Good and Bad Maximal functions

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Matrix weight

In this lecture, we discuss maximal functions applied to vector valued functions in spaces involving matrix weights.

For the purpose of this talk, a (scalar-valued) weight is a positive, locally integrable function.
A matrix weight is a locally integrable function that takes values in positive semidefinite matrices of size d.

Doob and Hardy-Littlewood

Maximal functions are scalar valued operators. One considers either

Dyadic Hardy-Littlewood's competitors	$\langle \lvert f \rvert \rangle_I$
Doob's competitors	$\lvert \langle f \rangle_I \rvert$

When f is scalar valued these are obviously equivalent in L^2. If f is vector-valued they still are equivalent.

What if the norm depends on a point?

The beginnings

Some history:

- (unweighted) dimensional growth of Carleson Embedding, Paraproducts, Commutators... (Pisier, Katz, Volberg, Nazarov, Treil, Pott, Gillespie, P. ...)
- weighted bounds of singular operators, A_2 and A_p theory (Nazarov, Treil, Volberg, Christ, Goldberg...)

Example of the unweighted theory: Carleson Embedding Theorem

The dyadic scalar statement is:
TFAE

$$\forall K : \frac{1}{|K|} \sum_{I \in K} \alpha_I \leq C_1,$$

$$\sum_I \alpha_I \langle |f| \rangle_I^2 \leq C_2 \|f\|_{L^2}^2.$$

When α are positive definite matrices of size d and f vector valued, then this theorem still holds with a constant d^2 by taking trace.

With extra effort this becomes $log^2 d$ but no better than that.

The angle, A_2

The A_2 theory was developed by Treil-Volberg in 'The angle between past and future' giving a beautiful motivation and full characterisation for the desired estimate of the form

$$\left(\int_{\mathbb{R}}\langle WHf, Hf\rangle_{\mathbb{C}^d}\right)^{1/2} \leq C_W\left(\int_{\mathbb{R}}\langle Wf, f\rangle_{\mathbb{C}^d}\right)^{1/2} \quad \left(\int_{\mathbb{R}}|Hf|^2 w\right)^{1/2} \leq C_w\left(\int_{\mathbb{R}}|f|^2 w\right)^{1/2}$$

where f is vector valued and the Hilbert transform H acts componentwise.

Stationary processes

Given a stationary sequence of vectors $e(n), n \in \mathbb{Z}$ in a Hilbert space, that is,

$$\langle e(n), e(m)\rangle = \langle e(0), e(m-n)\rangle.$$

For example $L^2(\mathbb{T})$ with $e(n) = z^n$. It can be shown that this example is enough by adjusting the measure μ:

$$\langle e(n), e(m)\rangle = \int z^{n-m} d\mu.$$

Stationary processes

The multivariate version is $e_j(n), n \in \mathbb{Z}, 1 \le j \le d$ with

$$\langle e_j(n), e_k(m) \rangle = \langle e_j(0), e_k(m-n) \rangle.$$

For example $z^m \vec{e}_j, m \in \mathbb{Z}$ and $\{\vec{e}_j : 1 \le j \le d\}$ orthonormal base.

This leads to considering matrix measure $M = (\mu_{i,j})$ and $L^2(\mathbb{T}, M)$.

Stationary processes

If we let $E(n) = span\{e_k(n) : 1 \leq k \leq d\}$ then the future is

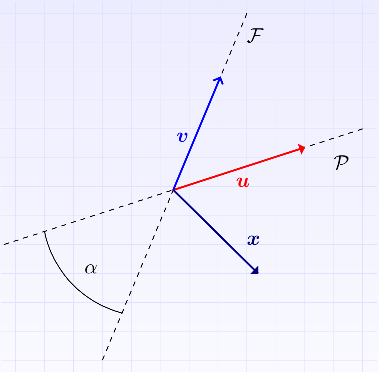
$$\mathcal{F} = span^{cl}\{E(n) : n \geq 0\}$$

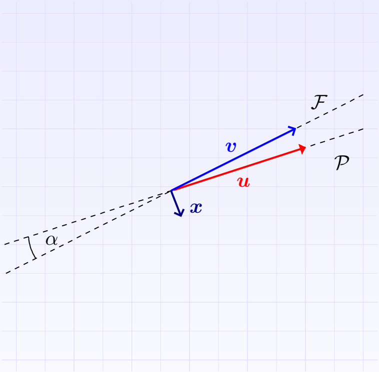
and the past

$$\mathcal{P} = span\{E(n) : n < 0\}.$$

The Hilbert transform (or rather its part P_+) answers questions about the angle between past and future.

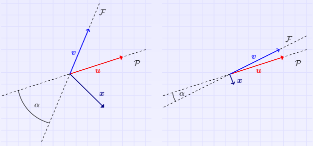
Stationary processes





$x = u - v$

Stationary processes



We have in both cases u and v unit vectors such that $u \in \mathcal{P}$ and $v \in \mathcal{F}$.

Let P_+ the projection on $\mathcal{F}$. Take $x = u - v$. We have $P_+ x = -v$, $\|P_+ x\| = 1$, $\|x\|^2 = \|u - v\|^2 = 2 - 2\langle u, v\rangle = 2(1 - \cos\alpha)$. It follows

$$\|P_+\| \geqslant \frac{\|P_+ x\|}{\|x\|} = \frac{1}{\sqrt{2(1 - \cos\alpha)}}$$

is arbitrarily large when α is small.

Matrix Weights

It is known that for boundedness of P_+ it is necessary that $dM = W(x)dx$ with W selfadjoint and positive definite.

It is also known the the dimension has to be finite (and W a matrix).

By the theorem of Treil - Volberg the necessary and sufficient condition for boundedness of

$$\|H\|_{L^2(W)\to L^2(W)}$$

is the matrix A_2 condition

$$Q_2(W) = sup_I \|\langle W\rangle_I^{1/2}\langle W^{-1}\rangle_I^{1/2}\|^2 < \infty$$

Some 90s impressions....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STOP

Main open question

Despite the advances in the scalar theory, the following remains a puzzling open question:

$$\|H\|_{L^2_W \to L^2_W} \leq C[W]_{A_2}^{?}$$

In the scalar case $? = 1$ (P.)

Improvements and Failures, singular operators

- $[W]_{A_2}^{3/2}\log^2(1+[W]_{A_2})$ for the Hilbert transform (Bickel-P.-Wick) simplified 'the angle' (2014)
- $[W]_{A_2}^{3/2}$ for all CZO (Nazarov-P.-Treil-Volberg) presented a sparse domination by convex bodies (2017)
- $[W]_{A_2}^{1}$ for the square function (Hytönen-P.-Volberg) (2017)

The first only uses the 'angle' technology plus a trick in P.-Pott (2003), the others use very recent weighted technology, the pointwise sparse domination. Interestingly, for the square function, the 'older' proofs of sparse norm domination instead of pointwise domination is used!

A_p

An A_p theory was begun in the 90s by Nazarov-Treil and further developed by many (Volberg, Goldberg, Christ, Pott, Roudenko, Frazier...)

Embeddings and maximal functions

A natural tool for this (and interesting in its own right) are questions about Carleson Lemmata and Maximal inequalities in a matrix weighted setting.

From scalar unweighted to scalar weighted CET

TFAE

$$\forall K : \frac{1}{|K|} \sum_{I \in K} \alpha_I \leq C_1$$

$$\sum_I \alpha_I \langle |f| \rangle_I^2 \leq C_2 \|f\|_{L^2}^2$$

Just replace Lebesgue measure by w everywhere and rewrite it a bit

$$\forall K : \frac{1}{|K|} \sum_{I \in K} \langle w \rangle_I^2 \alpha_I \leq C_1 \langle w \rangle_K$$

$$\sum_I \alpha_I \langle |wf| \rangle_I^2 \leq C_2 \|f\|_{L^2_w}^2$$

Embedding theorems

In the scalar setting, the weighted Carleson Embedding theorem and the unweighted one are indistinguishable.

In case of a matrix Carleson sequence, this is not the case: the theorem follows trivially from the scalar case when there is no weight, but it does not when a matrix weight is present.

The weighted case had been an open problem for quite some time, solved around 2015.

The unweighted matrix case has a logarithmic dimensional growth, the one in the weighted case is probably worse (but unknown).

Weighted Embedding theorem

Culiuc-Treil (2015) proved that the matrix weighted Carleson Lemma holds: TFAE

$$\forall K : \sum_{I \in K} \langle W \rangle_I A_I \langle W \rangle_I \le C_1 \langle W \rangle_K$$

$$\sum_I \| A_I^{1/2} \langle Wf \rangle_I \|^2_{\mathbb{C}^d} \le C_2 \| f \|^2_{L^2_W}$$

Notice that here, we do not want to assume $W \in A_2$ and that the first inequality is in the sense of operators!

Only the Bellman argument works so far - the boundedness of the maximal operator paired with this Embedding theorem has no 'independent' proof.

Weighted Embedding theorem, a hint at a proof

The scalar Bellman proof uses the variable $0 \leq M \leq w$ and the derivative estimate

$$\partial_M \left(-\frac{x^2}{w+M}\right) = \frac{x^2}{(w+M)^2} \geq \frac{x^2}{4w^2}$$

But ?

$$\partial_M \left(-\langle (W+M)^{-1} x, x \rangle_{\mathbb{C}^d}\right)$$

$(W+M)^{-1} \Delta M (W+M)^{-1}$ cannot be estimated from below to make the M disappear.

$$\forall K : \frac{1}{|K|} \sum_{I \in K} \alpha_I \leq 1$$

$$\sum_I \langle w \rangle_I^{-2} \alpha_I \langle |wf| \rangle_I^2 \leq 4 \|f\|_{L^2_w}^2$$

Weighted Embedding theorem, a hint at a proof

Terms always disappear if we multiply them by 0.

$$\partial_M(-\langle (W+tM)^{-1}x, x\rangle_{\mathbb{C}^d})$$

involves the term

$$t(W+tM)^{-1}\Delta M(W+tM)^{-1}.$$

On one hand $t=0$ gives us what we want but the penalty t up front ruins everything. Complex function theory sometimes gives a relation about growth of functions and their size at the point 0.

A bilinear embedding theorem

P.-Pott-Reguera showed a weak bilinear version (used crucially in a classical scalar proof) and Domelevo-P.-Skreb showed that it cannot in any way be improved. (2019)

$$\forall K : \sum_{I \in K} \alpha_I \leq C_1$$

$$\Rightarrow \sum_I |\langle \alpha_I^{1/2} \langle W \rangle_I^{-1} \langle Wf \rangle_I, \alpha_I^{1/2} \langle W^{-1} \rangle_I^{-1} \langle W^{-1} g \rangle_I \rangle_{\mathbb{C}^d}| \leq C_2 \|f\|_{L^2_W} \|g\|_{L^2_{W^{-1}}}$$

Weak means α_I scalar and $|\langle \cdot, \cdot \rangle_{\mathbb{C}^d}|$ instead of $\| \cdot \|_{\mathbb{C}^d} \| \cdot \|_{\mathbb{C}^d}$.

Maximal functions

The study of certain matrix weighted maximal functions was initiated by Christ-Goldberg.

We show a striking difference in the behaviour of otherwise equivalent maximal functions.

Dyadic Hardy-Littlewood's competitors	$\langle \vert f \vert \rangle_I$
Doob's competitors	$\vert \langle f \rangle_I \vert$

When f is scalar valued, then these have the same behavior, also if we pass to general measures (or scalar weights).
When f is vector valued and if there is no weight, then these still have the same behavior.

Two maximal functions

But:

The reinterpretation in a matrix-weighted space of
Doob's inequality holds
(P.-Pott-Reguera) (2019)
but the
dyadic Hardy-Littlewood maximal function is not bounded
(Nazarov-P.-Skreb-Treil) (2021).

Two maximal functions

From $L^2_W \to L^2$ consider Doob type and Hardy-Littlewood type

$$M_W f(x) = \sup_{I : x \in I} \| W^{1/2}(x) \langle W \rangle_I^{-1} \langle W f \rangle_I \|_{\mathbb{C}^d} \quad \sup_{I:x\in I} w^{1/2}(x) \frac{1}{w(I)} \left| \int_I fw \right|$$

$$M^c_W f(x) = \sup_{I : x \in I, \varphi_I} \| W^{1/2}(x) \langle W \rangle_I^{-1} \langle \varphi_I W f \rangle_I \|_{\mathbb{C}^d} \quad \sup_{I:x\in I} w^{1/2}(x) \frac{1}{w(I)} \int_I |f| w$$

The supremum over $\varphi_I : I \to [-1, 1]$ replaces $|\cdot|$ for $d = 1$ (compare $\varphi_I = \mathrm{sign} f$) and resembles the convex body average

$$\langle\langle f \rangle\rangle_I = \{ \langle \varphi_I f \rangle_I : \quad \varphi_I : I \to [-1, 1] \}$$

used in the sparse domination by Nazarov-P.-Treil-Volberg (2017).

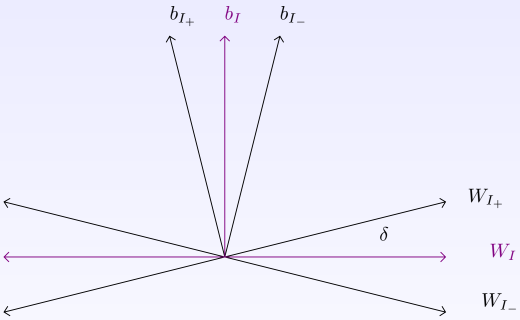
Convex body Carleson Embedding Theorem

The following are not equivalent

$$\forall K : \sum_{I \in K} \langle W \rangle_I A_I \langle W \rangle_I \leq C_1 \langle W \rangle_K$$

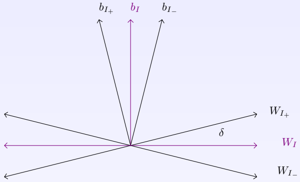
$$\sum_I \| A_I^{1/2} \langle \varphi_I W f \rangle_I \|^2_{\mathbb{C}^d} \leq C_2 \| f \|^2_{L^2_W}$$

Construction



$$W_{I_0} = \alpha_0 a_{I_0} a_{I_0}^* + \beta_0 b_{I_0} b_{I_0}^* \ldots$$

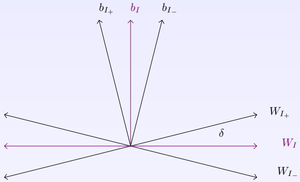
Construction



Build a dyadic martingale weight like in the picture with increasing eccentricity. Choose A_I with increasing norm into the weak direction b.

$$\forall K : \sum_{I \in K} \langle W \rangle_I A_I \langle W \rangle_I \le C_1 \langle W \rangle_K$$

Construction



Use $\varphi_I = \chi_{I_+}$ to induce a generation shift so that enough of the strong direction is grabbed to produce a blow up for example in the strong a direction.

$$\sum_I \|A_I^{1/2}\langle \varphi_I W a\rangle_I\|_{\mathbb{C}^d}^2 = \infty$$

Passing to the maximal function

Recall the linearisation trick. If S_I form a disjoint collection, then

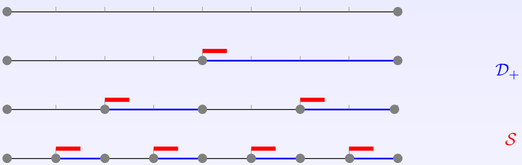
$$F(x) := \sum_I |\chi_{S_I}(x) W(x)^{1/2} \langle W \rangle_I^{-1} \langle \varphi_I W f \rangle_I|_{\mathbb{R}^2}$$

has for each x at most one summand and is dominated by the maximal function. The L^2 integrated form is

$$\sum_I |W(S_I)^{1/2} \langle W \rangle_I^{-1} \langle \varphi_I W f \rangle_I|_{\mathbb{R}^2}^2$$

and reminds of the convex body Carleson Embedding Theorem, which we know does not hold.

The new weight



The collection $\mathcal{S}$ is disjoint. For each I associate set S_I. (empty for $\mathcal{D}_-$ type intervals.)

$$W_s = W + s \sum_{I \in \mathcal{D}_+} \frac{\chi_{S_I}}{|S_I|} \tilde{A}_I, \quad \tilde{A}_I = C^{-1} \langle W \rangle_I A_I \langle W \rangle_I$$

$$W_s = W + s \sum_{I \in \mathcal{D}_+} \frac{\chi_{S_I}}{|S_I|} \tilde{A}_I, \quad \tilde{A}_I = C^{-1} \langle W \rangle_I A_I \langle W \rangle_I$$

$$I \in \mathcal{D}_+ \Rightarrow W_s(S_I) = W(S_I) + s\tilde{A}_I$$

By contradiction assume uniform bound for the maximal function and get

$$\lightning\ s \sum_I |\tilde{A}_I^{1/2} \langle W_s \rangle_I^{-1} \langle \varphi_I W_s a \rangle_I|^2 \lesssim (1+s) |\langle W \rangle_{I_0}^{1/2} a|^2$$

With $\varphi_I = \chi_{I_+}$ it makes the left hand side look like the (false) conclusion of the convex body Carleson Lemma if we could let $s = 0$.

$$\lightning \; s\sum_I |\tilde{A}_I^{1/2}\langle W_s\rangle_I^{-1}\langle \varphi_I W_s a\rangle_I|^2 \lesssim (1+s)|\langle W\rangle_{I_0}^{1/2} a|^2$$

Fact: If p is a polynomial and $|p(s)| \leq \frac{(1+s)^5}{s}$ for all $s>0$ then $|p(0)| \leq 25e^2$

Rational function:

$$R_I(s) = |\tilde{A}_I^{1/2}\langle W_s\rangle_I^{-1}\langle \varphi_I W_s a\rangle_I|^2.$$

Estimating its denominator and bringing it to the right hand side together with s would contradict via the value at $s=0$ the unbounded Embedding we derived earlier.

Matrix weighted product theory

Domelevo-Kakaroumpas-P.-Soler i Gibert develop the A_2 and A_p product theory.

Think of the A_2 class as what you would expect: in each variable separately.

In the case $p = 2$ this is fairly straightforward to work with, but relies on the ideas of Goldberg for other p, the reducing matrices, to get a workable quantity.

Journé operators

A Journé operator is a ‘biparameter Calderon-Zygmund operator’ as defined (in an iterative manner) by Journé in the 1980s.

For example, we (Domelevo-Kakaroumpas-P.-Soler i Gibert) can prove: If J is any Journé operator, then

$$\|J\|_{L^2_W \to L^2_W} \leq C[W]_{A_2}^5$$

We also have a result for $p \neq 2$ and we do not care about the exponent 5.

Journé operators

Trivial examples are tensor products of Calderon-Zygmund operators, such as $H_1 H_2$ whose Fourier symbol can be defined on $S^0 \times S^0$.

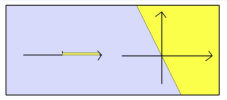
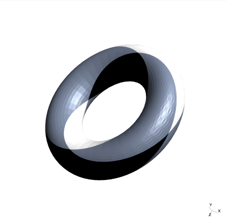


Figure: P_+ on S^0 and P_θ on S^1

Less trivial are 'twisted' Hilbert transforms defined via (smoothed) projections on $S^1 \times S^1$



dyadic Journé operators

Let $h_{I_1 \times I_2}$ be the tensor Haar function on the rectangle $I_1 \times I_2$.

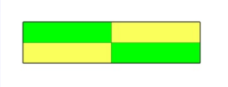


Figure: cancellative Haar function in $\mathbb{R} \times \mathbb{R}$

Dyadic shifts with decaying components such as

$$\alpha_{I_1 J_1 K_1 I_2 J_2 K_2} \langle f, h_{J_1 \times J_2} \rangle_{L^2} h_{K_1 \times K_2}$$

with 'sticky' coefficients $\alpha_{I_1 J_1 K_1 I_2 J_2 K_2}$ and the I are ancestors of the J, K are a good model for Journé operators. Indeed, Martikainen proved that they can represent them efficiently (2014).

Journé and matrix weights, proofs

The scalar weighted result is due to a series of (1980s) results by R. Fefferman.

In 2017 alternatives appeared, independently, Barron-Pipher and as part of the work on another subject in Holmes-P.-Wick.

For our matrix weighted work we had to develop the entire bi-parameter theory of weights, matrix weighted (shifted) square function and maximal function estimates (z.B. Khintchine).

We give alternative proofs, one with a sparse Square function domination (that is a bit weaker) and one using that correctly defined square functions do not 'see' the dyadic shift.