed set with the mask polynomial

$$\Delta(X) = X^a \prod_{i=1}^{K} (1 - X^{d_i}) \mod (X^N - 1)$$

where $a \in \mathbb{Z}_N$ and $(d_i, N) = N/p_i$.



Cuboids and cyclotomic divisibility

For $A \subset \mathbb{Z}_M$, and $\Delta(X) = X^a \prod_{i=1}^{K}(1 - X^{d_i})$ as above, define

$$\mathbb{A}^M[\Delta] := \sum_{(\epsilon_1,\ldots,\epsilon_K)\in\{0,1\}^K} (-1)^{\sum \epsilon_i} \mathbb{1}_A \left(a + \sum \epsilon_i d_i \right).$$

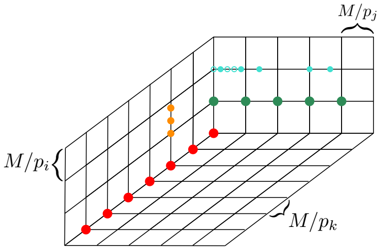
Then $\Phi_M | A$ if and only if $\mathbb{A}^M[\Delta] = 0$ for every such Δ.

This follows from structure results for vanishing sums of roots of unity (Rédei, de Bruijn, Schoenberg, Mann, Lam-Leung, ...), and has been used in the literature on that subject (Steinberger) and on Fuglede's conjecture (Malikiosis et al). We use this on various scales $N|M$.

Fibering

Let $N = p_1^{\alpha_1} \dots p_K^{\alpha_K}$. An *$N$-fiber in the p_i direction* is a set

$$\{a, a + N/p_i, a + 2N/p_i, \dots, a + (p_i - 1)N/p_i\} \subset \mathbb{Z}_N.$$



- A set $A \subset \mathbb{Z}_N$ is fibered in the p_i direction if it is a union of disjoint fibers in that direction.
- We use this on various scales $N|M$ (also for multisets, restricted to grids, etc).
- We use cuboids to get fibering results.

Plane bound: example of a counting argument

Let $A \oplus B = \mathbb{Z}_M$, where $M = (p_i p_j p_k)^2$, $|A| = p_i p_j p_k$, and p_i, p_j, p_k are distinct primes. Then for every $x \in \mathbb{Z}_M$ we have

$$|A \cap \Pi(x, p_i^2)| \leq p_j p_k,$$

where $\Pi(x, p_i^2)$ is the plane $\{x' \in \mathbb{Z}_M : \; p_i^2 \mid (x - x')\}$.

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This follows from the equidistribution property associated with prime power cyclotomic divisors.

Box product

For $m|M$ and $x \in \mathbb{Z}_M$, define

$$\mathbb{A}_m^M[x] = \#\{a \in A: \ (x-a, M) = m\},$$

and similarly for B. Define also the *box product*

$$\langle \mathbb{A}[x], \mathbb{B}[y] \rangle := \sum_{m|M} \frac{1}{\phi(M/m)} \mathbb{A}_m^M[x] \mathbb{B}_m^M[y].$$

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Theorem. $A \oplus B = \mathbb{Z}_M$ if and only if $|A||B| = M$ and

$$\langle \mathbb{A}[x], \mathbb{B}[y] \rangle = 1 \ \ \forall x, y \in \mathbb{Z}_M.$$

(From an unpublished 2001 preprint by Granville-Łaba-Wang.)

Saturating sets

For $x \in \mathbb{Z}_M$, let

$$A_x = \{a \in A : \ (x - a, M) = (b - b', M) \text{ for some } b, b' \in B\}$$

(the elements of A that contribute to $\langle \mathbb{A}[x], \mathbb{B}[b] \rangle$ with $b \in B$).

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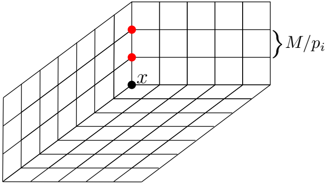
(the elements of A that contribute to $\langle \mathbb{A}[x], \mathbb{B}[b] \rangle$ with $b \in B$).

- If $a \in A$, then $A_a = \{a\}$. (Divisor exclusion.)
- If $x \notin A$, then $(x-a', M) \neq (a-a', M)$ for all $a \in A$ and $a' \in A_x$. This leads to geometric restrictions on A_x.



$A_x \subset \Pi(x, p_i^2)$

Cofibered structures and fiber shifting

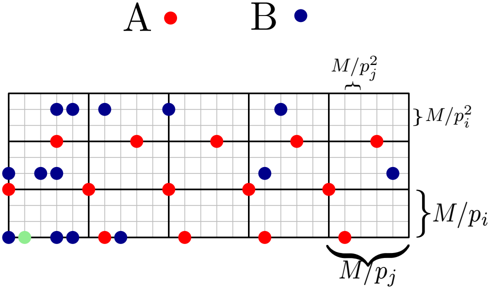
Lemma (special case). Let $M = (p_i p_j p_k)^2$, and assume that $A \oplus B = \mathbb{Z}_M$. Suppose that A and B have the following *cofibered structure*:

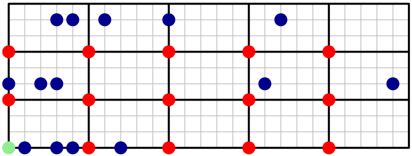
- A contains an M-fiber in the p_i direction,
- B is fibered in the p_i direction on the scale M/p_i.

Let A' be the set obtained from A by shifting the M-fiber by M/p_i^2 in the p_i-direction. Then $A' \oplus B = \mathbb{Z}_M$, and A' satisfies T2 if and only if A does.

We use this to reduce our original tiling to simpler ones.

Cofibered structures and fiber shifting





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We use this to reduce our original tiling to simpler ones.
But how do we get such structure?

One-dimensional saturating spaces

If there is an $x \in \mathbb{Z}_M \setminus A$ such that A_x is contained in the line in the p_i direction through x, then A, B have a cofibered structure similar to that in the lemma.

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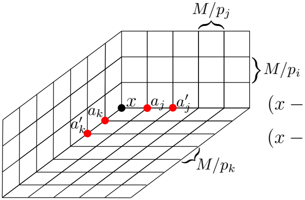
This is implied for example by the following configuration.

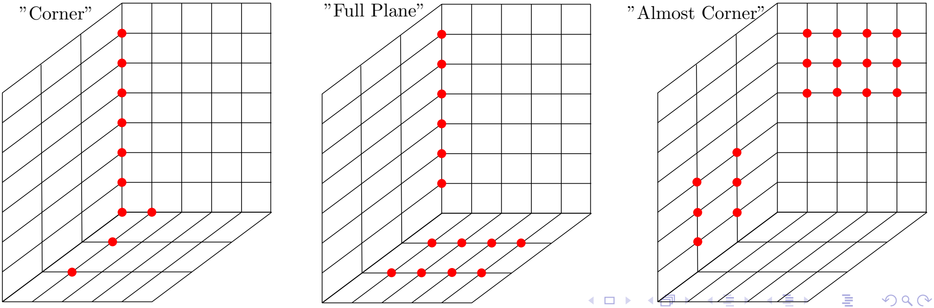


$$(x - a_j, M) = (x - a'_j, M) = (a_j - a'_j, M) = M/p_j$$
$$(x - a_k, M) = (x - a'_k, M) = (a_k - a'_k, M) = M/p_k$$

Putting it together

Let $M = (p_i p_j p_k)^2$, $|A| = |B| = p_i p_j p_k$.

- Assume $\Phi_M | A$. Use cuboids to get structure results: either A is fibered on all $p_i p_j p_k$-grids, or there is an $p_i p_j p_k$-grid Λ such that $A \cap \Lambda$ has one of the "special structures" we can classify.



Putting it together

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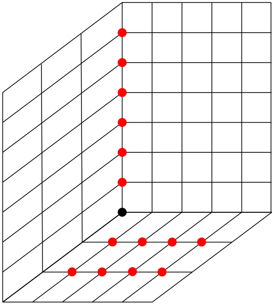
- In the fibered case, try to go down to a lower scale. (Caution: different grids may be fibered in different directions.)
- For the "special structures", we use saturating spaces and fiber shifting to reconstruct the rest of the tiling. If we can reduce to the case where one of the sets is the standard tiling complement, we are done.

Resolving a special structure

The “special structures” do not contain configurations that would immediately imply one-dimensional saturating spaces. Therefore we have to build up more complicated arguments.



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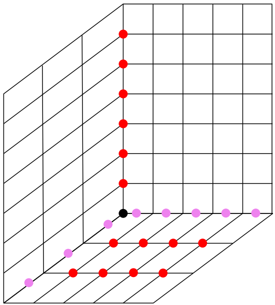

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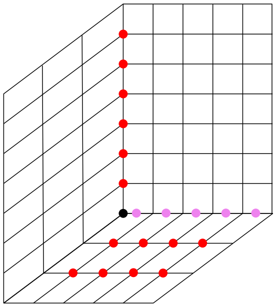

- A contains an M fiber in the p_j direction and B is M/p_j fibered in the p_j direction

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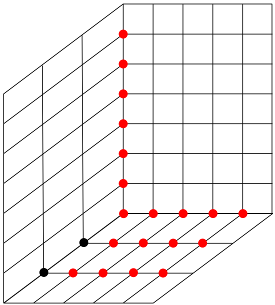

- A contains an M fiber in the p_j direction and B is M/p_j fibered in the p_j direction

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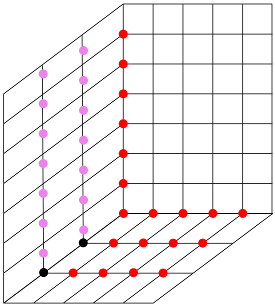

- A contains an M fiber in the p_j direction and B is M/p_j fibered in the p_j direction
- A contains M fibers in the p_i direction and B is M/p_i fibered in the p_i direction

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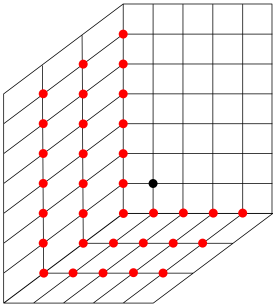

- A contains an M fiber in the p_j direction and B is M/p_j fibered in the p_j direction
- A contains M fibers in the p_i direction and B is M/p_i fibered in the p_i direction

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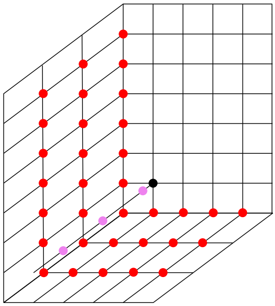

- A contains an M fiber in the p_j direction and B is M/p_j fibered in the p_j direction
- A contains M fibers in the p_i direction and B is M/p_i fibered in the p_i direction
- A contains M fibers in the p_i direction and B is M/p_k fibered in the p_k direction

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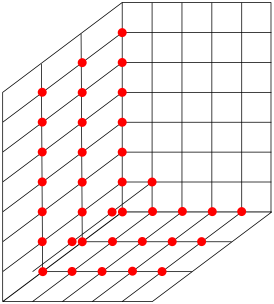

- A contains an M fiber in the p_j direction and B is M/p_j fibered in the p_j direction
- A contains M fibers in the p_i direction and B is M/p_i fibered in the p_i direction
- A contains M fibers in the p_i direction and B is M/p_k fibered in the p_k direction

Thank you!