Explicit and nonlinear variants of Bourgain's projection theorem

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The "dimension of projections" problem

Problem

Let $A \subset \mathbb{R}^2$ be a Borel of Hausdorff dimension $\alpha \in (0,2)$ and let $E \subset S^1$ be a Borel set of directions of Hausdorff dimension $\eta \in (0,1]$.

What can we say about $\dim_H(P_\theta A)$ for "typical" $\theta \in E$?

In other words, we want to compute

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- Trivial: $s(\alpha, \eta) \leq \max(\alpha, 1)$, since projections do not increase Hausdorff dimension and the dimension of a subset of the line is ≤ 1.
- Also trivial: $s(\alpha, \eta)$ is non-decreasing in α and η.

Classical results on dimensions of projections

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$s(\alpha, \eta) \geq \alpha/2$ *(if* $\eta > 0$*).*

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Observation

All three estimates yield $s(1, 1/2) \geq 1/2$.

Bourgain's projection theorem, $\dim_H$ version

Theorem (Bourgain 2010)

If $\eta > 0$ *and* $\alpha \in (0, 2)$, *then*

$$s(\alpha, \eta) \geq \frac{\alpha}{2} + c(\alpha, \eta)$$

for some $c(\alpha, \eta) > 0$. *In particular,*

$$s(1, 1/2) \geq 1/2 + c.$$

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Remark

The conjecture, based partly on Szemerédi-Trotter heuristics, is that s is linear in η; in particular, $s(1,1/2) = 3/4$.

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In fact, this is immediate from Bourgain's projection theorem: since $\eta(\dim_{\mathrm{H}}(B \times B), \dim_{\mathrm{H}}(B)) > \dim_{\mathrm{H}}(B)$, *there is $b \in B$ such that*

$$\dim_{\mathrm{H}}(B + bB) > \dim_{\mathrm{H}}(B).$$

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  - Estimates on exponential sums, Fourier decay in nonlinear dynamics (J. Bourgain, S. Dyatlov).
  - Discretized Elekes-Rónyai and dimension-expanding polynomials (O. Raz, J. Zahl).

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- What can we say about $|\pi_\theta A|_\delta$ for a typical $\theta \in E$?
- Without additional assumptions, not much!

δ-discretized projections: the theorem

Theorem (Bourgain 2010)

Given $\eta > 0$, $0 < \alpha < 2$, there exist $\varepsilon(\alpha, \eta) > 0$ and $\tau(\alpha, \eta) > 0$ such that the following holds.
Let $A \subset B^2(0,1)$, $|A|_\delta = \delta^{-\alpha}$, and let $E \subset S^1$ satisfy

$$|A \cap B(x,r)|_\delta \le \delta^{-\tau} r^\eta |A|_\delta \qquad \text{for all } x \in \mathbb{R}^2, r \in [\delta, 1],$$
$$|E \cap B(\theta,r)|_\delta \le \delta^{-\tau} r^\eta |E|_\delta \qquad \text{for all } \theta \in S^1, r \in [\delta, 1].$$

Then there is $\theta \in E$ such that for any set $A' \subset A$ with $|A'|_\delta \ge \delta^\tau |A|_\delta$, one has

$$|P_\theta A'|_\delta \ge \delta^{-\alpha/2-\varepsilon} = \delta^{-\varepsilon} |A|_\delta^{1/2}.$$

In fact, this holds for all $\theta \in E$ outside of a set E' with $|E'|_\delta \le \delta^\tau |E|_\delta$.

δ-discretized projections: remarks

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- The non-concentration assumptions on A and E rule out the possibility that they are too concentrated in a square/interval of size $\delta^{1/2}$ (in fact, they say that they are not too concentrated in a square/interval of size δ^{ρ}, where $\rho > 0$ is not explicit). It seems plausible that $\rho = 1/2$ should work. Spoiler alert: $\rho = 1/2$ works for A.

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- On the other hand, the non-concentration assumption on A is still quite weak: the set A doesn't have to look like the discretization of a fractal.

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- The proof seems to depend strongly on the linear nature of the projections, but some well-known problems would benefit from a non-linear analog.

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Remark

M. Hochman-E. Lindenstrauss-P. Varjú have work in progress that yields, using totally different methods, explicit estimates for a related problem (using entropy instead of dimension). They have very good qualitative dependence on the parameters, but the constants are still tiny.

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- We use some of the ideas of Guth-Katz-Zahl and combine them with a small part of Bourgain's argument. A new idea is a bootstrapping argument to show that projections of A satisfy the non-concentration estimates needed to apply a generalized sum-product estimate derived from (the proof of) Guth-Katz-Zahl.

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- This approach is fully quantitative but the resulting estimates are very poor unless $\eta \approx \alpha/2$.

An effective projection theorem for $\alpha \approx 2\eta$

Theorem (P.S. and H. Wang 2021?, special case)

We have

$$s(1, 0.4996) \geq 1/2 + 1/500,$$
$$s(1, 1/2) \geq 1/2 + 1/250.$$

In other words, if $A \subset \mathbb{R}^2$ is a Borel set with $\dim_H(A) \geq 1$ and $E \subset S^1$, then

$$\dim_H(E) > 0.4996 \Longrightarrow \dim_H(P_\theta A) \geq \frac{1}{2} + \frac{1}{500}$$
$$\dim_H(E) > 1/2 \Longrightarrow \dim_H(P_\theta A) \geq \frac{1}{2} + \frac{1}{250}$$

for some (in fact “nearly all”) $\theta \in E$.

Remarks on effective projection theorem

- Recall that $s(1, 1/2) \geq 1/2$ follows from three different classical results and $s(1, 1/2) \geq 1/2 + \varepsilon$ from Bourgain's projection theorem. We provide an explicit value $\varepsilon = 1/250$.

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- Arguably the more interesting result is $s(1, 0.4995) > 1/2 + 1/500$ since in particular it covers the case when the set of projections has dimension $= 1/2$ which is known to be much trickier (and an explicit interval to the left of $1/2$).

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- We also get (worse but still quantitative) estimates for $s(\alpha,\eta)$ when $\alpha \approx 2\eta$ and η is small.
- I stated the Hausdorff dimension version for simplicity, but in fact this is a corollary of a fully effective version of Bourgain's projection theorem under the original assumptions.

Some ideas in the proof I

- We say that X is a (δ, s)-set if

$$|X \cap B_r|_\delta \lesssim r^s |X|_\delta$$

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- By Bourgain's projection theorem, there is $x \in E$ such that $E + xE$ contains a $(\delta, 1/2 + c)$-set F.
- Now $(-E) \times (-E)$ is a $(\delta, 1)$-set and $E \times F$ is a $(\delta, 1 + c)$-set. By a theorem of T. Orponen, the set of directions spanned by $(-E \times -E)$ and $E \times F$ has Lebesgue measure $\gtrsim 1$.

Some ideas in the proof II

- Now fix a $(\delta, 1/2)$-set B. By Kaufman's (or Marstrand's) Theorem, there is a direction θ spanned by $(-E) \times (-E)$ and $F \times E$ such that

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- Since $F\subset E+xE$, there are $y_1, y_2, y_3, y_4, y_5\in E$ such that

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$$|(xy_1 + y_2 + y_3)B + (y_4 + y_5)B|_\delta \gtrsim \delta^{-1}.$$
- Plünnecke-Ruzsa gives the claim for product sets $A = B \times B'$; the general case is obtained by following Bourgain's argument.

Nonlinear Bourgain: the problem

Question

Let $\{\Delta_\omega : \mathbb{R}^2 \to \mathbb{R}\}_{\omega\in\Omega}$ be a family of smooth maps, where the index set Ω is a metric space. What assumptions on the family give rise to an analog of Bourgain's projection theorem, that is, to the statement that for ω outside of a set of small dimension η,

$$\dim_H(\Delta_\omega A) \geq \frac{\dim_H(A)}{2} + \varepsilon(\dim_H(A), \eta) \text{ ?}$$

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Remark

Unlike e.g. Kaufman's/Marstrand's projection theorem, that has been generalized to quite general families satisfying a "transversality" assumption, Bourgain's proof appears to be intrinsically linear.

Nonlinear Bourgain: motivation

Motivation

An important problem in analysis/geometric measure theory/combinatorics is "counting patterns" in a set in terms of its Hausdorff dimension.

Key examples are distances and radial projections corresponding to the families:

$$\Delta_y(x) = |x-y|,$$
$$\Delta_y(x) = N(x-y), \quad N \text{ some smooth norm},$$
$$\Delta_y(x) = \frac{x-y}{|x-y|}.$$

(In all these cases, $\Omega = \mathbb{R}^2$.)

Nonlinear Bourgain's Projection Thm, $\dim_H$ version

Theorem (P.S. 2020)

Given $0 < \alpha < 2$, $0 < \eta < 1$, *there is* $\varepsilon(\alpha, \eta) > 0$ *such that the following holds:*
Let $\{\Delta_\omega : \mathbb{R}^2 \to \mathbb{R}\}_{\omega \in \Omega}$ *be a family of* C^2 *maps with no singular points. Let*

$$\theta_x(\omega) = \mathrm{dir}(\Delta'_\omega(x)) : \Omega \to S^1.$$

Let $A \subset \mathbb{R}^2$ *be a Borel with* $\dim_H(A) = \alpha$, *and suppose that there exists* $\nu \in \mathcal{P}(\Omega)$ *such that*

$$\nu(\theta_x^{-1}(B(\theta, r))) \le C_x r^\eta$$

for all $x \in A$. *Then for* ν*-almost all* $\omega \in \Omega$,

$$\dim_H(\Delta_\omega A) \ge \frac{\dim_H(A)}{2} + \varepsilon(\alpha, \eta).$$

Nonlinear Bourgain's Projection Thm: remarks

- When Δ_ω is projection in direction ω, θ_x is the identity map and this reduces back to Bourgain's projection theorem.

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- When Δ_ω is projection in direction ω, θ_x is the identity map and this reduces back to Bourgain's projection theorem.
- There is a corresponding discretized single-scale analog, which again should be considered as the base result.
- In fact, in the discretized version I am able to weaken the non-concentration assumption on A to the natural one: recall that in Bourgain's thm, the assumption was (essentially)

$$|A \cap B_{\delta^\rho}|_\delta \le \delta^\eta |A|_\delta$$

where η is any positive number and $\rho = \rho(\eta) > 0$ is not explicit. I need to assume

$$|A \cap B_{\delta^{1/2}}|_\delta \le \delta^\eta |A|_\delta,$$

with the "dimension gain" ε obviously depending on η.

A few words about the proof

The proof relies on:

- A lower bound for the entropy of $\Delta_\omega \mu$ based on projected entropies of a multi-scale decomposition of μ (M.Hochman-P.S, T. Orponen, T.Keleti-P.S.)

A few words about the proof

The proof relies on:

- A lower bound for the entropy of $\Delta_\omega \mu$ based on projected entropies of a multi-scale decomposition of μ (M.Hochman-P.S, T. Orponen, T.Keleti-P.S.)
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- A new way of choosing the scales in the multi-scale decomposition of Moran measures, so that “all the conditional measures are almost Frostman measures, with no entropy loss”.
- Applying Bourgain’s discretized projection theorem (and also Kaufman’s and Falconer’s classical theorems on exceptional projections) to this multiscale decomposition.

The Falconer distance set problem

Conjecture (Originating in Falconer 1985)

Let $A \subset \mathbb{R}^2$ *be a Borel set with* $\dim_H(A) = 1$. *Then* $\dim_H(\Delta(A)) = 1$ *where* $\Delta(A) = \{|x - y| : x, y \in A\}$.

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Theorem (Katz-Tao 2001+Bourgain 2003)

If $A \subset \mathbb{R}^2$ *is a Borel set with* $\dim_H(A) = 1$*, then* $\dim_H(\Delta(A)) = 1/2 + c$*, where* $c > 0$ *is universal.*

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Stronger Conjecture

Let $A \subset \mathbb{R}^2$ *be a Borel set with* $\dim_H(A) = 1$. *Then there is* $x \in A$ *such that* $\dim_H(\Delta^x(A)) = 1$ *where* $\Delta^x(A) = \{|x - y| : y \in A\}$.
Moreover, this also holds if the Euclidean norm is replaced by any C^∞ *norm with positive curvature everywhere.*

Falconer's problem: recent progress for $\dim_H(A) > 1$

Theorem (T. Keleti, P.S. (2018))

If $\dim_H(A) > 1$ *then there is* $x \in A$ *such that*

$$\dim_H(\Delta^x A) > 2/3 + 1/42.$$

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If $\dim_H(A) > 5/4$ *then there is* $x \in A$ *such that* $|\Delta^x A| > 0$.

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Remark

The proofs of these results rely on a spherical projection theorem of Orponen that requires $\dim_H(A) > 1$ in an essential way.

Nonlinear Bourgain to Falconer's problem

Theorem (P.S. 2020)

Given $0 < \alpha < 2$, $0 < \eta < 1$, *there is* $\varepsilon(\alpha, \eta) > 0$ *such that the following holds: if* $A \subset \mathbb{R}^2$ *is a Borel set of dimension* α, *then*

$$\dim_H(\Delta_y A) \geq \frac{\dim_H(A)}{2} + \varepsilon(\alpha, \eta)$$

for all $y \in \mathbb{R}^2$ *outside of a set of exceptions of dimension* $\leq \eta$.

The same holds if the Euclidean norm is replaced by a C^2 *norm whose unit circle has everywhere positive Gaussian curvature.*

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- Very recently, and relying on the non-linear Bourgain projection thm (plus many other ideas), O. Raz and J. Zahl proved an ε-improvement for the discretized distance set problem considering only 3 non-collinear vantage points x.

An improvement on Kaufman's projection theorem?

- Both Kaufman's and Falconer's classic bounds yield $s(1,3/4) \geq 3/4$ (in other words, the projection of a set of dimension 1 has dimension at least $3/4$, outside of a set of exceptional directions of dimension $\leq 3/4$).

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Theorem* (P.S. work in progress)

If $\eta \in (0, 1)$, then $s(1, \eta) > \eta + \varepsilon(\eta)$.

The end

Many thanks!!!