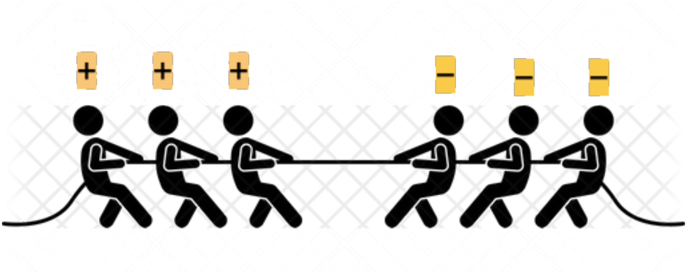
Sign Uncertainty

Felipe Gonçalves



Overview

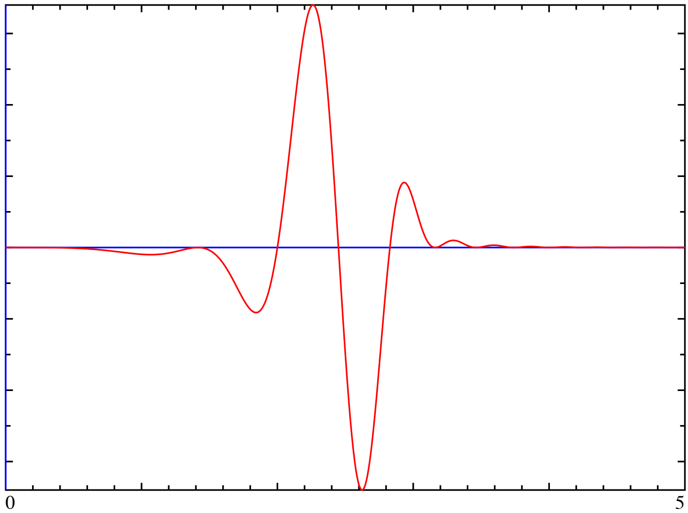
- ☐ Sign uncertainty principle
- ☐ Linear programming bounds for sphere packings
- ☐ Very recent results

We say $f : \mathbb{R}^d \to \mathbb{R}$ is eventually nonnegantive if

$$f(x) \geqslant 0, \quad |x| \gg 1.$$

Let

$$r(f) = \text{last sign change of f.}$$



If $f : \mathbb{R}^d \to \mathbb{R}$ is integrable we define its Fourier Transform by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \xi x} dx$$

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$$\mathcal{A}_+(d) = \begin{cases} f, \widehat{f} \in L^1(\mathbb{R}^d), \text{even and real-valued} \\ f(x) \geqslant 0 \text{ if } |x| \gg 1 \\ \widehat{f}(x) \geqslant 0 \text{ if } |x| \gg 1 \\ f(0) \leqslant 0, \ \widehat{f}(0) \leqslant 0 \end{cases}$$

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Can f and $\widehat{f}$ both have mass $\leqslant 0$ and be nonnegative outside a small ball $B_\varepsilon(0)$?

Can we make $r(f)r(\widehat{f})$ arbitrarily small?

Thm [Bourgain et al., Ann. Inst. Fourier, 2010]

If $f \in \mathcal{A}_+(d)$ then

$$\sqrt{r(f)r(\widehat{f})} \geqslant 0.241\sqrt{d}$$

Moreover, there is $f \in \mathcal{A}_+(d)$ so that

$$\sqrt{r(f)r(\widehat{f})} \leqslant 0.398\sqrt{d}$$

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$$\mathbb{A}_+(d) := \inf_{f \in \mathcal{A}_+(d)\setminus\{0\}} \sqrt{r(f)r(\widehat{f})}$$

$$\boxed{\mathbb{A}_+(d) \asymp \sqrt{d}}$$

Thm [Cohn & G., Invent. Math., 2019]

We have

$$\mathbb{A}_+(12) = \sqrt{2},$$

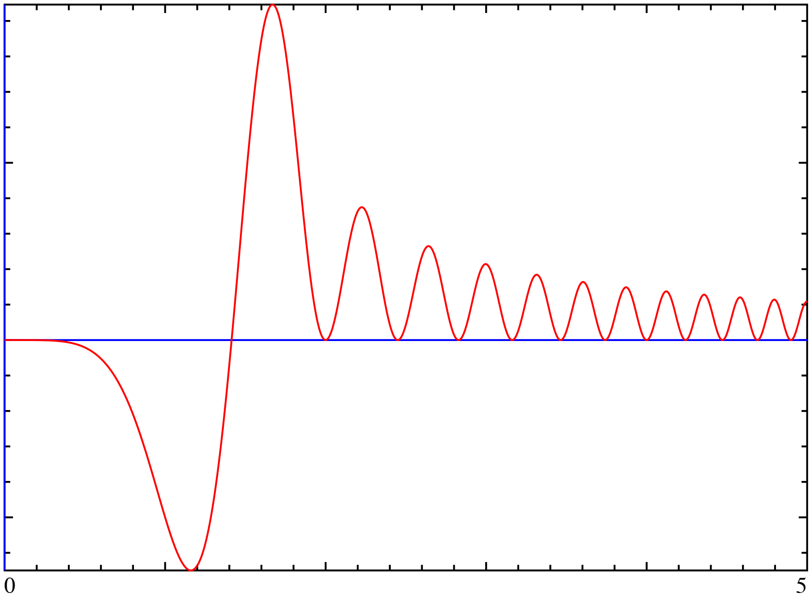
and there is $f \in \mathcal{A}_+(12) \cap \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^{12})$ with

$$\widehat{f} = f, \quad f(0) = 0 \text{ and } r(f) = \sqrt{2}.$$

Moreover, any such f must satisfy

$$f(\sqrt{2n}) = 0 \quad (n \geqslant 0).$$

Optimal f



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$$\begin{aligned} \mathbb{A}_+(d) = \inf r(f) \quad : \quad & f \in L^1_{rad}(\mathbb{R}^d) \setminus \{0\} \\ & \widehat{f} = f \quad (\textit{self-dual}) \\ & f(0) = 0 \\ & f(x) \geqslant 0 \text{ if } |x| \gg 1 \end{aligned}$$

Lower Bounds

- Assume $d \equiv 4 \mod 8$.

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Thm [Cohn & G., Invent. Math., 2019]

Then

$$\mathbb{A}_+(d) \geqslant \sqrt{2k}$$

where a_k is the first non-zero coefficient.
In particular, the Eisenstein series E_6 shows that $\mathbb{A}_+(12) \geqslant \sqrt{2}$.

Thm [G. & Oliveira e Silva & Steinerberger, J. Math. Anal. Appl., 2016]

Existence: $\exists f \in \mathcal{A}_+(d) \setminus \{0\}$ such that

$$r(f) = \mathbb{A}_+(d),$$

we can assume f radial, $\widehat{f} = f$ and $f(0) = 0$.

Multiple Roots: $f(|x|)$ has infinitely many roots for $|x| > r(f)$.

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Sphere Packing Problem

What is the most dense arrangement of non-overlapping equal spheres in $\mathbb{R}^d$? $\rightsquigarrow$ $\Delta(d) =$ largest packing density

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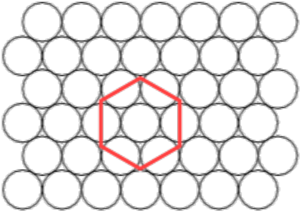
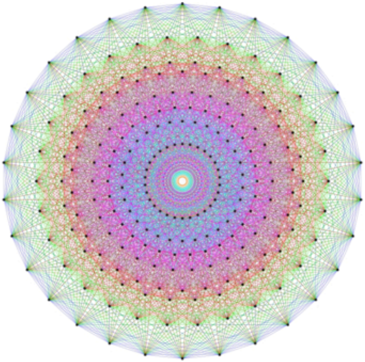
(Thue, 1910) (Hales, 1998)



Honeycomb $\Delta = 91\%$

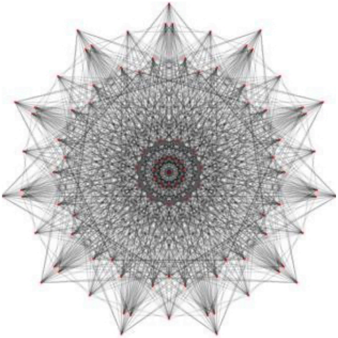
Cannonball $\Delta = 74\%$

(Viazovska, 2016)



E8 Lattice Δ=25%

(CKMRV, 2016)



Leech Lattice Λ_{24} Δ=0.2%

$$\mathcal{A}_{LP}(d) = \begin{cases} f, \widehat{f} \in L^1(\mathbb{R}^d), \text{ both radial and real-valued} \\ f(x) \leqslant 0 \text{ if } |x| \gg 1 \\ \widehat{f} \geqslant 0 \\ f(0) = \widehat{f}(0) = 1 \end{cases}$$

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Thm [Cohn & Elkies, Ann. of Math., 2003]

$$\mathrm{Dens}(P) \leqslant vol(\tfrac{1}{2}B^d)r(f)^d$$

for any sphere packing $P \subset \mathbb{R}^d$ and any $f \in \mathcal{A}_{LP}(d)$.

$$\mathbb{A}_{LP}(d) := \inf_{f \in \mathcal{A}_{LP}(d)} r(f) \quad \Longrightarrow \quad \Delta(d) \leqslant vol(\tfrac{1}{2}B^d)\mathbb{A}_{LP}(d)^d$$

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$$\mathbb{A}_{LP}(8) \geqslant \sqrt{2} \quad \longleftarrow \quad \text{E8 lattice}$$

$$\mathbb{A}_{LP}(24) \geqslant \sqrt{4} \quad \longleftarrow \quad \text{Leech lattice}$$

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Viazovska (Ann. of Math., 2017) proved these bounds are attained by constructing the magic function $f \in \mathcal{A}_{LP}(d)$

Ansatz: $$f(r) = sin^2(\pi r^2/2) \int_0^{i\infty} R(z) e^{\pi i z r^2} dz$$

We realized that there is a underlying -1 uncertainty principle.

$$\mathcal{A}_{-}(d) = \begin{cases} f, \widehat{f} \in L^1(\mathbb{R}^d), \text{even and real-valued} \\ f(x) \geqslant 0 \text{ if } |x| \gg 1 \\ -\widehat{f}(x) \geqslant 0 \text{ if } |x| \gg 1 \\ -f(0) \leqslant 0, \ \widehat{f}(0) \leqslant 0 \end{cases}$$

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Thm [Cohn & G., Invent. Math., 2019]

$$0.241\sqrt{d} \leqslant \mathbb{A}_-(d) \leqslant 0.319\sqrt{d}$$

$$\mathbb{A}_-(d) = \inf\{r(f) : \widehat{f} = -f, f(0) = 0, f(x) \geqslant 0 \text{ if } |x| \gg 1\}$$

- $f \in \mathcal{A}_{LP}(d)$ then $b := \widehat{f} - f \in \mathcal{A}_{-}(d)$ and $r(b) \leqslant r(f)$.

$$\boxed{\mathbb{A}_{LP}(d) \geqslant \mathbb{A}_{-}(d).}$$

In fact, we conjecture equality for all d.

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- In 2019, Cohn & G. used these ideas to show

$$\mathbb{A}_+(12) = \sqrt{2}$$

$$f(r) = \sin^2(\pi r^2/2) \int_0^{i\infty} \frac{(\theta_3(z)^4 + \theta_2(z)^4)\theta_4(z)^{12}}{\eta(z)^{12}} e^{\pi i z r^2} dz$$

d	Best Packing	$\mathbb{A}_{LP}(d)$	$\mathbb{A}_{-}(d)$	$\mathbb{A}_{+}(d)$
1	$\mathbb{Z}$	1	1	????????
2	Honeycomb	$? = (4/3)^{\frac{1}{4}}$	$?= (4/3)^{\frac{1}{4}}$	?
8	$E8$	$\sqrt{2}$	$\sqrt{2}$	?
12	?	?	?	$\sqrt{2}$
24	Leech	$\sqrt{4}$	$\sqrt{4}$	?

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12	?	?	?	$\sqrt{2}$
24	Leech	$\sqrt{4}$	$\sqrt{4}$	?

There is numerical evidence for the

$$\mathbb{A}_{LP}(d) = \mathbb{A}_{-}(d) \sim \mathbb{A}_{+}(d) \sim c\sqrt{d}. \quad \text{(conjecture)}$$

New results related with modular bootstrap in CFTs propose $c = \frac{1}{\pi} = 0.318\ldots$

So far $\mathbb{A}_{+}(d)$ does not have an associated geometrical problem. $\rightsquigarrow$ CFTs ?

New Sign Uncertainty Principles

(G., Oliveira e Silva, Ramos, 2020)

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- $\boxed{\text{Discrete Fourier and Hankel Transf..}}$
- Functions on the Hamming cube ($\rightsquigarrow$ Complexity of Boolean Functions)

$$H_2\left(\frac{deg(f)}{N}\right) > 1 - \frac{\log_2(16\#\{f = 0\})}{N}$$

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$$H_2\left(\frac{deg(f)}{N}\right) > 1 - \frac{\log_2(16\#\{f = 0\})}{N}$$

- Hilbert Transf., Hankel Transf., and other Smooth Conv. Kernels.

For $f : \mathbb{Z}_Q \to \mathbb{C}$ we define the DFT

$$\widehat{f}(n) = \frac{1}{\sqrt{Q}} \sum_{|m| \leqslant \frac{Q-1}{2}} f(m) e^{-2\pi i\, m\, n/Q} \quad (Q \text{ Odd}).$$

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Thm [G., Oliveira e Silva, Ramos, 2020]

$$\mathcal{A}_{\pm}^{disc}[Q] = \begin{cases} f,\ \widehat{f} : \mathbb{Z}_Q \to \mathbb{R} \text{ both even} \\ \widehat{f}(0) \leqslant 0,\ \pm f(0) \leqslant 0. \end{cases}$$

$$\mathbb{A}_{\pm}^{disc}[Q] := \min \left\{ \sqrt{k(f)\, k(\pm \widehat{f})} \right\}$$

where $k(f) = \min\{ k > 0 : f(n) \geqslant 0 \text{ for } n \in [k, \frac{Q-1}{2}] \}$.
Then

$$\mathbb{A}_{\pm}^{disc}[Q] \gtrsim \sqrt{Q}.$$

Heuristically we expect

$$\frac{\mathbb{A}_{\pm}^{disc}[Q]}{\sqrt{Q}} \longrightarrow \mathbb{A}_{\pm}(1) \quad (Q \to \infty).$$

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Conjecture

$$\mathbb{A}_{+}(1) = \frac{1}{\sqrt{2GR}} = \frac{\sqrt{\sqrt{5}-1}}{2} = 0.555\ldots$$

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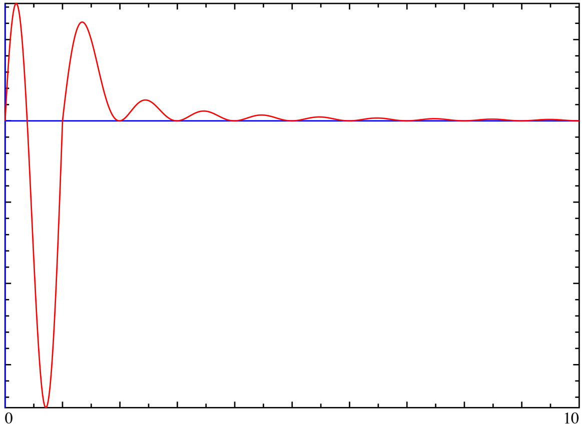
Good News: The discrete case we can use a Linear Programming solver! (Gurobi)

-1 Optimal anti-dual $(f(n))$ for $Q = 51^2$ and $k(f) = 51$.



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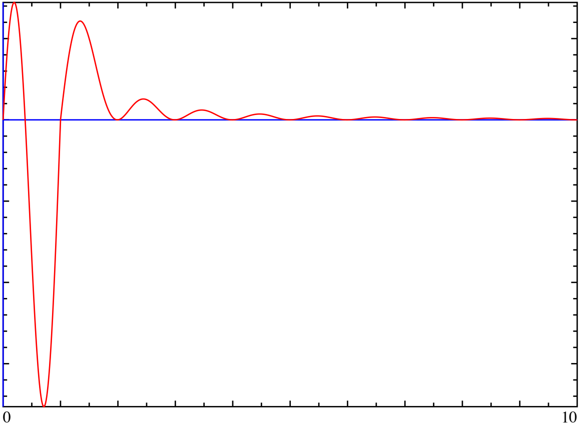
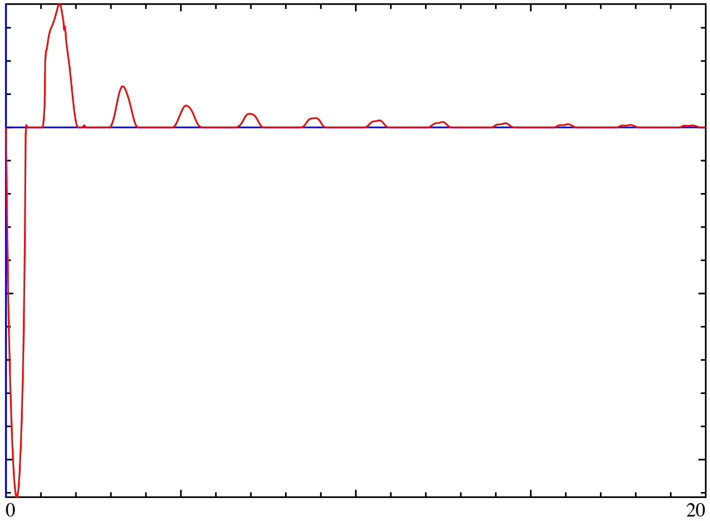


$$f(x) \approx |\sin(2\pi x)|\chi_{[-1,1]}(x) - \frac{2\sin^2(\pi x)}{\pi(1-x^2)}$$

$$\frac{\mathbb{A}_{-}^{disc}[Q]}{\sqrt{Q}} \approx \mathbb{A}_{-}(1) = 1$$

$+1$ Optimal self-dual $(f(n))$ for $Q = 7596$ and $k(f) = 49$.



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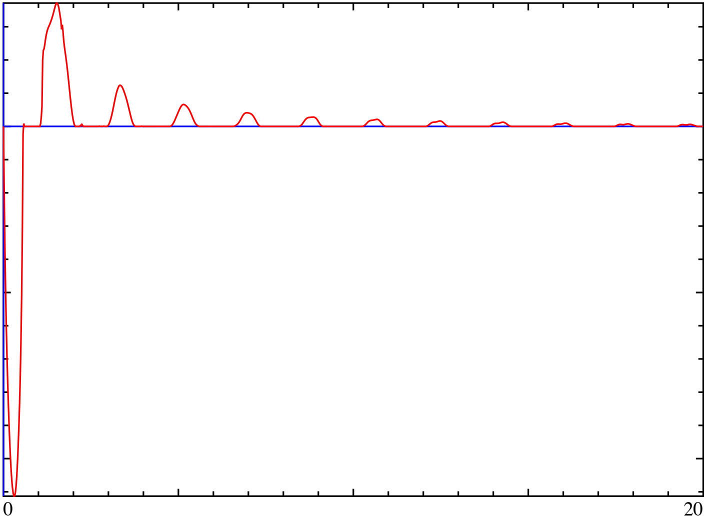


$$\frac{\mathbb{A}_{+}^{disc}[Q]}{\sqrt{Q}} \approx \frac{1}{\sqrt{2\,GR}} =^{?} \mathbb{A}_{+}(1)$$

8 GENERALIZED SIGN UNCERTAINTY

("NEW SIGN UNC. PRINCIPLES" OLIVEIRA E SILVA, RAMOS)

THM: LET $T : L^p(X,\mu) \to L^q(Y,\nu)$ BE BOUNDED AND INVERTIBLE. FOR $f \in L^p(X,\mu)$ AND

$\varepsilon_1 = \text{sign}(\int f)$, $\varepsilon_2 = \text{sign}(\int Tf)$.

$$\mu(\{\varepsilon_1 f > 0\})^{\frac{1}{p'}} \nu(\{\varepsilon_2 Tf > 0\})^{\frac{1}{q}} \geq C \frac{\|f\|_1}{\|Tf\|_\infty}$$

$$C = C(\|T\|, \|T^{-1}\|).$$

NOTE: T = FOURIER TRANSF., $p = q = 2$ WE RECOVER.

$\boxed{\text{APPLICATIONS}}$ $L^2(X,\mu)$ AND φ_n AN ORTHONORMAL BASIS.

$$T : f \in L^2(X,\mu) \to (\hat{f}(n))_{n=0}^{\infty} \in \ell^2(\mathbb{N})$$

WHERE $f = \sum_{n=0}^{\infty} \hat{f}(n) \varphi_n$. $\underline{\underline{f(\eta) = \sum \hat{f}(n) \|\varphi_n\|_\infty}}$

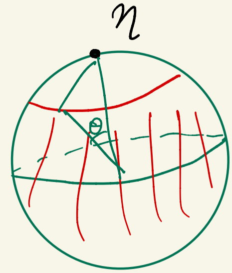
THM ASSUME $\exists \eta \in X$ SO THAT $\varphi_n(\eta) = \|\varphi_n\|_\infty$.

THEN FOR $\varepsilon_1 = \text{sign}(\int f)$, $\varepsilon_2 = \text{sign}(f(\eta))$

$$\mu(\{\varepsilon_1 f > 0\}) \sum_{n : \varepsilon_2 \hat{f}(n) > 0} \|\varphi_n\|_\infty^2 \geq \frac{1}{16}$$

THM $L^2(S^{n-1}) = \bigoplus_{k\geq 0} \mathcal{H}_k$ AND CHOOSE ORTHONORMAL BASIS $\{Y_{k,j}\}_{j=1}^{h_k}$ OF $\mathcal{H}_k$ SO THAT $Y_{k,j}(\eta) > 0$.

ASSUME $f : S^{n-1} \to \mathbb{R}$ IS SUCH THAT



1. $f(x) \geq 0$ IF $\operatorname{dist}(x, \eta) \geq \theta_f$
2. $f = \sum \hat{f}(k,j) Y_{k,j}$ THEN

   $\varepsilon\ \hat{f}(k,j) \geq 0$ IF $k \geq k_f$

3. $\varepsilon\ f(\eta) \leq 0, \quad \int_{S^{n-1}} f \leq 0$

THEN $$k_f \cdot (1 - \cos^2\theta_f)^{1/2} \geq \frac{1}{e} + \frac{c}{n}$$

- LET $P : S^{n-1} \to \mathbb{R}$ BE A POLYNOMIAL SATISFYING THE CONDITIONS OF THE THM ABOVE. WE SAY P IS <u>LOCALLY OPTIMAL</u> IF $K_P(1-\cos^2\theta_P)^{1/2}$ IS <u>MINIMAL</u> FOR ALL POLY. CLOSE TO P AND DEG NOT GREATER.

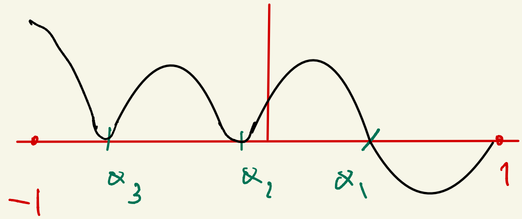
- SUPPOSE $\Omega \subset S^{n-1}$ IS A TIGHT SPHERICAL t-DESIGN. WITH COSINE DISTANCES $\{\alpha_m < \alpha_{m-1} < \cdots < \alpha_1\}$, $m = \lceil t/2 \rceil$

LET

$$\boxed{P(x) = (x-1)(x-\alpha_m)^a (x-\alpha_1) \prod_{j=2}^{m-1} (x-\alpha_j)^2}$$

$a = 2$ IF $\alpha_m > -1$

$a = 1$ IF $\alpha_m = -1$



THM P IS LOCALLY OPT. FOR ANY $\varepsilon = \pm 1$.

SIMPLEX AND CROSS-POLYTOPE ARE GLOBALLY OPTIMAL

COMPLEXITY OF BOOLEAN FUNCTIONS

NEW SIGN UNC. PRINC.

LET $f : \{\pm 1\}^N \longrightarrow \{0,1\}$ BE A BOOLEAN FUNCTION.

A LOT OF INTEREST IN BOUNDING $\deg(f)$

$$f(x) = \sum_{S \subset [N]} \hat{f}(S)\, x^S \quad , \deg(f) = \max\{\#S : \hat{f}(S) \neq 0\}$$

FOR INSTANCE, RECENTLY IT WAS SHOWN

$$\sqrt{s(f)} \leq \deg(f) \leq s(f)^2$$

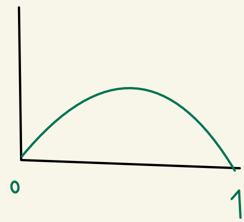
($s(f)$ = SENSITIVITY OF f)

$$\deg(f) \geq \log_2 N - O(\log\log N)$$

THM $\quad H_2\left(\frac{\deg(f)}{N}\right) \geq 1 - \frac{\log_2(16\,\Theta(f))}{N}$

$$\Theta(f) = \min\{\#\{f=1\},\ \#\{f=0\}\}$$

$$H_2(x) = -x\log_2 x - (1-x)\log_2(1-x)$$



1. $+1$ Sign Uncertainty [Bourgain et al., 2010.]
2. Dim 8 and 24 papers on optimal sphere Packings [Viazoviska et al., 2017]
3. Existence of optimizers [G. et al., 2017]
4. Interpolation on $\sqrt{n}$ [V. R. 2016]
5. Solution of $\mathbb{A}_+(12) = \sqrt{2}$ [Cohn & G., 2019]
6. First order interpolation on $\sqrt{2n}$ (and Gaussian energy minimization) [C.K.M.R.V, 2019]
7. Fourier uniqueness on $f(n^\alpha), \widehat{f}(n^\beta)$ [Ramos Sousa, 2020] [Nazarov and Sodin, 2020]
8. Perturbed interpolation at $\sqrt{n+\varepsilon_n}$ [Ramos et al., 2020]
9. Interpolation on the zeros of the zeta function [Bondarenko et al, 2020]
10. Mass concentration phenomena for sign uncertainty [G. et al., 2020]
11. Generalized sign Fourier uncertainty [Carneiro et al. 2020]
12. New sign uncertainty principles [G. et al., 2020]