Nonuniqueness in MCF and Ricci Flow

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Joint work with

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Motivation

find examples of nonuniqueness after a singularity occurs

find examples of nonuniqueness from nonsmooth initial data

both for MCF and RF

Mean Curvature Flow

A family of hypersurfaces parametrized by $X : \mathcal{M}^n \times (t_0, t_1) \to \mathbb{R}^{n+1}$ evolves by MCF if

$$V = H \quad \text{where} \quad V \stackrel{\text{def}}{=} \boldsymbol{X}_t \cdot \boldsymbol{\nu}, \quad H \stackrel{\text{def}}{=} g^{ij}(\nabla \boldsymbol{X}) \boldsymbol{\nu} \cdot \nabla_i \nabla_j \boldsymbol{X}.$$

and $g_{ij}(\nabla \boldsymbol{X}) = \nabla_i \boldsymbol{X} \cdot \nabla_j \boldsymbol{X}$

Theorem. *For any compact smooth initial immersed hypersurface* $X_0 : \mathcal{M} \to \mathbb{R}^{n+1}$ *there exist* $T > 0$ *and a smooth solution* $X : [0, T) \times \mathcal{M} \to \mathbb{R}^{n+1}$ *with* $X(0, p) = X_0(p)$.

Variations:

– if X_0 is proper and has bounded second fundamental form then there is still a proper solution with bounded curvatures, for a short time.

– prescribed boundary conditions

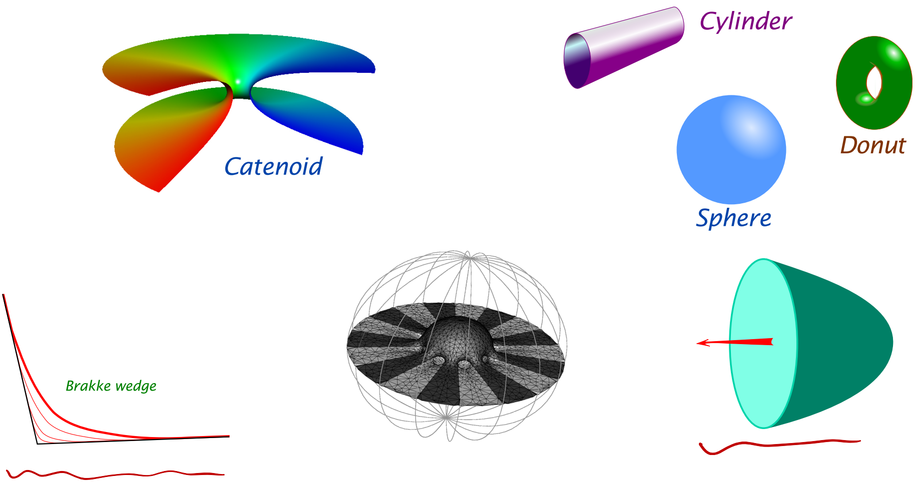
Shrinking and Expanding solitons

"Separate variables"

Self similar solutions:

type			equation for $\mathcal{N}$
Stationary:	$\mathcal{M}_t = \mathcal{N}$	$t \in \mathbb{R}$	$H = 0$
Translators:	$\mathcal{M}_t = \mathcal{N} + t\boldsymbol{v}$	$t \in \mathbb{R}$	$H + \boldsymbol{X} \cdot \boldsymbol{\nu} = 0$
Shinkers:	$\mathcal{M}_t = \sqrt{-t}\,\mathcal{N}$	$-\infty < t < 0$	$H + \frac{1}{2}\boldsymbol{X} \cdot \boldsymbol{\nu}$
Expanders:	$\mathcal{M}_t = \sqrt{t}\,\mathcal{N}$	$0 < t < \infty$	$H - \frac{1}{2}\boldsymbol{X} \cdot \boldsymbol{\nu} = 0$

Examples



Expanders from cones in 3D

A–Ilmanen–Chopp, 1994

C_α : round double cone with opening angle $\alpha \in (0, \frac{\pi}{2})$.

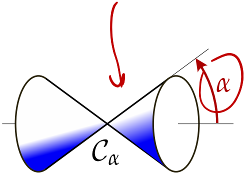
$\mathcal{M}_t^\alpha$ $(t > 0)$: the disconnected expander.

Theorem. *There is an* $\alpha_* \in (0, \frac{\pi}{2})$ *such that*

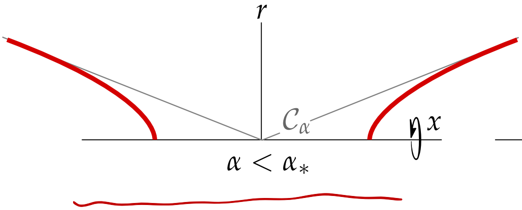
$\alpha < \alpha_* \implies$ $\mathcal{M}_t^\alpha$ *is the unique MCF starting with* C_α

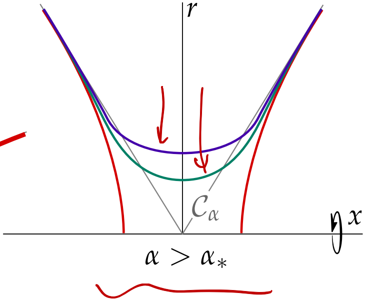
$\alpha > \alpha_* \implies$ *There are three distinct smooth self similar evolutions of* C_α.

Expanders from cones in 3D



$$\alpha_* \approx 58^\circ$$





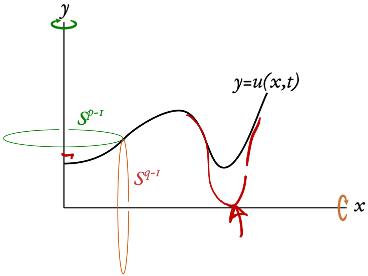
Shrinkers & expanders from cones in $\mathbb{R}^p \times \mathbb{R}^q$

Consider $O(p) \times O(q)$ symmetric hypersurfaces in $\mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^q$ of the formally

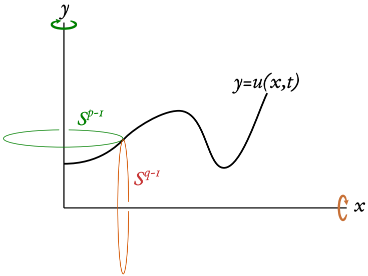
$$\mathcal{M}_t = \left\{(X, Y) \in \mathbb{R}^p \times \mathbb{R}^q : \|Y\| = u(\|X\|, t)\right\}$$

MCF is equivalent with

$$\frac{\partial u}{\partial t} = \frac{u_{xx}}{1 + u_x^2} + \frac{p-1}{x} u_x - \frac{q-1}{u}$$



Shrinkers & expanders from cones in $\mathbb{R}^p \times \mathbb{R}^q$



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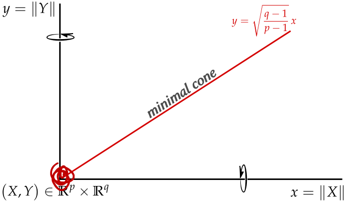


The cone $y = Ax$ with $A = \sqrt{\frac{q-1}{p-1}}$ is stationary.

$p = q = 4$: Simon's cone, minimizing

$4 \leq p + q \leq 7$ minimal cone, but not minimizing

$p \geq 2 \quad q \geq 2$

Shrinkers & expanders from cones in $\mathbb{R}^p \times \mathbb{R}^q$

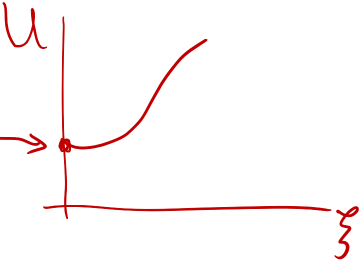
(equations)

$y = \sqrt{\pm t}\, U\left(\frac{x}{\sqrt{\pm t}}\right)$ is a shrinking $(-)$ or expanding $(+)$ soliton iff

$$\frac{U_{\xi\xi}}{1+U_\xi^2} + \left(\frac{p-1}{\xi} \pm \frac{\xi}{2}\right) U_\xi \mp \frac{1}{2} U - \frac{q-1}{U} = 0$$

Boundary condition at $\xi = 0$:

$$U_\xi(0) = 0, \qquad U(0) > 0$$

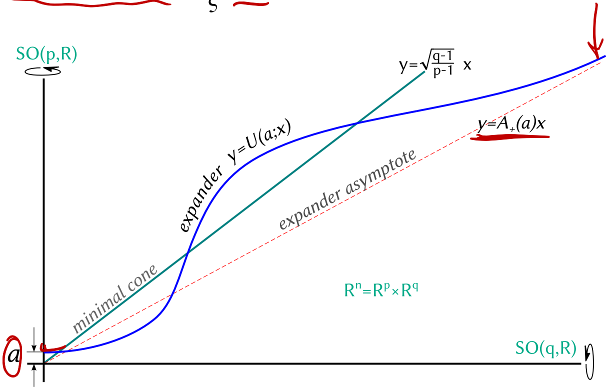


Shrinkers & expanders from cones in $\mathbb{R}^p \times \mathbb{R}^q$

Expanders Theorem (A-Ilmanen-Velázquez). *For each $a > 0$ there is a unique solution $U_+(a;\xi)$ of the expander ODE that is defined for all $\xi \geq 0$ and that satisfies $U_\xi(a;0) = 0$, $U(a;0) = a$.*

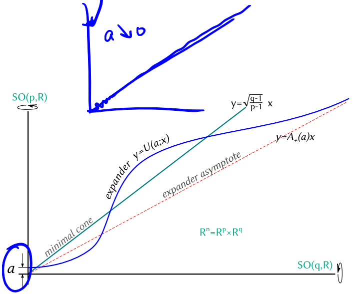
– *$\xi \mapsto U_+(a;\xi)$ is strictly increasing*

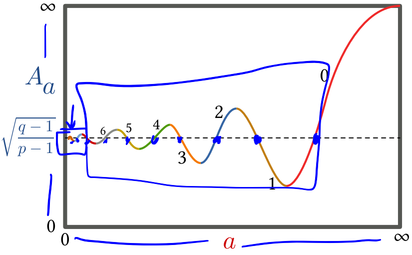
– *the asymptotic slope $A_+(a) = \lim_{\xi\to\infty} \dfrac{U_+(a;\xi)}{\xi}$ exists.*



Multiplicity of expanding solitons

$4 \leq p+q \leq 7 \qquad p, q, \geq 2$



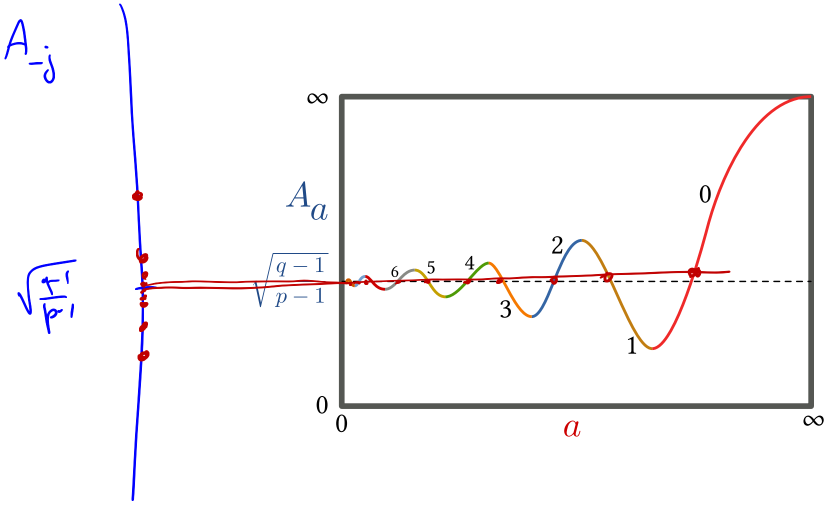


The asymptotic slope $A_+(a)$ is a continuous function of a

$$\lim_{a \searrow 0} A_+(a) = \sqrt{\frac{q-1}{p-1}}$$

$$\forall N \in \mathbb{N}\ \exists \epsilon_N > 0\ \forall A : \left|\sqrt{\tfrac{q-1}{p-1}} - A\right| < \epsilon$$

$$\implies \text{there exist } 0 < a_1 < \cdots < a_N \text{ with } A_+(a_j) = A$$



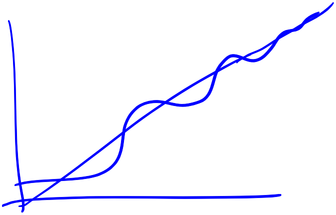
A_{-j}
∞
A_a
$\sqrt{\frac{q-1}{p-1}}$
0
1
2
3
4
5
6
0
0
a
∞

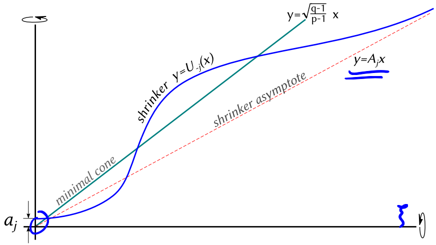
Shrinkers & expanders from cones in $\mathbb{R}^p \times \mathbb{R}^q$

Shrinker Theorem (A-Ilmanen-Velázquez). *There is a sequence of solutions $U_{-,j}(\xi)$ of the shrinker ODE that are defined for all $\xi \geq 0$ and that satisfy $U_\xi(0) = 0$, $U_{-,j}(0) = a_j \searrow 0$.*

– *the asymptotic slopes $A_{-,j} = \lim_{\xi\to\infty} \dfrac{U_{-,j(\xi)}}{\xi}$ exist.*

– $\lim_{j\to\infty} A_{-,j} = \sqrt{\frac{q-1}{p-1}}$





Shrinkers & expanders from cones in $\mathbb{R}^p \times \mathbb{R}^q$

Conclusion:

There is a sequence of shrinking solitons $\mathcal{N}_{-j}$ each of which is asymptotic to a cone with aperture A_{-j}.

$A_{-j} \to \sqrt{\frac{q-1}{p-1}}$ as $j \to \infty$

For each j there are K_j expanding solitons $\mathcal{N}_{+j}^{(1)}, \dots, \mathcal{N}_{+j}^{(K_j)}$ that have the same asymptotic cone as $\mathcal{N}_{-j}$.

$K_j \to \infty$ as $j \to \infty$

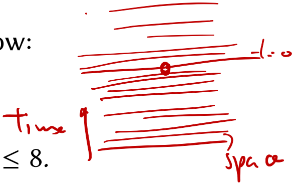
For each j the family of surfaces

$$\mathcal{M}_j^{(m)}(t) = \begin{cases} \sqrt{-t}\mathcal{N}_{-j} & (t < 0) \\ \sqrt{t}\mathcal{N}_{+j}^{(m)} & (t > 0) \end{cases}$$

is a varifold solution of MCF with one singular point at the origin, at time $t = 0$

Ricci flow

A similar construction can be carried out for Ricci flow:



Theorem (A - Knopf, 2019). *Assume* $p, q \geq 2$, $p + q \leq 8$.

For every $K \in \mathbb{N}$ *there is a shrinking soliton* $(G^{-}, \mathfrak{X}^{-})$ *on* $\mathbb{D}^{p+1} \times S^q$ *and there are* K *different expanding solitons* $(G_j^{+}, \mathfrak{X}_j^{+})$ *all of which are asymptotic to the same cone metric on* $(0, \infty) \times S^p \times S^q$.

Together the shrinking and expanding solitons form K *distinct Ricci flow spacetimes with one singular point, all of which coincide for* $t < 0$.

$$-2\mathrm{Rc} = \mathcal{L}_{\mathfrak{X}} g + \lambda g, \quad g = (ds)^2 + \varphi(s)^2 g_{S^p} + \psi(s)^2 g_{S^q}, \quad \mathfrak{X} = f(s)\frac{\partial}{\partial s}$$

Renormalized MCF and Huisken's functional

"Variation of constants"

Shrinking renormalized flow: $\mathcal{M}_t = \sqrt{-t}\,\mathcal{N}_{\log(-t)} \qquad (-\infty < t < 0)$

Evolution Equation: $V = H + \frac{1}{2}\mathbf{X}\cdot\boldsymbol{\nu}$

Huisken's Lyapunov functional: $\mathcal{H}(\mathcal{N}_\tau) = \int_{\mathcal{N}_\tau} e^{-\|\mathbf{X}\|^2/4}\, dH^n_{\mathcal{N}_\tau}$

Expanding renormalized flow: $\mathcal{M}_t = \sqrt{t}\,\mathcal{N}_{\log t} \qquad (0 < t < \infty)$

Evolution Equation: $V = H - \frac{1}{2}\mathbf{X}\cdot\boldsymbol{\nu}$

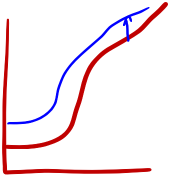
Huisken's Lyapunov functional: $\mathcal{H}(\mathcal{N}_\tau) = \int_{\mathcal{N}_\tau} e^{+\|\mathbf{X}\|^2/4}\, dH^n_{\mathcal{N}_\tau}$

$SO(p) \times SO(q)$ invariant expander flow

Let $\mathcal{N}_t = \{(X, Y) : \|Y\| = u(\|X\|, t)\}$. Then the renormalized expanding MCF is equivalent with

$$\frac{\partial u}{\partial t} = \frac{u_{xx}}{1 + u_x^2} + \left(\frac{p-1}{x} + \frac{x}{2}\right) u_x - \frac{1}{2}u - \frac{q-1}{u}$$

$$\begin{cases} u_x(0, t) = 0 \\ u(x, t) = Ax + o(1) \quad (x \to \infty) \end{cases}$$



Quasilinear parabolic pde of the form

$$u_t = a(x, u, u_x)u_{xx} + b(x, u, u_x)$$

If U is a given expanding soliton, then the IVP generates a real analytic local semiflow in the space

$$X = \{u = U(x) + e^{-\gamma x^2} f \mid x^2 f, x f_x, f_{xx} \in C^{0,\alpha}\}$$

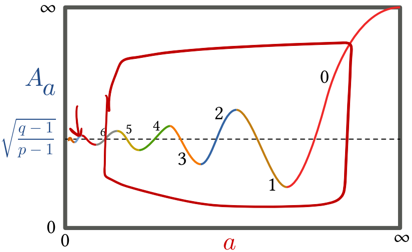
where γ depends on U.

Unstable manifolds

Unstable manifold theorem. *If U is an expanding soliton and if m is the Morse index of the linearization at U, i.e. the number of positive eigenvalues of the operator*

$$\mathcal{L} = \frac{d}{dx}\left(\frac{1}{1+U_x^2}\frac{d}{dx}\right) + \left(\frac{p-1}{x} + \frac{x}{2}\right)\frac{d}{dx} - \frac{1}{2} + \frac{q-1}{U(x)^2}$$

in the space X, then there is an m-dimensional real analytic family of ancient solutions $W(\mu_1, \ldots, \mu_m; x, t)$ to the expanding flow with $W(\mu_1, \ldots, \mu_m; \cdot, t) \in X$, and $W(\mu_1, \ldots, \mu_m; x, t) \to U(x)$ as $t \to -\infty$.



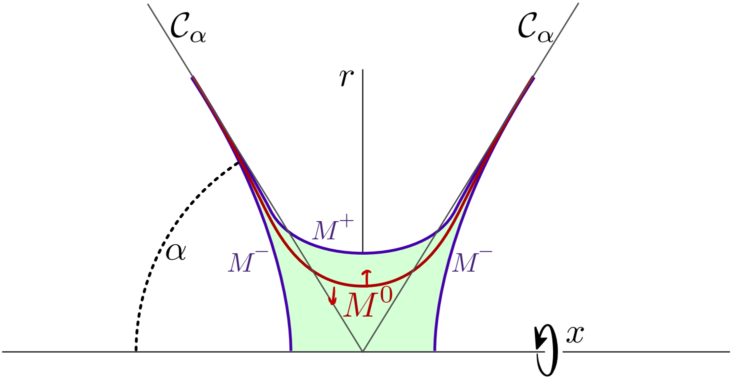
Back to $\mathbb{R}^3$

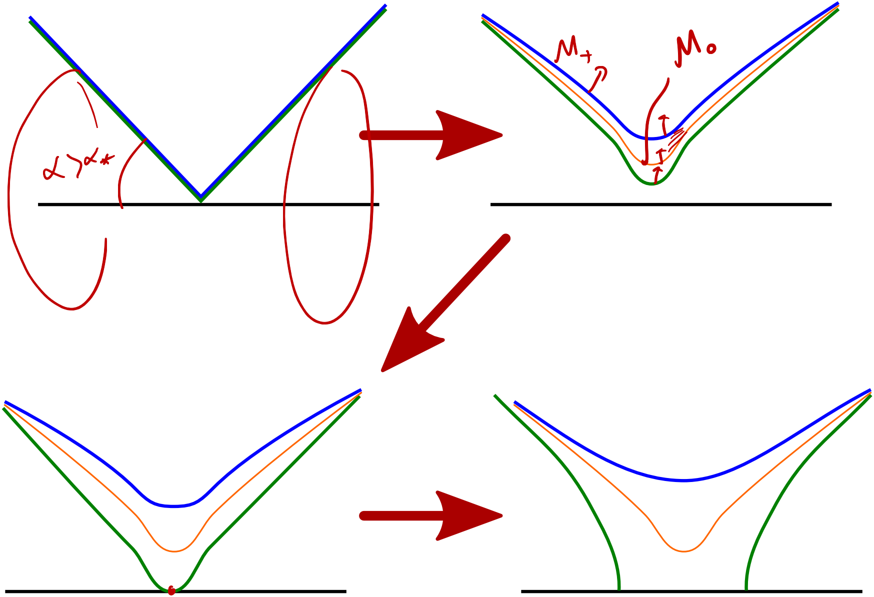
The Minimax Expander and Connecting Orbits

M_+ and M_- are both local minima of the renormalized Huisken functional.

The third expander M_1 between M_- and M_+ is not a local minimizer.

Linearization shows M_1 has Morse index 1.




α>α*
M+
M0

Connecting Orbits with $O(p) \times O(q)$ symmetry