Multilinear singular and oscillatory integrals and applications

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Caltech/UCLA joint analysis seminar
February 2, 2021

Let $n \geq 3$ be an integer, $0 < \delta \leq 1/2$. There is a positive integer N such that each set $S \subseteq \{0, 1, 2, \ldots, N-1\}$ with $|S| \geq \delta N$ contains a non-trivial arithmetic progression of length n.

- Proven by

$$n = 3: \quad \text{Roth '53}$$
$$n \geq 4: \quad \text{Szemerédi '69,'75}$$

- Smallest $N = N(n, \delta)$?

$$N(3, \delta) \leq \exp(\delta^{-C}) \quad \text{Heath-Brown '87}$$
$$N(4, \delta) \leq \exp(\delta^{-C}) \quad \text{Green and Tao '17}$$
$$N(n, \delta) \leq \exp(\exp(\delta^{-C_n})) \text{ if } n \geq 5 \quad \text{Gowers '01}$$

A variant of the averaging trick by Varnavides gives:

Theorem

Let $n \geq 3$ and $d \geq 1$ integers. Then there exists $C(n,d)$ such that for any $0 < \delta \leq 1/2$ and any measurable $A \subseteq [0,1]^d$ with measure at least δ one has

$$\int_{[0,1]^d} \int_{[0,1]^d} \prod_{i=0}^{n-1} 1_A(x+it)dtdx$$

$$\geq \begin{cases} \exp(\delta^{-C(n,d)})^{-1} & ; n = 3,4 \\ \exp(\exp(\delta^{-C(n,d)})^{-1}) & ; n \geq 5 \end{cases}$$

We start with 3-term arithmetic progressions

$$x, x+t, x+2t$$

in a set $A \subseteq [0,1]^d$.

Question: What can we say about the gaps

$$G(A) = \big\{ t \in [-1,1]^d : (\exists x \in [0,1]^d)\big(x, x+t, x+2t \in A\big) \big\}?$$

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A generalization of the Steinhaus theorem.

If $|A| > 0$, then $G(A)$ contains a ball around the origin.

- We may assume A is compact. Find an open set $U \supseteq A$ with $|U| \leq \frac{7}{6}|A|$.
- Set $\varepsilon := \frac{\text{dist}(A, \mathbb{R}^d \setminus U)}{3} \neq 0$.
- For any $t \in [-1, 1]^d, |t| < \varepsilon$,

$$A \cap (A - t) \cap (A - 2t) \subseteq U$$

  and occupies at least half of U, so it is non-empty.
- Take any point from the intersection and arrive at an a.p. $x, x + t, x + 2t$ in A. □

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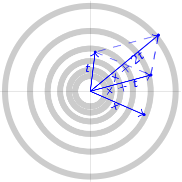
No lower bound on the radius ε of the ball contained in $G(A)$ that depends only on $|A|$. Such a bound is impossible!

$$\text{gaps}_3(A) = \{\lambda \in [0,\infty) : (\exists x, t)(x, x+t, x+2t \in A \text{ and } |t| = \lambda)\}$$

Does $\text{gaps}_3(A)$ contain an interval of length depending only on d and $|A|$? No!

- Consider $A = \{x \in [0,1]^d : (\exists m \in \mathbb{Z})(m - \frac{1}{10} < \frac{|x|}{\varepsilon} < m + \frac{1}{10})\}$



- Parallelogram law:

$$|x|^2 - 2|x+t|^2 + |x+2t|^2 = 2|t|^2$$

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- Therefore any interval contained in $\text{gaps}(A)$ has to be shorter than ε. However, $|A| \gtrsim 1$.

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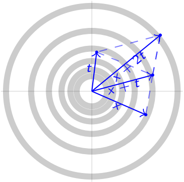

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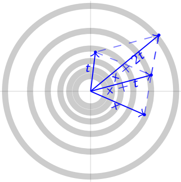

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- Cook, Magyar and Pramanik investigated sizes of gaps

$$\text{gaps}_{p,3}(A) = \big\{\lambda \in [0,\infty) : (\exists x, t)\big(x, x+t, x+2t \in A \text{ and } \|t\|_{\ell^p} = \lambda\big)\big\}$$

of three-term progressions in ℓ^p, $p \neq 2$ in subsets of $\mathbb{R}^d$ of positive Banach density.

- A consequence of their work is the following result in the unit box:

Theorem (Cook, Magyar, Pramanik, 2015)

If $p \neq 1, 2, \infty$, d sufficiently large, $0 < \delta \leq 1/2$, $A \subseteq [0,1]^d$ a measurable set of measure $|A| > \delta$. Then $\text{gaps}_{p,3}(A)$ contains an interval of length depending only on p, d, δ.

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What about longer progressions?

$$\text{gaps}_{p,n}(A) = \big\{\lambda \in [0,\infty) : (\exists x,t)\big(x, x+t, \ldots, x+(n-1)t \in A \text{ and } \|t\|_{\ell^p} = \lambda\big)\big\}$$

Theorem (D., Kovač 2020)

For every $n \geq 3$, $p \in [1,\infty) \setminus \{1,2,\ldots,n-1\}$, and dimension $d \geq D(n,p)$ there exists a constant $C(n,p,d)$ with the following property: if $0 < \delta \leq 1/2$ and $A \subseteq [0,1]^d$ is a measurable set with $|A| \geq \delta$, then the set $\text{gaps}_{p,n}(A)$ contains an interval I with

$$|I| \geq \begin{cases} \big(\exp(\exp(\delta^{-C(n,p,d)}))\big)^{-1} & \text{when } 3 \leq n \leq 4, \\ \big(\exp(\exp(\exp(\delta^{-C(n,p,d)})))\big)^{-1} & \text{when } n \geq 5. \end{cases}$$

- Sharp regarding the values of p.
- $D(n,p) = 2^{n+3}(n+p)$ works
- The lower bounds reflect the best known bounds in Szemeredi's theorem.

- Multilinear which detect progressions with gaps of size $\lambda > 0$ in a set $A \subseteq [0,1]^d$ of measure $|A| > \delta$:

$$\mathcal{N}_\lambda^0 = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \prod_{i=0}^{n-1} 1_A(x + it)\, \mathrm{d}\sigma_\lambda(t)\, \mathrm{d}x$$

where $d\sigma_\lambda$ surface measure on the ℓ^p sphere of radius λ

- Smoothened out version at a scale $0 < \epsilon < 1$:

$$\mathcal{N}_\lambda^\epsilon = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Big(\prod_{i=0}^{n-1} 1_A(x + it) \Big) (\sigma_\lambda * \varphi_{\epsilon\lambda})(t)\, \mathrm{d}t\, \mathrm{d}x$$

- For each λ we decompose

$$\mathcal{N}_\lambda^0 = \mathcal{N}_\lambda^1 + \left(\mathcal{N}_\lambda^0 - \mathcal{N}_\lambda^\epsilon\right) + \left(\mathcal{N}_\lambda^\epsilon - \mathcal{N}_\lambda^1\right)$$

- Sturctured part: lower bound uniform in λ. This is guaranteed by Szemeredi's theorem.
- Uniform part: controlled by Gowers uniformity norms. Small uniformly in λ. Follows from a bound on certain oscillatory integrals.
- Error part: Our goal is to find $j \in \{1, 2, \ldots, J\}$ such that for each $\lambda \in (2^{-j}, 2^{-j+1}]$ the error term is small.

In all of the above, "small" and J chosen suitably.

Combining the three estimates gives a lower bound on $\mathcal{N}_\lambda^0$.

Error part: How to find one such interval $(2^{-j}, 2^{-j+1}]$? Pigeonholing!

If this doesn't happen, then for each $j \in \{1, \ldots, J\}$ there exist $\lambda_j \sim 2^{-j}$ such that the error term large. Contradicts

$$\sum_{j=1}^{J} |\mathcal{N}^{\epsilon}_{\lambda_j} - \mathcal{N}^{1}_{\lambda_j}| \leq \epsilon^{-F(n,p,d)} o(J)$$

Left-hand side equals

$$\sum_{j=1}^{J} \Big| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Big(\prod_{i=0}^{n-1} 1_A(x + it) \Big) (\sigma_{\lambda_j} * \varphi_{\epsilon\lambda_j} - \sigma_{\lambda_j} * \varphi_{\lambda_j})(t) \, \mathrm{d}t \, \mathrm{d}x \Big|$$

- $n = 3$: bilinear Hilbert transform (Lacey, Thiele, 1997, 1999)

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f_0(x) f_1(x + t) f_2(x + 2t) \frac{dt}{t} dx$$

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- $n \geq 4$: multilinear Hilbert transform, open problem.

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f_0(x) f_1(x+t) f_2(x+2t) \dots f_{n-1}(x+(n-1)t) \frac{dt}{t} dx$$

Cancellation estimates suffice. Tao (2015), Zorin-Kranich (2015), D., Kovač, Thiele (2016).

Patterns in sets $A \subseteq \mathbb{R}^d$ of positive upper Banach density

$$\overline{\delta}_d(A) := \limsup_{N\to\infty} \sup_{x\in\mathbb{R}^d} \frac{|A \cap (x + [0, N]^d)|}{|x + [0, N]^d|}$$

- Falconer and Marstrand 1986, Furstenberg, Katznelson, and Weiss 1978: the distance/gap set

  $$\{|t| : x, x + t \in A\}$$

  where $A \subseteq \mathbb{R}^2$ of positive density contains all large numbers.
- Bourgain 1986: Isometric copies of all large dilates of a fixed simplex in $\mathbb{R}^n$ whose affine span $n - 1$ dimensional.
- 3AP: Cook, Magyar, Paramanik 2015 measured in ℓ^p. Longer progressions: open.
- Product of simplices in $\mathbb{R}^d$: Lyall and Magyar (2018).

- A particular case of Lyall and Magyar: Let $n \geq 1$, $A \subseteq ([0,1]^2)^n$ with $|A| > \delta$: there exists an interval I with

$$|I| \geq (\exp(\exp(\cdots \exp(C(n)\delta^{-3\cdot 2^n})\cdots)))^{-1}$$

such that for any $\lambda \in I$ one finds a box

$$(x_1 + r_1 t_1, x_2 + r_2 t_2, \ldots, x_n + r_n t_n), \quad r_i \in \{0,1\}$$

in A with $|t_i| = \lambda$ for each i.

- D, Kovač 2020: Quantitative improvement to

$$|I| \geq (\exp(\delta^{-C(n)}))^{-1}$$

Simpler than 3AP. Twisted paraproducts (vs the multilinear HT).

- Harder: corners

$$(x_1, x_2, \ldots, x_n), (x_1 + t, x_2, \ldots, x_n), \ldots, (x_1, x_2, \ldots, x_n + t)$$

$n = 3$, D. Kovač, Rimanič 2016 in sets of positive density, in ℓ^p.
$n \geq 4$ analogous result in $([0,1]^2)^n$ but bad quantitative estimates.
Leads to the simplex Hilbert transform – its boundedness open, cancellation estimates sufficient.

What about non-linear progressions, say,

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Theorem (Bourgain 1988)

For every $A \subset [0,1]$ with Lebesgue measure at least δ there exist $t > \exp(-\exp(\delta^{-c}))$ and x with $x, x+t, x+t^2 \in A$.

- D., Guo, Roos 2016: polynomials.
- Bourgain's argument is based on a structural/frequency decomposition of functions and a pigenoholing argument.
- One key estimate for an error term:

Let $\widehat{g}$ be supported near $\lambda \gg 1$. The there exist $C, \sigma > 0$ such that

$$\Big\| \int_{\mathbb{R}} f(x+t) g(x+t^2) \eta(x,t) dt \Big\|_1 \leq C\lambda^{-\sigma} \|f\|_2 \|g\|_2$$

where η is a smooth compactly function supported in $\mathbb{R} \times \mathbb{R} \setminus \{0\}$.

- Investigated by Christ 2020.

- Bilinear Hilbert transfrom along curves

$$T(f,g)(x) = \int_{\mathbb{R}} f(x+t)g(x+t^2)\frac{dt}{t}$$

Li '13, Lie '15, Li and Xiao '16, Lie '18. Previous estimate and paraproducts, splitting motivated by stationary phase considerations.

- Multiplier analysis:

$$T(f,g)(x) = \int_{\mathbb{R}^2} \widehat{f}(\xi)\widehat{g}(\eta)\Big(\int_{\mathbb{R}} e^{2\pi i(\xi t+\eta t^2)}\frac{dt}{t}\Big)e^{2\pi i x(\xi+\eta)}d\xi d\eta$$

Derivative of the phase: $\xi - 2\eta t$. Decompose $\widehat{f}$, $\widehat{g}$, $1/t$ into Littlewood-Paley blocks. If $|t| \sim 1$:

– $|\xi|$ and $|\eta|$ small: no oscillation (paraproducts)

– $|\xi| \ll |\eta|$ or $|\eta| \ll |\xi|$: rapid decay (paraproducts)

– $|\xi| \sim |\eta|$ large: critical points (local inequality)

- Longer progressions $x, x+t, x+t^2, x+t^3, \dots, x+t^n$ open
- A pattern which is in difficulty betweem three- and four term progressions: "non-linear corners" in $A \subseteq [0,1] \times [0,1]$

$$(x,y),(x+t,y),(x,y+t^2)$$

- Imply the result by Bourgain.
- Key estimate:

Theorem (Christ, D., Roos 2020)

Let $\widehat{g}(\xi_1,\xi_2)$ be supported near $|\xi_2| \sim \lambda$. Then there is $C>0$ and $\sigma>0$ such that for all $f,g \in \mathcal{S}(\mathbb{R}^d)$

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- More fundamental motivation: Natural object from the point of view from oscillatory integrals. The first natural higher-dimensional case *not* treated by Christ 2020.

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- The key estimate can be to obtain L^p bound the triangular Hilbert transfrom along curves:

Theorem (Christ, D., Roos 2020)

The following bound holds

$$\Big\| \int_{\mathbb{R}} f(x+t,y)g(x,y+t^2)\frac{dt}{t}\Big\|_r \leq C\|f\|_p\|g\|_q$$

where $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$, $1<p,q<\infty$, $1\leq r<2$

- Implies bounds for the bilinear HT along curves and oscillatory integrals of Stein and Wainger 1995, 2001
- Key ingredient #2 for the SIO: bounds for the twisted paraproduct

$$\int_{\mathbb{R}^2} f(x+t,y)g(x,y+s)K(s,t)dsdt$$

  where K two-dimensional CZ kernel with parabolic scaling. First investigated by Kovač (2010, 2020 – connection with anisotropic boxes), Bernicot 2010. Range of exponents!

- Bound for the corresponding maximal function.
- Continuous averages studied by Austin 2012:

$$\frac{1}{n}\int_0^n f_1(S^t x) f_2(T^{t^2} x)dt$$

where $S^t x = (t, 0) \cdot x$ and $T^t x = (0, t) \cdot x$ with a given $\mathbb{R}^2$ action on a probability space $(X, \mathcal{F}, \mu)$

– a.e. convergence as $n \to \infty$: Christ, D., Kovač, Roos 2020

- Ergodic averages

$$\frac{1}{n}\sum_{i=0}^{n-1} f(S^i x) g(T^{i^2} x)dt$$

where $S, T : X \to X$ commuting m.p. transformation on a probability space X

– a.e. convergence as $n \to \infty$ is open in general.

– $S = T$: Krause, Mirek, Tao 2020.

- Enough to prove a bound on $\mathrm{L}^\infty \times \mathrm{L}^\infty$ by $\mathrm{L}^{3/2} \to \mathrm{L}^3$ improving for the parabola (Strichartz '70) and interpolation.
- **Spatial decomposition** of f_1, f_2 at scale $\lambda^{-1/2} < \lambda^{-\gamma} < \lambda^{-1}$.
- **Cauchy-Schwarz** in x, y to compensate a highly singular situation: bound the L^1 norm of the localized operator by

$$\lambda^{-2\gamma} \int_{\mathbb{R}^4} f_1(x+t+s, y)\overline{f_1(x+t, y)} \\ f_2(x, y+(t+s)^2)\overline{f_2(x, y+t^2)}\zeta(x, y, t, s)dx\,dy\,dtds$$

where ζ is a smooth non-negative function compactly supported in a cube with side lengths $O(\lambda^{-\gamma})$

- Replace (localized) f_1, f_2 by

$$f_1(x+t+s,y)\overline{f_1(x+t,y)} = D_s^{(1)} f_1(x+t,y)$$
$$f_2(x,y+(t+s)^2)\overline{f_2(x,y+t^2)} = D_{2s\bar{t}}^{(2)} f_2(x,y+t^2) + O(\lambda^{-2\delta})$$

where

$$D_s^{(1)} f(x,y) = f(x+s,y)\overline{f(x,y)}$$
$$D_s^{(2)} f(x,y) = f(x,y+s)\overline{f(x,y)}.$$

and regard s as a parameter. (Ratio $\frac{2\cdot 2}{3} < 2$ vs $\frac{4\cdot 2}{4} = 2$)

- Taking $D_s^{(j)}$, high frequency can be converted into low frequencies: e.g. if $f(x,y) = e^{ia(y)x}$, then $D_s^{(1)} f(x,y) = e^{ia(y)s}$ independent of x.

- Gain if "$D_s^{(j)} f_j$ has large frequencies for most s", i.e.

$$\int_{\mathbb{R}} \int_{\mathbb{R}^2} 1_{|\xi_j| \leq R} |\widehat{D_s^{(j)} f_j}(\xi)|^2 \, d\xi \, ds \lesssim \varrho \|f\|_2^4.$$

  for suitable R, ϱ and at least one j.

- **Structural decomposition:** $f = f_\sharp + f_\flat$
- $f_\flat$ good in the above sense, $f_\sharp$ admits a structure

$$f_\sharp(x) = \sum_{n=1}^{\mathcal{N}} h_n(x) e^{i\alpha_n x}$$

  with $\mathcal{N} = O(\varrho^{-1})$, h_n smooth and supported in $[-R, R]$, $\alpha_n \in \mathbb{R}$.
- Apply this decomposition to the fibers of f_1 and f_2.
- Terms with $f_\sharp$ lead to a **sublevel set estimate**.

Lemma

Let $K \subset \mathbb{R}^2 \times (0, \infty)$ *be a compact set and* α, β *measurable functions* $\mathbb{R}^2 \to \mathbb{R}$*. Suppose that either* $|\alpha| \asymp 1$ *or* $|\beta| \asymp 1$*. Then there exist* $\sigma, C \in (0, \infty)$ *such that for all* $\epsilon \in (0, 1]$*,*

$$\left|\{(x, y, t) \in K : |\alpha(x + t, y) - 2t\beta(x, y + t^2)| \leq \epsilon\}\right| \leq C\epsilon^{\sigma}.$$

The constants C *and* σ *only depend on* K *and not on the measurable functions* α, β*.*

Thank you!