Extreme values of the argument of the zeta function

Alex Dobner

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- $\zeta(s)$ has a meromorphic continuation to all of $\mathbb{C}$
- most interesting behavior is in the *critical strip* $0 \leq \operatorname{Re}(s) \leq 1$
- $\zeta(s)$ has lots of zeros in the critical strip. We know most are near the *critical line* $\operatorname{Re}(s) = 1/2$

Logarithm of zeta

For $Re(s) > 1$,

$$\log \zeta(s) = -\sum_{p} \log(1 - p^{-s}) = \sum_{n \geq 1} \frac{\Lambda(n)}{(\log n) n^s}$$

where

$$\Lambda(n) := \begin{cases} \log p & \text{if } n = p^k \\ 0 & \text{otherwise.} \end{cases}$$

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Heuristic

$$\log \zeta(s) \approx \sum_p \frac{1}{p^s}$$

Riemann-von Mangoldt formula

Denote nontrivial zeros of ζ by $\rho = \beta + i\gamma$.

$$N(T) \coloneqq \text{number of zeros with } 0 < \gamma < T$$

Theorem (Riemann-von Mangoldt)

For $T \geq 1$,

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O(T^{-1}),$$

where $S(T) \coloneqq \frac{1}{\pi} \operatorname{Im} \log \zeta(\frac{1}{2} + iT)$.

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- It is known that $S(T) = O(\log T)$. Hence $N(T) \sim \frac{T}{2\pi}\log T$.
- Average spacing between zeros at height T is $\frac{2\pi}{\log T}$ but on short intervals they can be irregularly spaced

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Upper bounds:

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Ω theorems:

- (Selberg, 1946) $S(t) = \Omega\left(\frac{(\log t)^{1/3}}{(\log\log t)^{7/3}}\right)$
- (Tsang, 1986) $S(t) = \Omega\left(\left(\frac{\log t}{\log\log t}\right)^{1/3}\right)$

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- (Tsang, 1986) $S(t) = \Omega\left(\left(\frac{\log t}{\log\log t}\right)^{1/3}\right)$
- (Montgomery, 1977) On RH, $S(t) = \Omega\left(\left(\frac{\log t}{\log\log t}\right)^{1/2}\right)$
- (Bondarenko-Seip, 2018) On RH,
  $S(t) = \Omega\left(\left(\frac{\log t \log\log\log t}{\log\log t}\right)^{1/2}\right)$

Behavior of $\log \zeta(s)$ in the critical strip

Theorem (Selberg central limit theorem)

For any fixed Δ,

$$\frac{1}{T}\,\mathrm{meas}\left\{ t \in [T, 2T] \colon \frac{\operatorname{Im} \log \zeta(\frac{1}{2} + it)}{\sqrt{\frac{1}{2} \log \log T}} \geq \Delta \right\} = \frac{1}{\sqrt{2\pi}} \int_{\Delta}^{\infty} e^{-u^2/2}\, du + o_{T \to \infty}(1).$$

This also holds for $\operatorname{Re} \log \zeta(\frac{1}{2} + it)$.

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This also holds for $\operatorname{Re} \log \zeta(\frac{1}{2} + it)$.

Restatement: Choosing t randomly from $[T, 2T]$, the random variable $\operatorname{Im} \log \zeta(\frac{1}{2} + it)/\sqrt{\frac{1}{2} \log \log T}$ converges in distribution to $\mathcal{N}(0, 1)$.

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Heuristic

In the range $[T, 2T]$, $S(t)$ has Gaussian distribution with mean 0 and variance $\frac{1}{2\pi^2}\log\log T$.

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$$\text{meas}\{t \in [T, 2T] \colon S(t) \geq V\} \gg T\exp(-cV^2/\log\log T)$$

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- (Soundararajan, 2008) For $\operatorname{Re}\log\zeta$ instead of $\operatorname{Im}\log\zeta$, the conjecture holds for $V \ll (\log T)^{1/2-\varepsilon}$
- (Radziwiłł, 2011) For $V \ll (\log\log T)^{1/2+1/10-\varepsilon}$, the asymptotic predicted by Selberg CLT holds for both real and imaginary parts

Large deviations of $S(t)$

Theorem (D.)

Let $0 < a < 1/3$. There exist constants $\kappa, c > 0$ depending on a such that for all T sufficiently large and all V in the range $(\log T)^a \le V \le \kappa\left(\frac{\log T}{\log\log T}\right)^{\frac{1}{3}}$ we have

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Differences between $\operatorname{Re}\log\zeta$ and $S(t)$:

- To get large values of $\operatorname{Re}\log\zeta(\frac{1}{2} + it)$, may work with $|\zeta(\frac{1}{2} + it)|$ instead
- Possibility of zeros off the critical line can only help you for finding large values of $\operatorname{Re}\log\zeta(\frac{1}{2} + it)$

Steps of proof

Step 1: Get a rigorous version of $\log \zeta(\frac{1}{2} + it) \approx \sum_p \frac{1}{\sqrt{p}} p^{-it}$ using convolution formula of Selberg. This will give $S(t)$ in terms of primes and zeros.

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The crossover point where the measure bound from step 2 is worse than step 3 occurs at $V = \kappa \left(\frac{\log T}{\log \log T} \right)^{1/3}$

Step 1

Let λ be a parameter. Convolving $\log \zeta(\frac{1}{2} + it)$ with a smooth function of width λ^{-1} cuts out large prime frequencies $\log p \geq \lambda$. This gives a formula like

$$\text{smoothed } S(t) \approx \operatorname{Re} \sum_{\log p \leq \lambda} \frac{1}{\sqrt{p}} p^{-it} + Z(t)$$

where $Z(t)$ is a contribution coming from zeta zeros off the critical line. $Z(t)$ is dominated by zeros $\rho = \beta + i\gamma$ for which $|t - \gamma| \leq \lambda^{-1}$.

Step 2

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$$Z(t) \ll (\lambda^2 \theta_t^2)(\lambda^{-1} \log T) = \lambda \theta_t^2 \log T$$

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Apply a *zero density estimate* to see that this holds except on a set of measure

$$\leq T \exp\left(-c'\left(\frac{V \log T}{\lambda}\right)^{1/2}\right).$$

Step 3

Want to understand how often $\mathrm{Re} \sum_{\log p \leq \lambda} \frac{1}{\sqrt{p}} p^{-it} \gg V$.
Terms coming from $p \leq V$ (say) are insignificant because $\sum_{p \leq V} \frac{1}{\sqrt{p}} \ll \sqrt{V}$.

Heuristic

The sum

$$\sum_{\log V \leq \log p \leq \lambda} \frac{1}{\sqrt{p}} p^{-it}$$

for $t \in [T, 2T]$ behaves like

$$\sum_{\log V \leq \log p \leq \lambda} \frac{X_p}{\sqrt{p}}$$

where X_p are iid random variables.

Step 3

So we may predict that $\operatorname{Re}\sum_{\log V\leq\log p\leq\lambda}\frac{1}{\sqrt{p}}p^{-it}$ behaves like a Gaussian with variance

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So we expect it is $\gg V$ on a set of measure

$$\gg T\exp(-c''V^2/(\log\lambda-\log\log V)).$$

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So we expect it is $\gg V$ on a set of measure

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Compare this with upper bound on exceptional set from zeros:

$$\le T \exp\left(-c'\left(\frac{V \log T}{\lambda}\right)^{1/2}\right).$$

To get largest range of V, optimal choice of λ is $\lambda \asymp \log\log T$. This makes the variance in the prime sum $\asymp 1$, which is why our bound in the theorem is $T\exp(-cV^2)$.

Rigorous proofs of steps 2 and 3

Selberg proved the bound $Z(t) \ll \lambda\theta_t^2 \log T$ for every $t \in [T, 2T]$ when $\lambda \asymp \log\log T$. Surprisingly this is unconditional on RH even though it requires control of zeros on intervals of length $1/\log\log T$.

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For step 3, we use Soundararajan's *resonance method*

Resonance method

Let $D(t) = \operatorname{Re} \sum_{p \le \log T} \frac{1}{\sqrt{p}} p^{-it}$ (or any Dirichlet polynomial we want to study).

Pick a *resonator* $R(t) = \sum_{n \le N} f(n) n^{-it}$, and let

$$M_1(R, T) := \int |R(t)|^2 \Phi(\tfrac{t}{T})\, dt$$
$$M_2(R, T) := \int D(t) |R(t)|^2 \Phi(\tfrac{t}{T})\, dt$$

where Φ is a smooth bump function supported in $[1, 2]$.

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Pick R to make this ratio is large as possible!

Resonance method

To get a bound on the measure of $F_V := \{D(t) \geq V\}$, note that

$$\begin{aligned} M_2(R,T) &\leq V\, M_1(R,T) + \int_{F_V} D(t)|R(t)|^2 \Phi\big(\tfrac{t}{T}\big)\, dt \\ &\leq V\, M_1(R,T) + \log T \int_{F_V} |R(t)|^2 \Phi\big(\tfrac{t}{T}\big)\, dt \\ &\leq V\, M_1(R,T) + \log T\, \big(\operatorname{meas}(F_V)\big)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} |R(t)|^4 \Phi\big(\tfrac{t}{T}\big)\, dt\right)^{\frac{1}{2}} \end{aligned}$$

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Then if $M_2(R,T) \geq 2M_1(R,T)$, rearranging gives

$$\operatorname{meas}(F_V) \gg \frac{M_2(R,T)^2}{(\log T)^2} \left(\int |R(t)|^4 \Phi\big(\tfrac{t}{T}\big)\, dt \right)^{-1}$$

If the resonator coefficients are real valued, and N is small relative to T

$$M_1(R,T) \approx T\hat{\Phi}(0) \sum_{n \le N} f(n)^2$$

$$M_2(R,T) \approx T\hat{\Phi}(0) \sum_{\substack{mp=n\le N \\ p \le \log T}} \frac{f(m)f(n)}{\sqrt{p}}$$

$$\int |R(t)|^4 \approx \Phi(\tfrac{t}{T})\, dt \approx T\hat{\Phi}(0) \sum_{\substack{a,b,c,d \le N \\ ab=cd}} f(a)f(b)f(c)f(d)$$

Choose f to be a multiplicative function supported on square free numbers with

$$f(p) = \frac{V}{\sqrt{p}}, \text{ for } (\log N)^{2/3} \leq p \leq (\log N)^{5/6}$$

Can show that the l^2 mass of $f(n)$ is mostly supported on number $n < N$ with at least V prime divisors.

$$\frac{1}{\sum f(i)^2} \sum_{\substack{mp=n\leq N \\ p\leq \log T}} \frac{f(m)f(n)}{\sqrt{p}} \gg V$$

$$\sum_{\substack{a,b,c,d \leq N \\ ab=cd}} f(a)f(b)f(c)f(d) \ll \exp\big(V^2\big)$$

Thanks for listening!