Equidistribution of affine random walks on some nilmanifolds.

Based on joint works with Weikun He and Elon Lindenstrauss

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Outline

Random walk associated to a group action

Consider an action $G \curvearrowright X$.
Let $\mu \in \mathcal{P}(G)$ be a probability measure. Let $x \in X$.

Definition

The random walk on X induced by μ and starting at x is the sequence of random variable $(g_n g_{n-1} \cdots g_1 x)_{n \geq 1}$ where $(g_n)_{n \geq 1}$ is i.i.d. of law μ.

The law of $g_n g_{n-1} \cdots g_1 x$ is $\mu_n := \mu^{*n} * \delta_x$, i.e. for any function $f \colon X \to \mathbb{C}$,

$$\mathbb{E}[f(g_n g_{n-1} \cdots g_1 x)] = \int_X f \, \mathrm{d}(\mu^{*n} * \delta_x).$$

We are interested in the convergence in law, i.e. the convergence of $\mu^{*n} * \delta_x$ in the weak-$*$ topology on X.

Let $X = N/\Lambda$ be a compact nilmanifold. That is,

1. N is a connected simply-connected nilpotent Lie group,
2. $\Lambda \subset N$ is a lattice in N, i.e. it is a discrete subgroup of N and
3. the Haar measure on N induces a finite N-invariant measure on N/Λ.

We normalize this measure to be a probability measure and denote it by m_X.

Group of affine transformations

On $X = N/\Lambda$, we consider the action of its automorphism group

$$\mathrm{Aut}(X) = \{\, \gamma \in \mathrm{Aut}(N) \mid \gamma(\Lambda) = \Lambda \,\} = \mathrm{Aut}(\Lambda)$$

and that of its affine transformations

$$\mathrm{Aff}(X) = \mathrm{Aut}(X) \ltimes N,$$

Here, for $\gamma \in \mathrm{Aut}(X)$ and $n \in N$, $(\gamma, n) \in \mathrm{Aut}(X) \ltimes N$ is the map

$$\forall x \in N, \quad x\Lambda \mapsto n\gamma(x)\Lambda.$$

For $g = (\gamma, n)$, we call γ the automorphism/linear part and denote $\theta(g) = \gamma$.

Examples

1. Let $d \geq 1$, $N = \mathbb{R}^d$ and $\Lambda = \mathbb{Z}^d$. Then $X = \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, $\mathrm{Aut}(X) = \mathrm{GL}_d(\mathbb{Z})$.
2. Let N be the Heisenberg group, $N = \mathbb{R}^3$ endowed with the law

$$(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + xy').$$

Let $\Lambda = \{ (x, y, z) \in N \mid x, y, z \in \mathbb{Z} \}$.
Then $\mathrm{Aut}(X) = \mathrm{GL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ is the set of transformations

$$(x, y, z) \mapsto \left(ax+by, cx+dy, \det(g)(z-\frac{xy}{2})+\frac{1}{2}(ax+by)(cx+dy)+\alpha x+\beta y\right)$$

where $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})$ and $(\alpha, \beta) \in \frac{1}{2}\mathbb{Z}^2$ satisfies some parity condition.

Motivating question on equidistribution in law

Question

Let X be a compact nilmanifold. Given $\mu \in \mathcal{P}(\mathrm{Aff}(X))$ and $x \in X$,

1. *does $\mu^{*n} * \delta_x \rightharpoonup^* \mathrm{m}_X$?*
2. *If it does, how fast is the convergence?*

Expected anwser : Yes, unless there is obvious obstruction.

Remark (Example of obstruction)

Let $H = \langle \mathrm{Supp}(\mu) \rangle$, the subgroup generated by the support.
If $\overline{Hx} \neq X$, then $\mu^{*n} * \delta_x \not\rightharpoonup^* \mathrm{m}_X$.

Orbit closures, case of a torus

Theorem (Guivarc'h-Starkov & Muchnik)

Let Γ be a subgroup of $\mathrm{GL}_d(\mathbb{Z}) = \mathrm{Aut}(\mathbb{T}^d)$. Assume

1. *the action of the Γ on $\mathbb{Q}^d$ is strongly irreducible.*
2. *the Zariski closure Γ in $\mathrm{GL}_d(\mathbb{R})$ is semisimple without compact factor.*

Then for every $x \in \mathbb{T}^d$, the orbit Γx is either finite or dense.

Definition

We say Γ acts strongly irreducibly on $\mathbb{Q}^d$ if it does not preserve any finite nontrivial union of proper subspaces of $\mathbb{Q}^d$.

Orbit closures, nilmanifold case

Note that $\mathrm{Aff}(X) \curvearrowright X$ is transitive. Hence $X = \mathrm{Aff}(X)/(\mathrm{Aut}(X) \ltimes \Lambda)$ is a homogeneous space. Let $\mathfrak{g}$ be the Lie algebra of $\mathrm{Aff}(X)$.

Theorem (Benoist-Quint)

Let $H \subset \mathrm{Aff}(X)$ be a subgroup and $x \in X$.
Assume that the Zariski closure of $\mathrm{Ad}(H)$ in $\mathrm{GL}(\mathfrak{g})$ is semisimple without compact factor.
Then $\overline{Hx}$ is a finite homogeneous union of affine submanifolds.

Equidistribution in law, torus, linear

Theorem (Bourgain-Furman-Lindenstrauss-Mozes, He-Saxcé)

Let $\mu \in \mathcal{P}(\mathrm{GL}_d(\mathbb{Z}))$ *having a finite* β*-exponential moment for some* $\beta > 0$. *Let* $\Gamma = \langle \mathrm{Supp}(\mu) \rangle$. *Assume that the action of the* Γ *on* $\mathbb{R}^d$ *is strongly irreducible.*
Then for any $x \in \mathbb{T}^d$, $\mu^{*n} * \delta_x \rightharpoonup^* \mathrm{m}_{\mathbb{T}^d}$ *unless* $x \in \mathbb{Q}^d/\mathbb{Z}^d$ *(i.e.* Γx *is finite).*

Definition

For some $\beta > 0$, *we say* μ *has a finite* β*-exponential moment if*

$$C_\beta = \int \mathrm{Lip}_X(g)^\beta \, \mathrm{d}\mu(g) < +\infty,$$

where $\mathrm{Lip}_X(g) = \sup_{x,x' \in X,\, x \neq x'} \frac{\mathrm{d}_X(gx,gx')}{\mathrm{d}_X(x,x')}$.

Equidistribution in law, torus, affine

Recall that $\theta\colon \operatorname{Aff}(X) \to \operatorname{Aut}(X)$ denotes the projection.

Theorem (He-Lindenstrauss-L.)

Let $\mu \in \mathcal{P}(\mathrm{GL}_d(\mathbb{Z}) \ltimes \mathbb{R}^d)$ having a finite support.
Let $H = \langle \operatorname{Supp}(\mu) \rangle$ and $\Gamma = \theta(H)$.
Assume Γ satisfies the assumptions in the BFLM theorem.
Then for any $x \in \mathbb{T}^d$, $\mu^{*n} * \delta_x \rightharpoonup^* \mathrm{m}_{\mathbb{T}^d}$ unless Hx is finite.

Very similar result was obtained by Boyer, under different assumptions.

Equidistribution in law, Heisenberg nilmanifold

Let $X = N/\Lambda$ with N being the $(2k+1)$-dimensional Heisenberg group. Note that $[N, N]$ is the one dimensional center and $N/[N, N]\Lambda$ is a $2k$-dimensional torus.
Denote by $\pi\colon X \to N/[N, N]\Lambda$ the projection.

Theorem (H-Lindenstrauss-L.)

Let $\mu \in \mathcal{P}(\mathrm{Aff}(X))$ with a finite support.
Let $H = \langle \mathrm{Supp}(\mu) \rangle$ and $\Gamma = \theta(H)$.
Assume that the action of Γ on $N/[N, N]$ satisfies the assumptions in the BFLM theorem.
Then for any $x \in X$, $\mu^{*n} * \delta_x \rightharpoonup^* \mathrm{m}_X$ unless $\pi(Hx)$ is finite.

If $\mu \in \mathcal{P}(\mathrm{Aut}(X))$, then finite support can be relaxed to having a finite exponential moment.

Outline

Wasserstein distance

We fix a Riemannian distance on $X = N/\Lambda$.
For $\alpha \in (0,1)$, let $\mathcal{C}^{0,\alpha}(X)$ denote the space of α-Hölder continuous functions on X, equipped with the norm

$$\|f\|_{0,\alpha} = \|f\|_\infty + \sup_{x \neq y \in X} \frac{|f(x) - f(y)|}{d(x,y)^\alpha}.$$

Definition

Let ν and η be Borel measures on X. The α-Wasserstein distance between them is

$$\mathcal{W}_\alpha(\nu, \eta) = \sup_{f \in \mathcal{C}^{0,\alpha}(X) : \|f\|_{0,\alpha} \leq 1} \left| \int_X f \, \mathrm{d}\nu - \int_X f \, \mathrm{d}\eta \right|.$$

We use $\mathcal{W}_\alpha(\mu^{*m} * \delta_x, \mathrm{m}_X)$ as a function of m to measure the rate of equidistribution.

Top Lyapunov exponent

Definition

Denote by $\lambda_{1,N/(N,N)}(\theta_*\mu)$ *the top Lyapunov exponent of the random walk induced by* $\theta_*\mu$ *on the Euclidean space* $N/(N,N)$*.*

$$\lambda_{1,N/(N,N)}(\theta_*\mu) = \lim_{n\to+\infty} \frac{1}{n} \int \log\|\theta(g)\|_{N/(N,N)} \, \mathrm{d}\mu^{*n}(g)$$

where $\|\cdot\|_{N/(N,N)}$ *denotes any operator norm on* $\mathrm{End}(N/(N,N))$*.*

Quantitative statement, Heisenberg nilmanifold

Theorem (He-Lindenstrauss-L.)

Assume $N = \mathbb{R}^d$ or a Heisenberg group. Assume that the action of Γ on $N/(N,N)$ satisfies the assumptions in the BFLM theorem.
Given $\lambda \in (0, \lambda_{1,N/(N,N)}(\mu))$ and $\alpha \in (0,\beta)$, there exists $C \geq 1$ such that (C,λ,α)-quantitative equidistribtion holds, i.e:

If for some $x \in X$, $t \in (0, \frac{1}{2})$ and $m \geq C \log \frac{1}{t}$,

$$\mathcal{W}_\alpha(\mu^{*m} * \delta_x, \mathrm{m}_X) > t,$$

then there exists $x' \in X$ and a finite set $F \subset \mathrm{Aff}(X)$ such that

1. $d_X(x,x') \leq e^{-\lambda m}$,
2. *for any $g \in \mathrm{Supp}(\mu)$ we have $d_{\mathrm{Aff}(X)}(g,F) \leq e^{-\lambda m}$ (unnecessary in linear case),*
3. *and the projection of the orbit $\langle F \rangle x'$ in $N/(N,N)\Lambda$ is finite of cardinality less than t^{-C}.*

Outline

Growth rate

Let H be a group and μ a probability measure on H.

Definition

Consider an action of H on a group Z by automorphisms. Let $\theta_Z \colon H \to \mathrm{Aut}(Z)$ denote the homomorphism. Define

$$\tau_Z(\mu) = \lim_{m \to +\infty} \frac{1}{m} \log \#\theta_Z(\mathrm{Supp}(\mu^{*m})).$$

This definition needs to be modified when μ is not finitely supported, but the idea is similar.

Spectral radius

Definition

If $(X, \mathrm{m}_X) \to (Y, \mathrm{m}_Y)$ is a factor map of a H-spaces (H acts equivariantly on X, Y by probability measure preserving transformations), define

$$\sigma_{X,Y}(\mu) = -\lim_{m\to+\infty} \frac{1}{m} \log \|P(\mu)^m\|_{L^2(X,\mathrm{m}_X)\ominus L^2(Y,\mathrm{m}_Y)},$$

where P is the Markov operator.

In this definition we look at $L^2(Y, \mathrm{m}_Y)$ as the subspace in $L^2(X, \mathrm{m}_X)$ of functions constant on fibers of the factor map.

Height of a torus subgroup

Example (Motivating example)

Let $T = \mathbb{R}^2/\mathbb{Z}^2$*. The subgroup of* T *generated by* $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$ *and* $\begin{pmatrix} 1/2 \\ 1/3 \end{pmatrix}$ *has* *height* ≤ 6*.*

Formally, let $T = V/\Delta$ be some torus of dimension d. We choose an identification $\mathbb{Z}^d$ with its group of unitary characters via some isomorphism $a \mapsto \chi_a$. Each closed subgroup L of T is uniquely determined by its dual

$$L^* = \{\, a \in \mathbb{Z}^d \mid L \subset \ker \chi_a \,\}.$$

Definition

A closed subgroup L *of a torus* T *is said to have* *height* $\leq h$ *if its dual* $L^* \subset \mathbb{Z}^d$ *can be generated by integer vectors of norm* $\leq h$*.*

This definition depends on the choice of basis for $\mathbb{Z}^d$, but not in a way that would matter to us if we just fix some basis for this.

Quantitative equidistribution

For a nilmanifold $X = N/\Lambda$ as before, denote by $T = N/[N,N]\Lambda$ the maximal factor torus and by $\pi : X \to T$ the canonical projection.

Definition (Quantitative equidistribution, general case)

Let $\lambda > 0$, $C > 1, \alpha \in (0,1]$, $\mu \in \mathcal{P}(\mathrm{Aut}(X))$, *and let* $H = \langle \mathrm{Supp}(\mu) \rangle$. *We say that the* μ*-induced random walk on* X *satisfies* (C, λ, α)*-quantitative equidistribution if the following holds for any integer* $m \geq 1$ *and any* $t \in (0, \frac{1}{2})$*. Assume*

$$m \geq C \log \frac{1}{t} \quad \text{and} \quad \mathcal{W}_\alpha(\mu^{*m} * \delta_x, \mathrm{m}_X) > t.$$

Then there exists a point $x' \in X$ *such that*

1. $d(x, x') \leq e^{-\lambda m}$
2. $\pi(Hx')$ *lies in a proper closed* H*-invariant subset of* T *of height* $\leq t^{-C}$.

Quantitative equidistribution - remarks

1. For $\mu \in \mathcal{P}(\mathrm{Aff}(X))$, the definition can be modified like in the Heisenberg special case we showed.
2. For X torus or Heisenberg nilmanifold, this coincides with our previous definition.
3. If we have (C, λ, α)-quantitative equidistribution for some $C > 1$, we say that we have (λ, α)-quantitative equidistribution.

Quantitative statement, general case

Theorem (He-Lindenstrauss-L.)

Let $\mu \in \mathcal{P}(\mathrm{Aut}(X))$ be with a finite β-exponential moment, Γ as before. Assume that there exists a rational Γ-invariant connected central subgroup $Z \subset N$ such that

$$\tau_Z(\mu) < 2\sigma_{X,Y}(\mu)$$

where $Y = N/(\Lambda Z)$. The following holds:
If the μ-induced random walk on Y satisfies (λ, α)-quantitative equidistribution for some $\lambda > 0$ and $\alpha \in (0, \beta]$
then the μ-induced random walk on X satisfies (λ', α)-quantitative equidistribution for any $\lambda' \in (0, \lambda)$.

For a general nilmanifold, if e.g the ascending central sequence $1 = Z_0 \subset Z_1 \subset \cdots \subset Z_l = N$ satisfies the conditions of this theorem in every step, then we can get from the quantitative equidistribution on the torus that BFLM/HS/HLL gives us quantitative equidistribution on the nilmanifold.

Examples

1. N is a step 2 nilpotent group and the action of Γ on its center $Z(N)$ is virtually nilpotent, such as the Heisenberg nilmanifold we discussed.
2. Action of $\Gamma \subset \mathrm{SL}_{2d}(\mathbb{Z})$ on $\mathbb{T}^{2d}$ consisting of $d \times d$-block triangular matrices. For example, let μ be the law of

$$\left(\begin{array}{c|c} A & I_d \\ \hline 0 & D \end{array}\right).$$

where A and D are independent random variables. Given any A, we can choose D to ensure

$$\tau_{\mathbb{R}^d \oplus 0}(\mu) < 2\sigma_{\mathbb{T}^{2d}, 0 \oplus \mathbb{T}^d}(\mu).$$

Thank you for listening!