Radon-like Transforms, Geometric Measures, and Invariant Theory

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Geometric Structures in Harmonic Analysis

Fourier Restriction

Highlights

Outgrowth of work ~50 years ago on the Bochner-Riesz Conjecture; has extensive applications to PDEs and is a pillar of an interconnected web of core conjectures in harmonic analysis.

L^p-Improving Radon-Like Transforms

Highlights

Implied by Fourier restriction inequalities but not generally the other way around; now richly developed for curves and much other work for surfaces; intermediate cases not well-understood.

Decoupling Theory

Highlights

Recent development initiated by Bourgain and Demeter; has deep and not fully understood connections to efficient congruencing in number theory; one important tool here is Brascamp-Lieb inequalities.

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- For L^p-improving estimates, a complicated game of inflation maps gets replaced by a less complicated game of finding useful invariant polynomials.
- Decoupling sees geometry in a fundamentally different way than restriction and L^p-improving inequalities.

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- You think of the action of the representation as being geometrically trivial and consequently would really like to consider all vectors along a given orbit to be different expressions of the same underlying geometric object (think coordinate changes).
- The most natural thing to do is to form an equivalence relation, but this is a bit too naive. Problems occur when the zero vector is in the (Zariski) closure of an orbit.

Hilbert Comes to the Rescue

The set $\boldsymbol{N}$ of vectors w/orbit closures containing $\mathbf{0}$ is called the nullcone.

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$$\mathrm{BL}^{-1}(\pi, N) \int_H \prod_{j=1}^{m} (f_j(\pi_j))^{\frac{N_j d}{N d_j}} \leq \prod_{j=1}^{m} \left(\int_{H_j} f_j \right)^{\frac{N_j d}{N d_j}}$$

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Let $\Pi_N : H^N \times H_1^{N_1} \times \cdots \times H_m^{N_m} \to \mathbb{R}$ be

$$\Pi_N(x^{(1)}, \ldots, x^{(N)}, y_1^{(1)}, \ldots, y_1^{(N_1)}, \ldots, y_m^{(N_m)}) := \left\langle \pi_1 x^{(1)}, y_1^{(1)} \right\rangle_{H_1} \cdots \left\langle \pi_1 x^{(N_1)}, y_1^{(N_1)} \right\rangle_{H_1} \cdots \left\langle \pi_m x^{(N)}, y_m^{(N_m)} \right\rangle_{H_m}$$

and let $G := \mathrm{SL}(H) \times \mathrm{SL}(H_1) \times \cdots \times \mathrm{SL}(H_m)$. Then

$$\left[\mathrm{BL}^{-1}(\pi, N)\right]^{\frac{N}{d}} = C \inf_{M \in G} |||\rho_M \Pi_N|||,$$

ρ_M is action of $M \in G$, $||| \cdot |||$ is Hilbert-Schmidt.

Brascamp-Lieb Example

L^p-Improving Inequalities: Two-Step Process

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- The Kakeya-Brascamp-Lieb Inequality
- Geometric Nonconcentration Inequalities

Part 1: Kakeya-Brascamp-Lieb

General Setup

- We have a defining function $\rho(x,y)$ on $\mathbb{R}^n \times \mathbb{R}^n$ mapping into $\mathbb{R}^{n-k}$. For each x, ${}^x\Sigma$ is the zero set of $\rho(x,\cdot)$. Likewise Σ^y is the zero set of $\rho(\cdot,y)$.
- Assume that all ${}^x\Sigma$ and Σ^y are algebraic varieties of bounded degree.
- We have derivative matrices $D_x\rho$ and $D_y\rho$.
- On each ${}^x\Sigma$ there is a natural measure $d\sigma$ *a la* the coarea formula:

$$\int_{{}^x\Sigma} f d\sigma := \int_{{}^x\Sigma} f(y) \frac{dH^k(y)}{\det(D_y\rho(x,y)(D_y\rho(x,y))^T)^{1/2}},$$

- We will study the operator T given by

$$Tf(x) := \int_{{}^x\Sigma} f d\sigma.$$

Theorem

For any nonnegative Lebesgue measurable $f_1, \ldots, f_m$ *on* $\mathbb{R}^n$*,*

$$\int_{\mathbb{R}^n} \left[\int_{x_\Sigma} \cdots \int_{x_\Sigma} \left[\mathrm{BL}(D_x \rho) \right]^{-\frac{m(n-k)}{n}} \prod_{j=1}^{m} f_j(y_j) d\sigma(y_1) \cdots d\sigma(y_m) \right]^{\frac{n}{m(n-k)}} dx$$

$$\lesssim \prod_{j=1}^{m} ||f_j||_{L^1(\mathbb{R}^n)}^{\frac{n}{m(n-k)}}.$$

History

Based directly on an inequality of Zhang [Analysis&PDE 2018] which extends multilinear Kakeya [Bennett, Carbery, Tao (Acta 2006); Guth endpoint (Acta 2010)] so that families of tubes are replaced by families of slabs.

Part 1: Kakeya-Brascamp-Lieb

$$\int_{\mathbb{R}^n} \left[\int_{x\Sigma} \cdots \int_{x\Sigma} \left[\mathrm{BL}(D_x \rho)\right]^{-p} \prod_{j=1}^{m} f_j(y_j)\, d\sigma(y_1) \cdots d\sigma(y_m) \right]^{\frac{1}{p}} dx \lesssim \prod_{j=1}^{m} ||f_j||_{L^1(\mathbb{R}^n)}^{\frac{1}{p}}$$

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Suppose Φ is some polynomial function from $(\mathbb{R}^n)^m$ into $\mathbb{R}^\ell$.

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Nonconcentration Inequalities

For a given Φ and s, find the "best possible" measure σ such that

$$I(E) := \int_{E^m} |\Phi(y_1, \ldots, y_m)| d\sigma(y_1) \cdots d\sigma(y_k) \gtrsim (\sigma(E))^{m+s}.$$

Product sets E^m cannot be degenerate (as measured by Φ) when $\sigma(E) > 0$.

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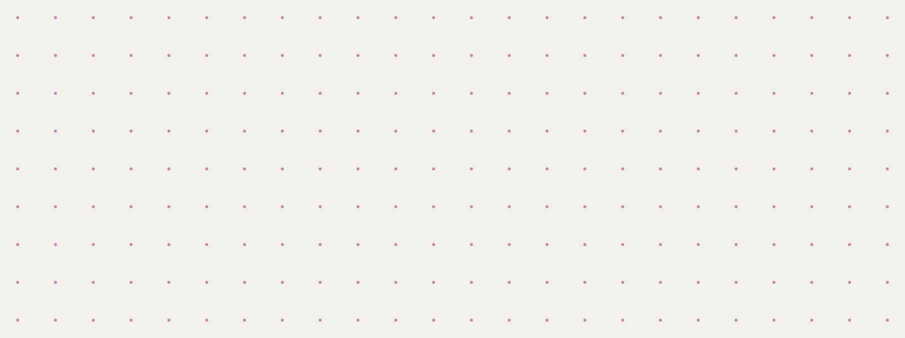

GIT Appears Again

Suppose $\Omega \subset \mathbb{R}^{n-k}$ is an open set and that $\Phi(t_1, \ldots, t_m)$ is a polynomial function of $t_1, \ldots, t_m \in \mathbb{R}^{n-k}$. Suppose $\partial_1^{\alpha_1} \cdots \partial_m^{\alpha_m} \Phi(t, \ldots, t) \equiv 0$ when $|\alpha_i| < c_i$ for some i. Let $s = (c_1 + \cdots + c_m)/(n-k)$. Then

$$\omega(t) := \inf_{T \in \mathrm{SL}_{n-k}} \max_{|\alpha_1| = c_1, \ldots, |\alpha_m| = c_m} \left| (T^* \partial)_1^{\alpha_1} \cdots (T^* \partial)_m^{\alpha_m} \Phi(t, \ldots, t) \right|^{\frac{1}{s}}$$

If σ is any nonnegative Borel measure which is absolutely continuous with respect to Lebesgue measure such that

$$\frac{d\sigma}{dt}(t) \leq \omega(t)$$

at each point $t \in \Omega$, where $\frac{d\sigma}{dt}$ is the Radon-Nikodym derivative of σ with respect to Lebesgue measure, then for any Borel set $F \subset \Omega$,

$$\int_F \cdots \int_F |\Phi(t_1, \ldots, t_m)| d\sigma(t_1) \cdots d\sigma(t_m) \gtrsim [\sigma(F)]^{m+s}$$

with implicit constant depending only on $(n-k, m, \deg \Phi)$.

Part 2: Nonconcentration Inequalities

Submanifolds of dimension d in $\mathbb{R}^{d(d+1)}$: Let $x := (x_i, x_{ij})$ for $i, j \in \{1, \ldots, d\}$. Defining function is $(\rho)_{ij} = (x_{ij} - y_{ij}) - x_i y_j$.

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Finding Good Invariant Polynomials

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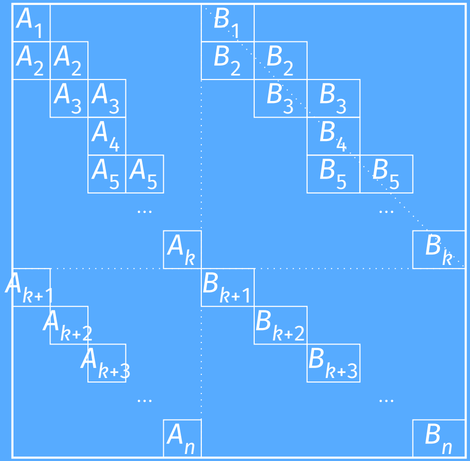
Block-form Matrices



Works well for cases when dimension of submanifold (k) exceeds codimension (n-k). Regard bottom rows as fixed.

Finding Good Invariant Polynomials

Cayley $\boldsymbol{\Omega}$

Think of “alternating contraction” on indices of a multilinear functional.

Decoupling is a Different Creature

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Joint with S. Guo, L. Pierce, J. Roos, P.-L. Yung

Given $\phi : \mathbb{N} \to \mathbb{Z}^n$ and an integer $s \geq 1$, consider the system of n equations $\phi(x_1) + \cdots + \phi(x_s) = \phi(x_{s+1}) + \cdots + \phi(x_{2s})$. For every finite set S of positive integers let $J_{s,\phi}(S)$ denote the number of solutions $(x_1, \ldots, x_{2s}) \in S^{2s}$ of the system. Fix N and consider an arbitrary subset $S \subseteq \{1, \ldots, N\}$.

$$\Big\| \sum_{j=1}^{N} a_j e^{2\pi i x \cdot \phi(j)} \Big\|_{L^{2s}([0,1]^n)} \leq C_{s,p,\phi,N} \Big(\sum_{j=1}^{N} |a_j|^p \Big)^{1/p}$$

implies the bound

$$J_{s,\phi}(S) \leq C_{s,p,\phi,N}^{2s} |S|^{2s/p}.$$

Decoupling is a Different Creature

Theorem

Suppose $\exists\theta = \theta(\phi, s) \in [s, 2s)$ and a constant $c = c(\phi, s) \in (0, \infty)$ such that for all $N \geq 1$ and for all subsets $S \subset \{1, \ldots, N\}$ we have the inequality

$$J_{s,\phi}(S) \leq c|S|^{\theta}.$$

Then the ℓ^p decoupling inequality for L^{2s} holds for $p = \frac{2s}{\theta} \in (1, 2]$: namely, there exists a constant c' such that for every $(a_j)_j \in \mathbb{C}^N$, we have

$$\Big\| \sum_{j=1}^{N} a_j e^{2\pi i x \cdot \phi(j)} \Big\|_{L^{2s}([0,1]^n)} \leq c'(1 + p^{-1}(\log N)^{\frac{1}{p'}}) \Big(\sum_{j=1}^{N} |a_j|^p \Big)^{1/p}. \quad (1)$$

Here we have $1/p + 1/p' = 1$, and we may take $c' = 2^{1/p} 4^{1/p'} c^{1/2s}$.

Decoupling is a Different Creature

Theorem

Suppose that $\gamma : [0,1] \to \mathbb{R}^n$ is a non-degenerate C^n curve. Then there exists a constant $C = C(\gamma, n) \in (0, \infty)$ such that the following holds: for each integer $1 \leq m \leq n$, for every $R \geq 1$ and every ball B of radius at least R^n, we have that for all $f \in L^{2m}(w_B)$,

$$\| E_{[0,1]} f \|_{L^{2m}(w_B)} \leq C \Big\| \Big(\sum_{|I| = R^{-1}} |E_I f|^2 \Big)^{1/2} \Big\|_{L^{2m}(w_B)} \tag{2}$$

where the summation is over intervals I belonging to a dissection of $[0,1]$ into intervals of length R^{-1}.

Thank You