Stationary random entire functions and related questions

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The talk is partly based on a joint work

Stationary random entire functions

Motivation

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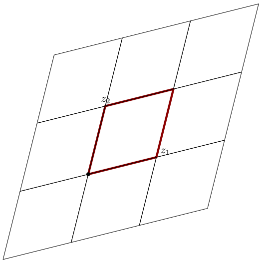
NO! Every entire function with two independent periods, is bounded and therefore constant.



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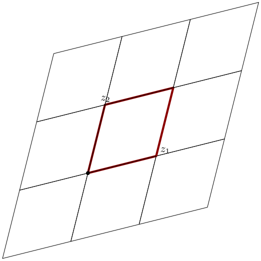

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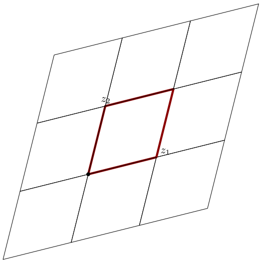

- In fact, there is no translation invariant metric defined on the space of entire functions.
- If there was, then for every n, $\rho(0, e^z) = \rho(0, e^{z-1}) = \rho(0, e^{z-n})$ implying that

$$0 = \lim_{n\to\infty} \rho(0, e^{z-n}) = \rho(0, e^z)$$
$$\Rightarrow e^z = 0.$$

Definitions

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- The group $\mathbb{C}$ acts on the space of entire functions, $\mathcal{E}$, by translations: for every $w \in \mathbb{C}$ and every entire function f,

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- A probability measure, λ, defined on $\mathcal{E}$ is called a **non-trivial translation invariant probability measure** if it is not supported on the constant functions and

$$\lambda(A) = \lambda \circ T_w^{-1}(A) := \lambda\left(\{T_{-w}f,\ f \in A\}\right),$$

for all measurable sets $A \subset \mathcal{E}$ and for every $w \in \mathbb{C}$.

Question

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- Non-formal: 'Is it possible to create a random entire function which is periodic by law?'

- Formal: Does there exist a non-trivial translation invariant probability measure on the space of entire functions?

Minimal Possible Growth

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- Do such measures exist?!!?!

YES! Many B.Weiss, 1997

Minimal Possible Growth

- Do such measures exist?!!?!

YES! Many B.Weiss, 1997

- **Question:** [Weiss] What is the minimal possible growth of functions in the support of such measures:

$$R \mapsto M_F(R) := \max_{|z|=R} |F(z)|, R \nearrow \infty?$$

Bounds on the Growth

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(A) For every non-trivial translation invariant probability measure on the space of entire functions

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(B) There exists a non-trivial translation invariant probability measure on the space of entire functions with

$$\limsup_{R\to\infty} \frac{\log\log M_f(R)}{\log^{2+\varepsilon} R} = 0,\ \forall\varepsilon > 0, \quad a.s.$$

Idea of Proof- Construction

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- The classical Krylov-Bogolyubov construction:
  Given a function f, and a sequence of sets $\{S_n\} \nearrow \mathbb{C}$ define the sequence of probability measures:

  $$\mu_n(A) = \frac{1}{m(S_n)} \int_{S_n} \mathbf{1}_A \left(T_w f\right) dm(w), \; m = \text{Lebesgue's measure}$$

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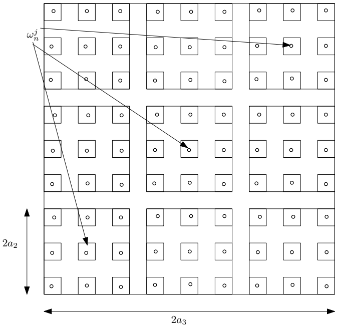
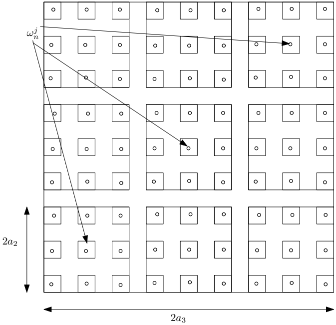
- To have a non-trivial limiting measure, the underlying function, f, has to be “self similar”.

Self similar function



Self similar function

- We constructed an 'inside out Cantor set':



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- We constructed an 'inside out Cantor set':
- $C_{k+1} = \bigcup_{j=0}^{8} T_{w_j} C_k$,
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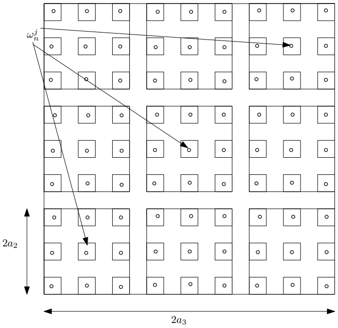

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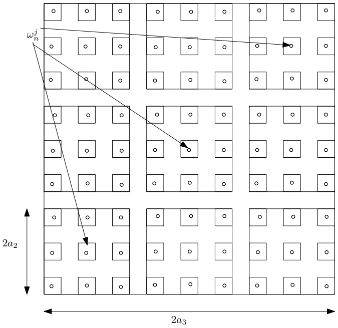
- We constructed an 'inside out Cantor set':
- $C_{k+1} = \bigcup_{j=0}^{8} T_{w_j} C_k$,
  $C = \bigcup_{k=1}^{\infty} C_k$.
- We constructed the function f so that it is almost periodic, looks almost the same on each copy of C_k inside C_{k+1}.



Recurrently Bounded Functions

Recurrent Sets

Recurrent Sets

- We say a set $E \subset \mathbb{R}^d$ is an (ε, R)-**recurrent set** if all $x \in \mathbb{R}^d$

$$\frac{m\left(B\left(x, R\left(|x|\right)\right) \cap E\right)}{m\left(B\left(x, R\left(|x|\right)\right)\right)} \geq m\left(B\left(x, \varepsilon\left(|x|\right)\right)\right), \; m \text{ Lebesgue's measure.}$$

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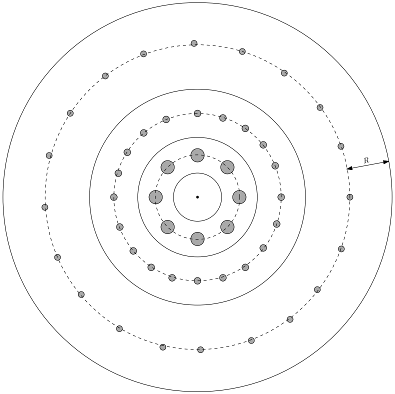

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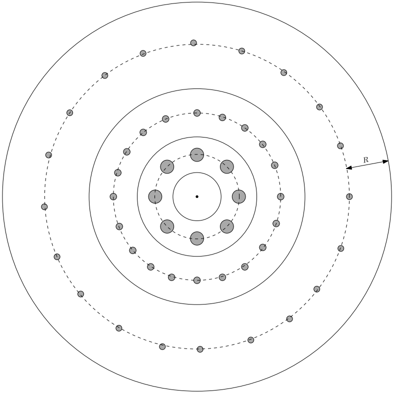

- **Question:** What it the minimal possible growth of subharmonic functions with (ε, R)-recurrent zero set?

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Recall that for every subharmonic function u,

$$u(x) \leq \frac{1}{m(B(x,R(|x|)))}\int_{B(x,R(|x|))} u(y)dm(y)$$

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Implying that

$$\Rightarrow e^{m(B(x,\varepsilon(|x|)))}\cdot u(x)\le M_u(B(x,R(|x|))).$$

Subharmonic Functions with Recurrent Zero Set

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Theorem (G. *Arxiv, 2019*)

Let $\varphi(t) = \dfrac{1}{R(t)\sqrt{-\mathcal{K}_{d-2}(\varepsilon(t))}}$, $\mathcal{K}_{d-2}(t) := \begin{cases} \log(t) & , d = 2 \\ \frac{-1}{t^{d-2}} & , d \geq 3 \end{cases}$.

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(A) Assume that $\limsup_{t\to\infty} \frac{1}{t\cdot\varphi(t)} < 1$, *and let* u *be a subharmonic function in* $\mathbb{R}^d$ *so that its zero set is* (ε, R)*-recurrent, and there exists* $x_0 \in \mathbb{R}^d$ *so that* $u(x_0) \geq 1$. *Then*

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(B) *If* $\frac{d}{dt}\left(\frac{1}{\varphi(t)}\right)$ *is bounded, then there exists a subharmonic function,* u *in* $\mathbb{R}^d$ *whose zero set is* (ε, R)*-recurrent, while* $u(0) \geq 1$ *and*

$$\limsup_{\rho\to\infty} \frac{\log\left(M_u(\rho)\right)}{\int_1^\rho \varphi(t)dt} < \infty.$$

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- There is still a gap, which grows with the dimension.

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$$\mathbb{P}\left(BM \text{ in } B_\rho \setminus E \text{ hits } \partial B_\rho\right) \le C \exp\left(-c\int_1^\rho \varphi(t)dt\right).$$

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The upper bound- dependence of layers

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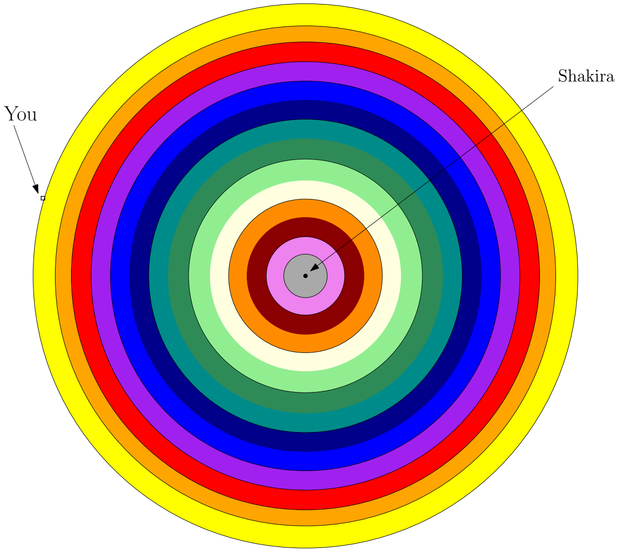
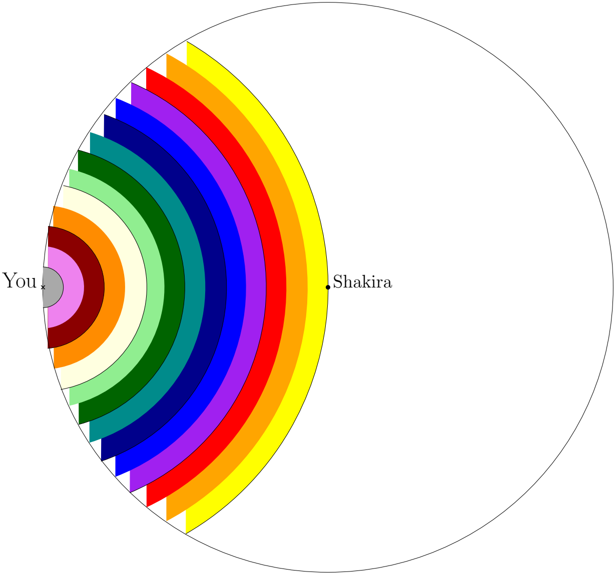


The upper bound- dependence of $\sim \sqrt{-\mathcal{K}_{d-2}(\varepsilon)}$ layers



Frequently Oscillating Functions

Stationary random entire functions Recurrently Bounded Functions Frequently Oscillating Functions

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- Given a subharmonic function u in $\mathbb{R}^d$ and a basic cube I, let:

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- A basic cube is *rogue* if it does not satisfy either (P_1) or (P_2).

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Observation

There exists a constant c_d so that for every subharmonic function u defined in a neighbourhood of the unit ball $B \subset \mathbb{R}^d$,

$$\lambda_{d-1}\left(\{u \le 0\} \cap \frac{1}{2}B\right) > \varepsilon > 0 \Rightarrow \sup_{y \in B} u(y) \ge u(0)e^{c_d \cdot \varepsilon}.$$

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Why is that true **(For the experts- scratching that itch...)**

$E \subset \frac{1}{2}B$ is compact $\Rightarrow \omega(0, E; B \setminus E) \gtrsim_d \lambda_{d-1}(E)$.

Let $E := \{u \le 0\} \cap \frac{1}{2}B$ and define $\Omega = B \setminus E$. Then

$$\begin{aligned} u(0) \le \int_{\partial\Omega} u(y) d\omega(0, y; \Omega) &\le M_u(B) \cdot \omega(0, \partial B; \Omega) \le \\ \le \cdots &\le M_u(B) e^{-\alpha_d \cdot \varepsilon}. \end{aligned}$$

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- **Definition:** Given a monotone non-decreasing function $f(t) \le t^d$, a subharmonic function u, is called f**-oscillating** if

$$\limsup_{N \to \infty} \frac{\#\left\{\text{rogue basic cubes in } [-N, N]^d\right\}}{f(2N)} < 1.$$

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- **Question:** What can we say about the minimal possible growth of f-oscillating subharmonic functions in general?

Optimal Bounds

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Theorem (G. *Arxiv, 2020*)

Let $f(t) = t^\alpha$, and define

$$\varphi_f(R) := \begin{cases} R & ,\alpha \le 1 \\ R^{\frac{d-\alpha}{d-1}} \log^{\frac{d}{d-1}}(R) & ,\alpha > 1 \end{cases}.$$

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Let $f(t) = t^\alpha$, and define

$$\varphi_f(R) := \begin{cases} R & , \alpha \le 1 \\ R^{\frac{d-\alpha}{d-1}} \log^{\frac{d}{d-1}}(R) & , \alpha > 1 \end{cases}.$$

(A) $\exists c_0$ so that if $f(t) \le c_0 t^d$ for all t large, then every f-oscillating subharmonic function u satisfies

$$\liminf_{R\to\infty} \frac{\log(M_u(R))}{\varphi_f(R)} > 0.$$

Optimal Bounds

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Theorem (Buhovsky, G., Logunov, and Sodin *Journal d'Analyse Mathematique, 2019.*)

(A) *For every non-trivial translation invariant probability measure on the space of entire functions*

$$\lim_{R\to\infty} \frac{\log\log M_f(R)}{\log^{2-\cancel{\varepsilon}} R} \cancel{= \infty,\ \forall\varepsilon > 0}, \quad a.s.$$

> 0

corollary

(B) *There exists a non-trivial translation invariant probability measure on the space of entire functions with*

$$\limsup_{R\to\infty} \frac{\log\log M_f(R)}{\log^{2+\varepsilon} R} = 0,\ \forall\varepsilon > 0, \quad a.s.$$

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The theorem above holds for every function $f(t) = t^\alpha \cdot g(t)$ with g a slowly varying function, with

$$\varphi_f(R) := \frac{R}{1 + \left(\frac{f(R)}{R}\right)^{\frac{1}{d-1}}} \log^{\frac{d}{d-1}} \left(2 + \frac{f(R)}{R}\right),$$

if either $\alpha < d$ or $\alpha = d$ and g is monotone non-increasing.

The lower bound

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Observation (Reminder)

$$\lambda_{d-1}\left(\{u \leq 0\} \cap \frac{1}{2}B\right) > \varepsilon > 0 \Rightarrow \sup_{y\in B} u(y) \geq u(0)e^{c_d\cdot\varepsilon}.$$

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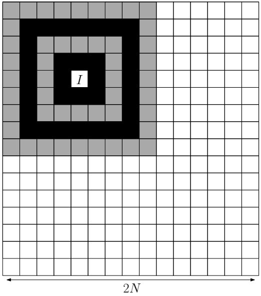

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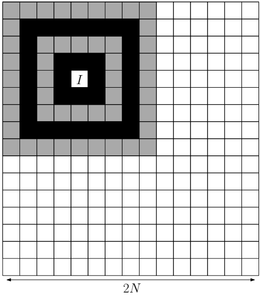

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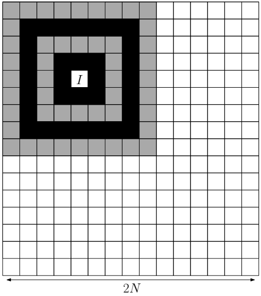

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    u_ξ is u translated by ξ, rescaled by r_ξ.

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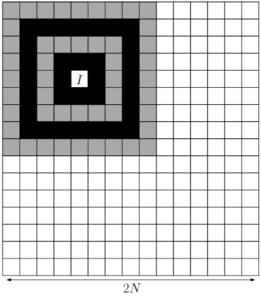

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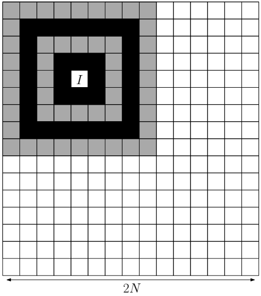

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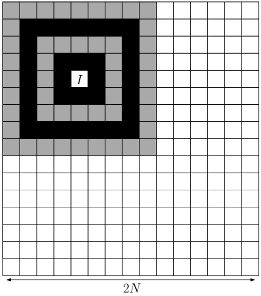

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- By induction $M_u(I)\cdot e^{\#\{C_j\}\delta_d}\le M_u([-N,N]^d)$.

Example

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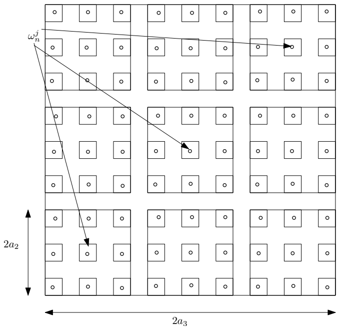

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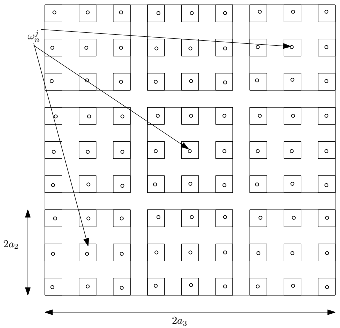
Could ‘self similarity’ help us here?

No! for 2 reasons:



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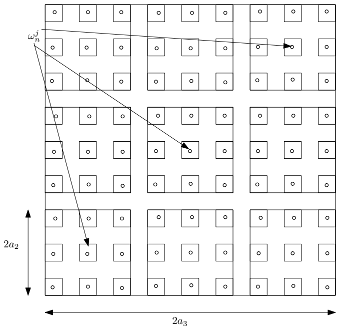

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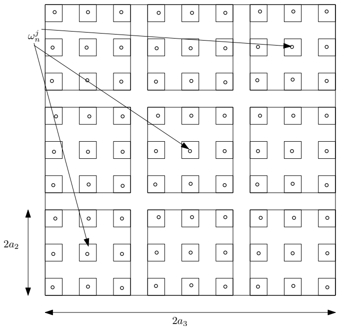

No! for 2 reasons:

1) ‘Self Similarity’ by definition means we accumulate rogue cubes from smaller scales. The accumulation is proportional to the volume.

2) We separate similar copies by hyperplanes. The dimension of a hyperplane is $(d-1)$, which will work for $d=2$ but for higher dimensions only if $f(t) \gtrsim t^{d-1}$.

Example: d=2

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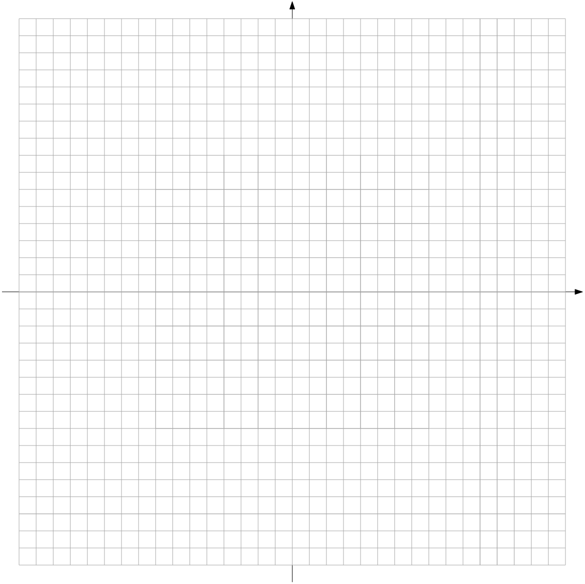

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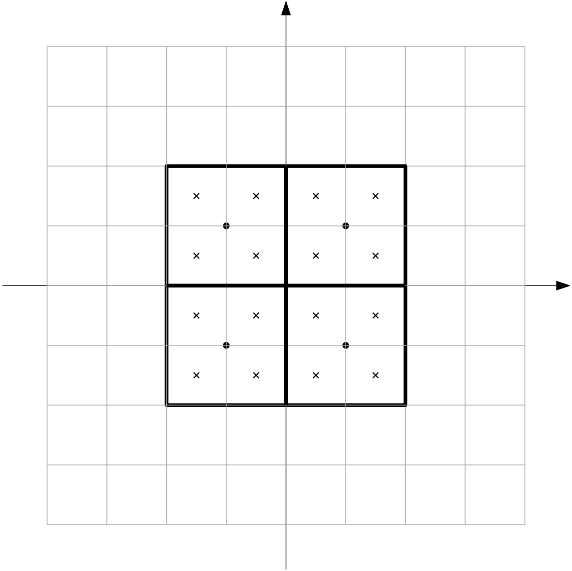

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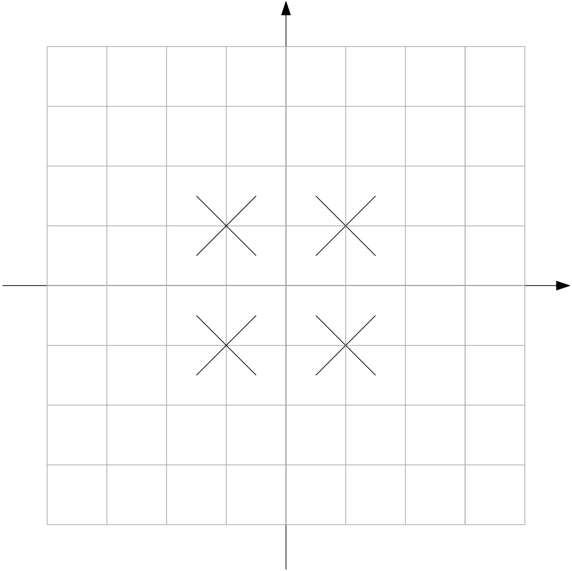

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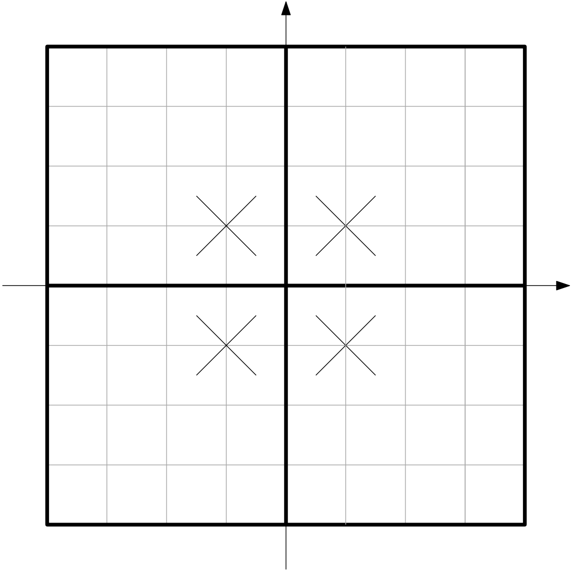

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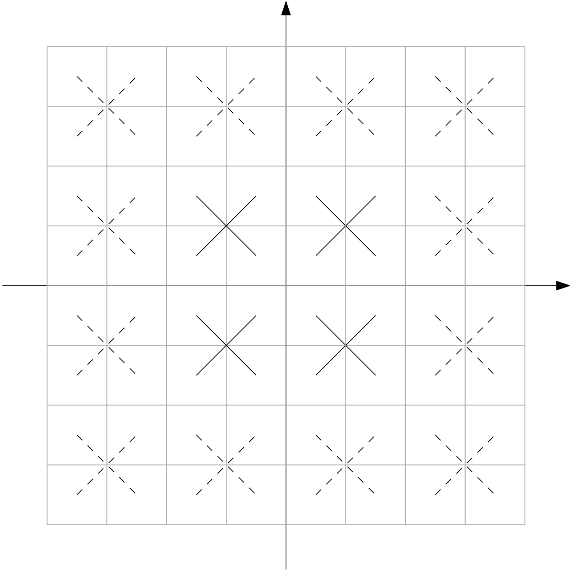

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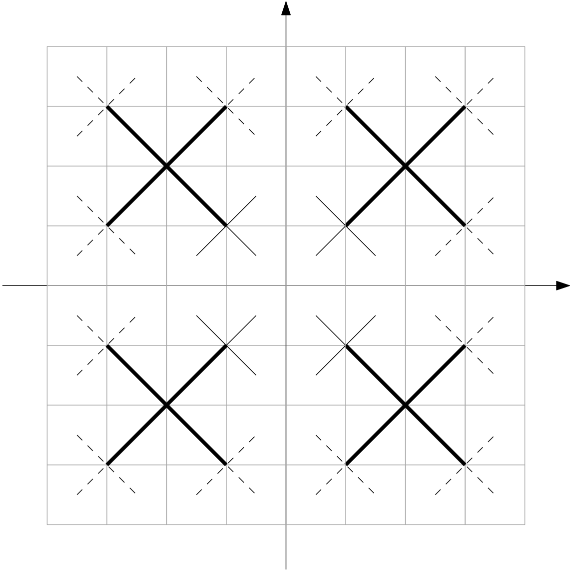

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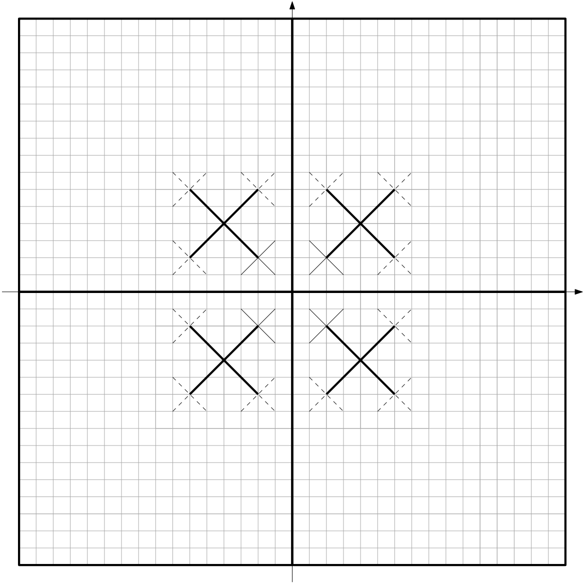

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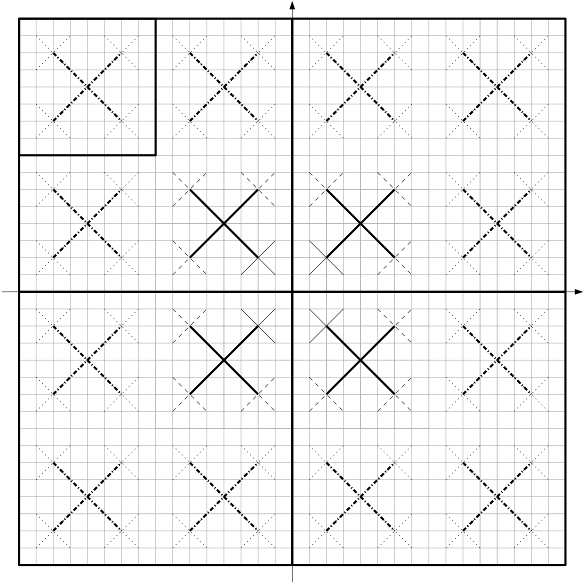

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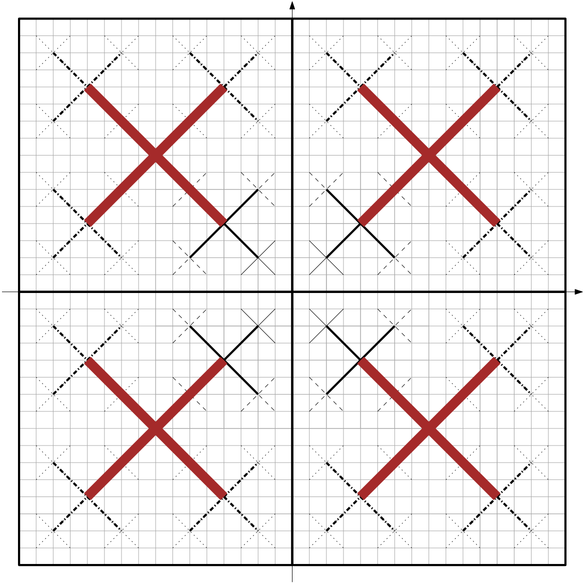
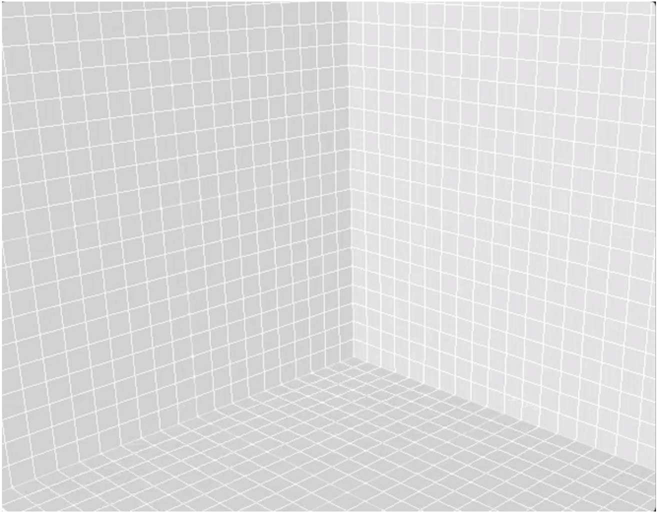


Example: d=3



Thank you!!!