Magnetic skyrmions in the conformal limit

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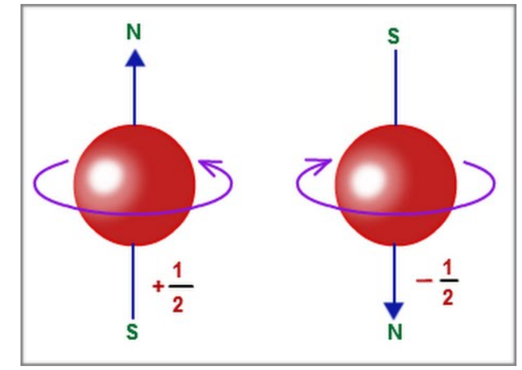
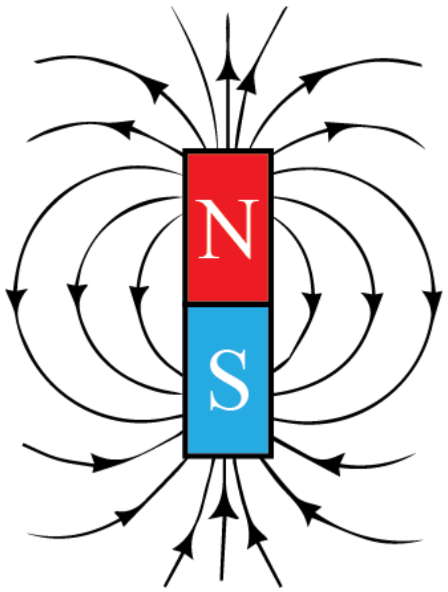
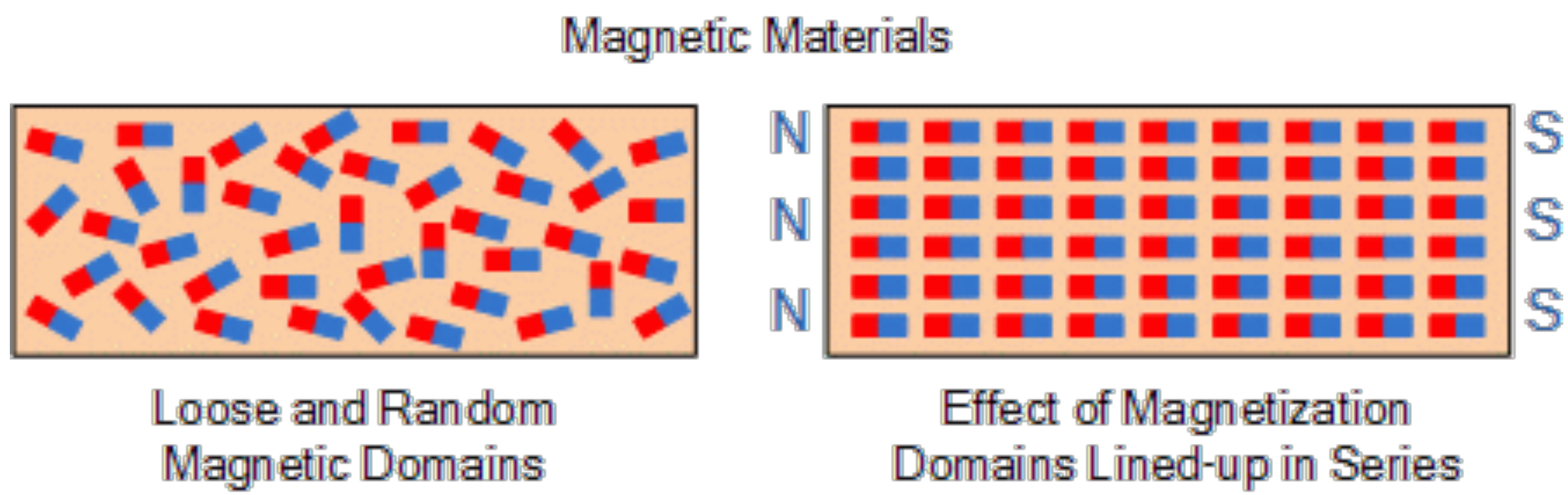
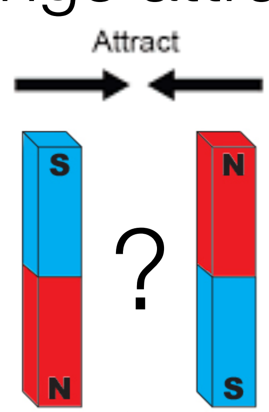
Magnetism and magnets







- spins act as tiny magnetic dipoles
- quantum-mechanical interaction between spins: *exchange*
- in transition metals below the *critical temperature*, exchange results in local spin alignment into the *ferromagnetic state*
- magnetic field mediates long-range attraction/repulsion between magnets



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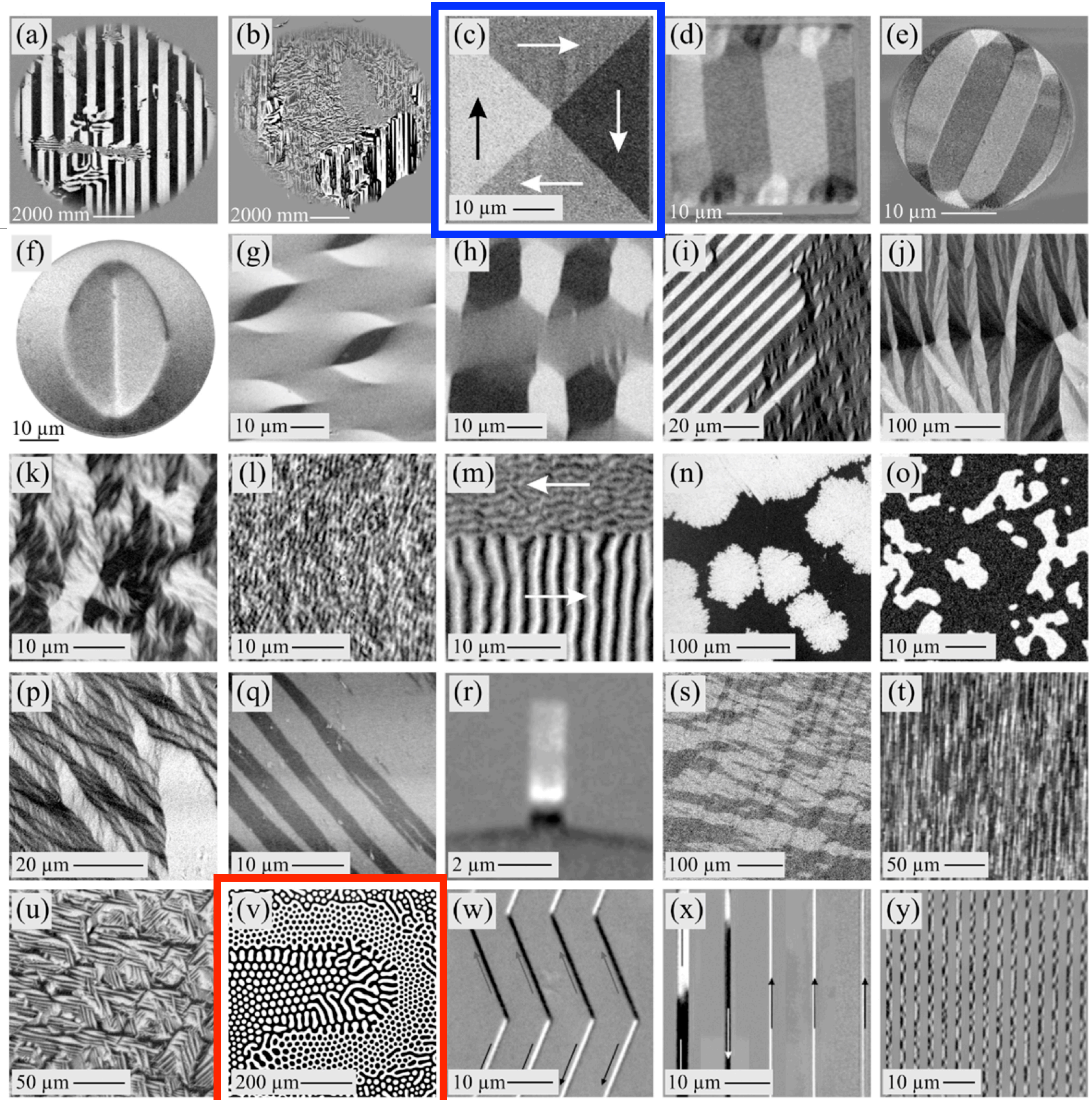
images borrowed from:
Tom Whyntie. (2016), zenodo.com
mammothmemory.net

Magnetic domains

- stray field frustrates the ferromagnetic order
- gives rise to a great variety of spin textures
- principle of pole avoidance



J. McCord, *J. Phys. D: Appl. Phys.* **48**, 333001 (2015)

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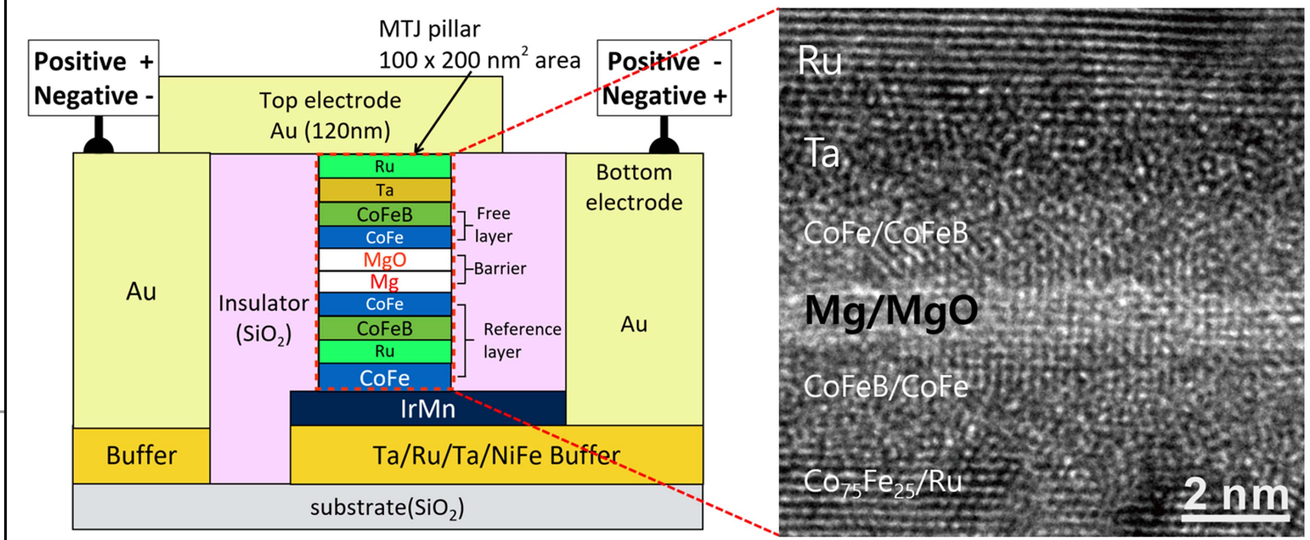
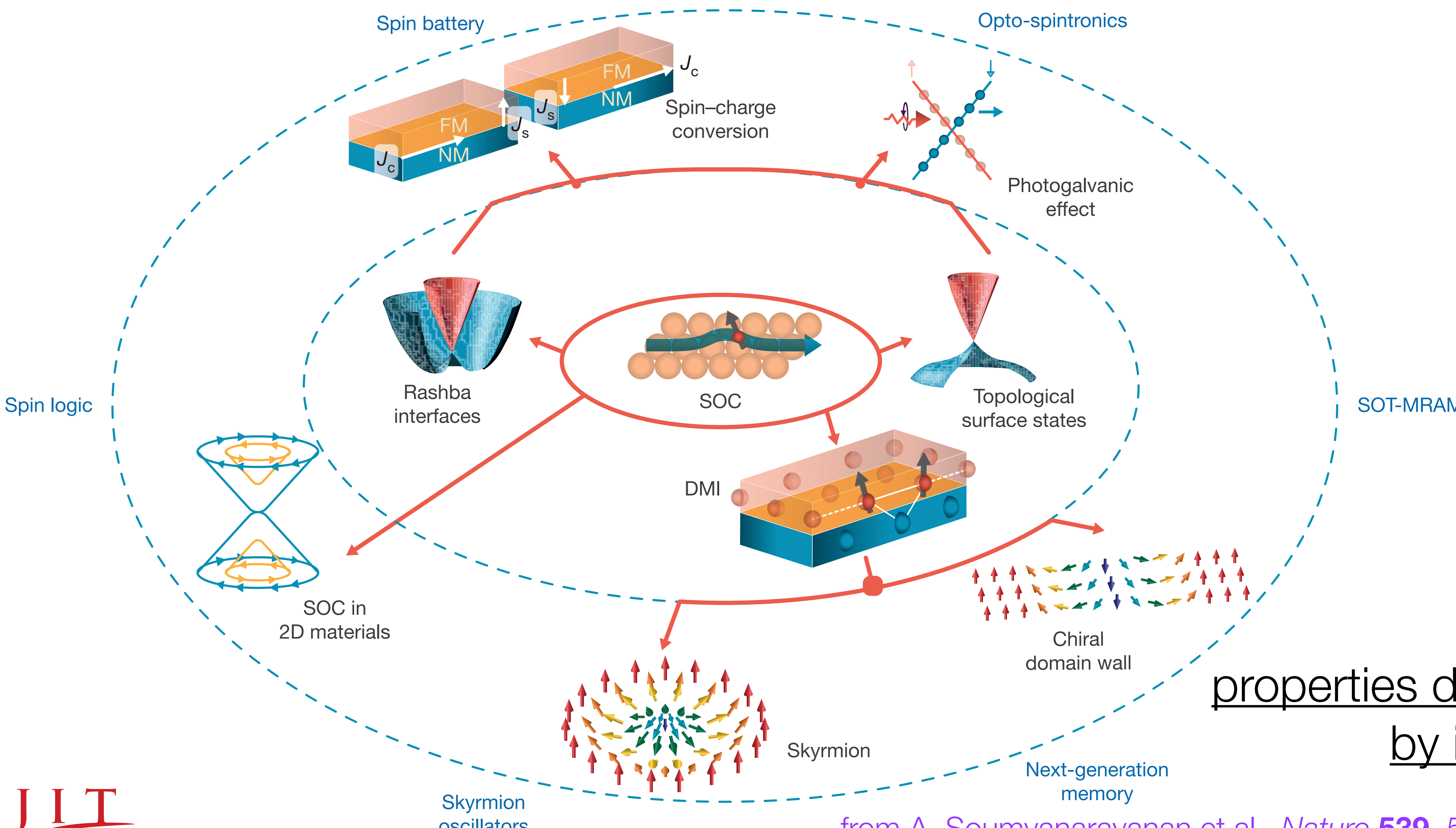
Next generation materials

atomically thin multilayers with strong spin-orbit coupling (SOC):



C.-M. Choi et al., *Semicond. Sci. Technol.* 32, 105007 (2017)



<u>properties dominated by interfaces</u>

from A. Soumyanarayanan et al., *Nature* **539**, 509-517 (2016)

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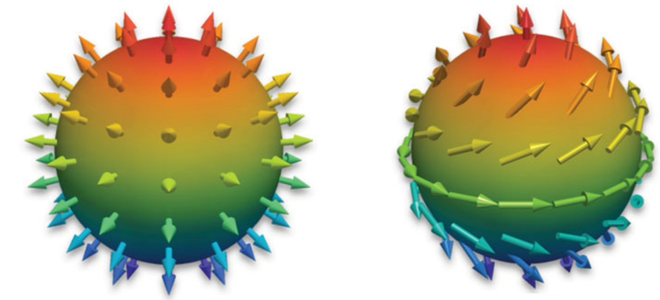


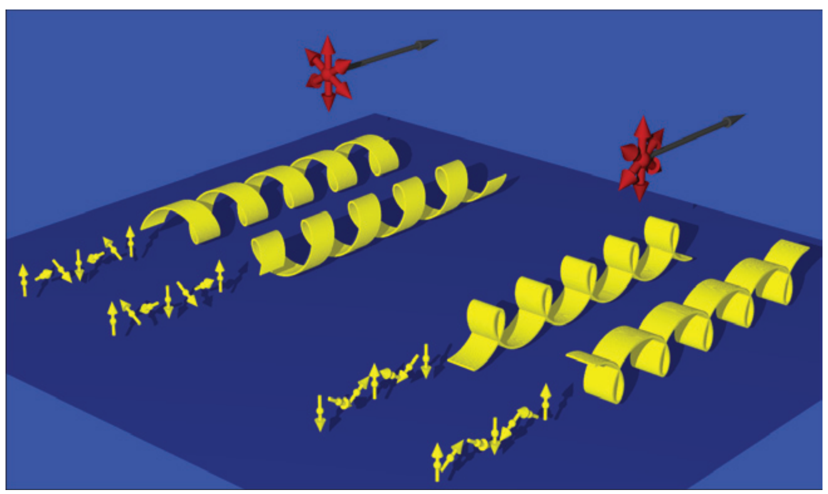
Topological spin textures

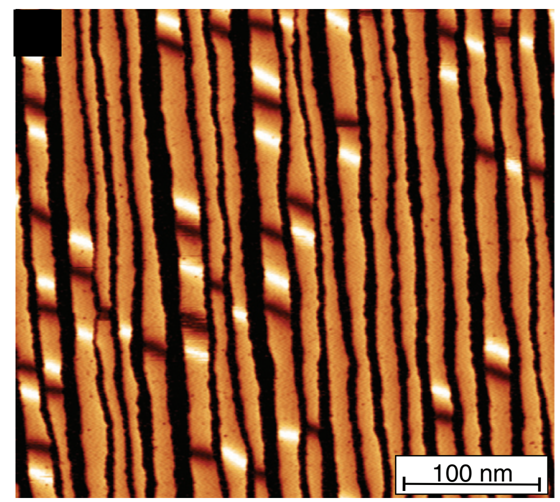
T. Lancaster, *Contemp. Phys.* **60**, 246-261 (2019)

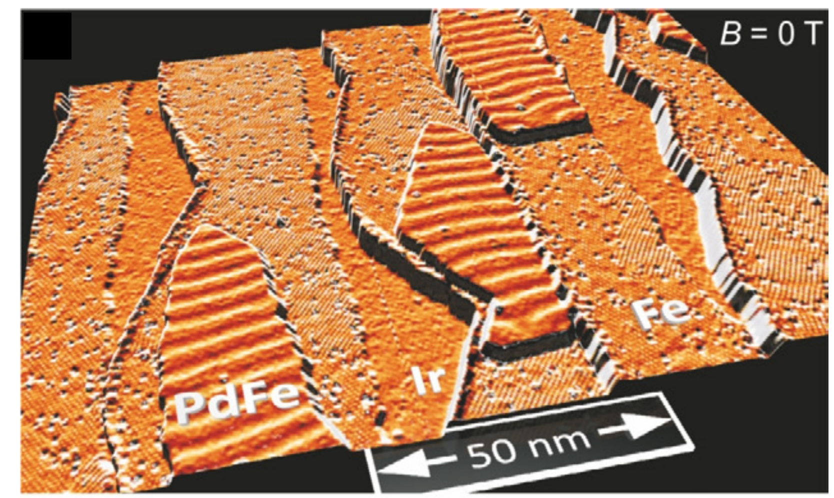
spin spirals and chiral domain walls:

2ML Fe on W(110)

Pd/Fe bilayer on Ir(111)

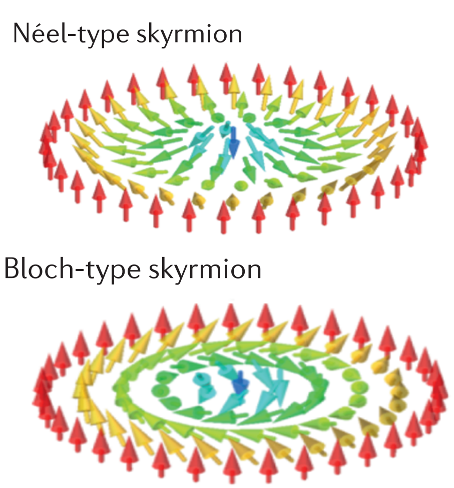


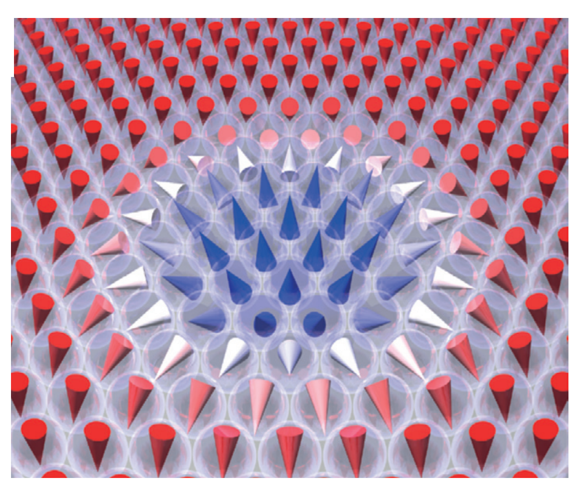


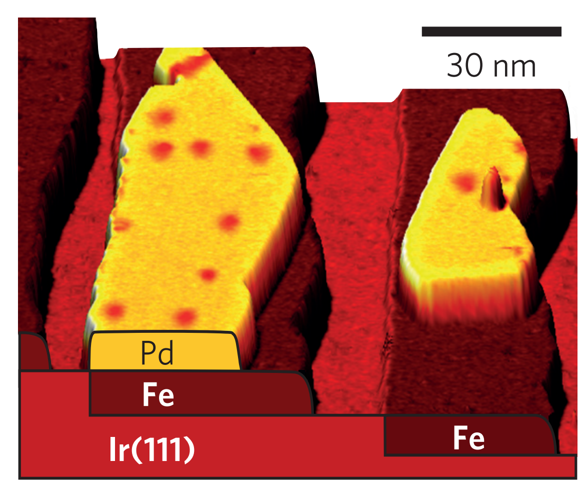


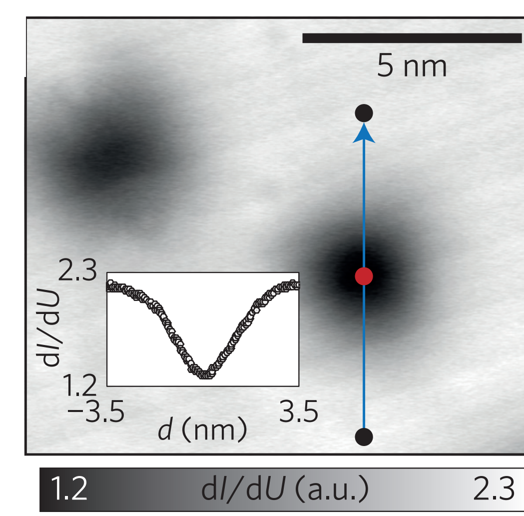
K. von Bergmann et al., J. Phys.: Condens. Matter **26**, 394002 (2014)

magnetic skyrmions:









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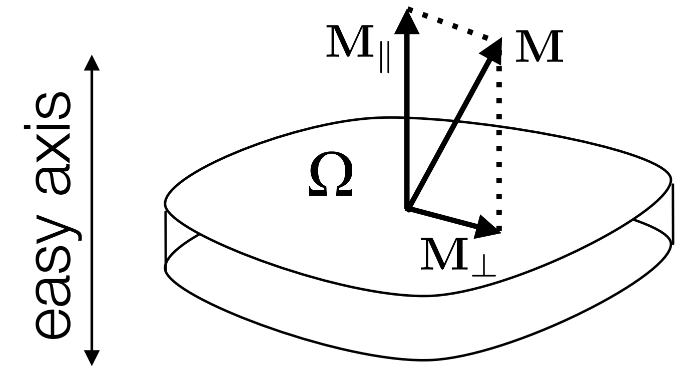
C. Hanneken et al., *Nature Nanotechnol.* **10**, 1039–1042 (2015)

Micromagnetic modeling framework



continuum theory (statics):

$\Omega \subseteq \mathbb{R}^2$ - film shape

$$E(\mathbf{M}) = \frac{A}{M_s^2} \int_{\Omega \times (0,d)} |\nabla \mathbf{M}|^2 \, d^3 r + \frac{K}{M_s^2} \int_{\Omega \times (0,d)} |\mathbf{M}_\perp|^2 \, d^3 r - \mu_0 \int_{\Omega \times (0,d)} \mathbf{M} \cdot \mathbf{H} \, d^3 r$$
$$+ \mu_0 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\nabla \cdot \mathbf{M}(\mathbf{r}) \nabla \cdot \mathbf{M}(\mathbf{r}')}{8\pi |\mathbf{r} - \mathbf{r}'|} \, d^3 r \, d^3 r' + \frac{Dd}{M_s^2} \int_{\Omega} (\bar{M}_\parallel \nabla \cdot \bar{\mathbf{M}}_\perp - \bar{\mathbf{M}}_\perp \cdot \nabla \bar{M}_\parallel) \, d^2 r.$$

M, Slastikov, 2016

Here $\mathbf{M} = (\mathbf{M}_\perp, M_\parallel)$, $\mathbf{M}_\perp \in \mathbb{R}^2$ $M_\parallel \in \mathbb{R}$ $|\mathbf{M}| = M_s$ in $\Omega \times (0,d) \subset \mathbb{R}^3$

<u>Parameters and their representative values</u>:

- exchange constant $A = 10^{-11} \mathrm{J/m}$
- anisotropy constant $K = 1.25 \times 10^6 \ \mathrm{J/m^3}$
- saturation magnetization $M_s = 1.09 \times 10^6 \ \mathrm{A/m}$
- DMI strength $D = 1 \ \mathrm{mJ/m^2}$

film thickness $d = 0.6 \ \mathrm{nm}$

lateral dimension:

$L \sim 100 \ \mathrm{nm}$

applied field strength $\mu_0 H = 100 \ \mathrm{mT}$

exchange length $\ell_{ex} = 3.66 \ \mathrm{nm}$

Need reduced micromagnetic models

analytically, the full 3D problem poses a **formidable challenge**:

- *vectorial*
- *nonlinear*
- *nonlocal*
- *multiscale*
- *topological constraints*

need a simplified model which is valid for the relevant parameter range and still captures quantitatively the physical features of the system

Solution: introduce reduced thin film models that are amenable to analysis

Use the tools from *rigorous asymptotic analysis* of calculus of variations

A. De Simone, R. V. Kohn, S. Müller, F. Otto, *Recent Analytical Developments in Micromagnetics*, 2006

Dimension reduction

$\mathbf{m} = (\mathbf{m}_\perp, m_\parallel)$

<u>assume</u> the magnetization $\mathbf{m} = \mathbf{M}/M_s$ does not vary significantly across the film thickness, measure lengths in the units of ℓ_{ex}, scale energy by Ad

$$E(\mathbf{m}) = \int_{\mathbb{R}^2} \left\{ |\nabla \mathbf{m}|^2 + (Q-1)|\mathbf{m}_\perp|^2 - 2\kappa\, \mathbf{m}_\perp \cdot \nabla m_\parallel \right\} d^2 r$$

$$+ \frac{1}{2\pi\delta} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{\sqrt{|\mathbf{r}-\mathbf{r}'|^2 + \delta^2}} - 2\pi \delta^{(2)}(\mathbf{r}-\mathbf{r}')\delta \right) m_\parallel(\mathbf{r}) m_\parallel(\mathbf{r}')\, d^2 r\, d^2 r'$$

$$+ \delta \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} K_\delta(|\mathbf{r}-\mathbf{r}'|) \nabla \cdot \mathbf{m}_\perp(\mathbf{r})\, \nabla \cdot \mathbf{m}_\perp(\mathbf{r}')\, d^2 r\, d^2 r'$$

Here:

$$Q = \frac{2K}{\mu_0 M_s^2}, \qquad \kappa = D\sqrt{\frac{2}{\mu_0 M_s^2 A}}, \qquad \ell_{ex} = \sqrt{\frac{2A}{\mu_0 M_s^2}}, \qquad \delta = \frac{d}{\ell_{ex}}$$

$$K_\delta(r) = \frac{1}{2\pi\delta} \left\{ \ln\left(\frac{\delta + \sqrt{\delta^2 + r^2}}{r} \right) - \sqrt{1 + \frac{r^2}{\delta^2}} + \frac{r}{\delta} \right\} \simeq \frac{1}{4\pi r} \qquad \delta \ll 1$$

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C. Garcia-Cervera, Ph.D. thesis (1999)

Reduced thin film energy

$\mathbf{m} = (\mathbf{m}_\perp, m_\parallel)$

regime $\delta \ll 1$:

Taylor-expand in Fourier space

$$E(\mathbf{m}) \simeq \int_{\mathbb{R}^2} \left\{ |\nabla \mathbf{m}|^2 + (Q-1)|\mathbf{m}_\perp|^2 - 2\kappa \mathbf{m}_\perp \cdot \nabla m_\parallel \right\} d^2r$$
$$+ \frac{\delta}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\nabla \cdot \mathbf{m}_\perp(\mathbf{r}) \nabla \cdot \mathbf{m}_\perp(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^2r\, d^2r' - \frac{\delta}{8\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(m_\parallel(\mathbf{r}) - m_\parallel(\mathbf{r}'))^2}{|\mathbf{r}-\mathbf{r}'|^3} d^2r\, d^2r'$$

M, Slastikov, 2016

the expression for the stray field energy is rigorously justified via Γ-expansion

Knüpfer, M, Nolte, 2019

for bounded 2D samples, extra boundary terms appear

Di Fratta, M, Slastikov (in preparation)

proper definition of the non-local terms is via Fourier:

$$\frac{1}{2} \int_{\mathbb{R}^2} |\mathbf{k}|\, |\widehat{m}_\parallel(\mathbf{k})|^2 \frac{d^2k}{(2\pi)^2} = \frac{1}{8\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(m_\parallel(\mathbf{r}) - m_\parallel(\mathbf{r}'))^2}{|\mathbf{r}-\mathbf{r}'|^3} d^2r\, d^2r', \quad \Big\} \text{ surface charges}$$

$$\frac{1}{2} \int_{\mathbb{R}^2} \frac{|\mathbf{k} \cdot \widehat{\mathbf{m}}_\perp(\mathbf{k})|^2}{|\mathbf{k}|} \frac{d^2k}{(2\pi)^2} = \frac{1}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\nabla \cdot \mathbf{m}_\perp(\mathbf{r})\, \nabla \cdot \mathbf{m}_\perp(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^2r\, d^2r'. \quad \Big\} \text{ volume charges}$$

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Reduced thin film energy (cont.)

$\mathbf{m} = (\mathbf{m}_\perp, m_\parallel)$

regime $\delta \ll 1$:

define $\overline{\mathbf{m}} = \frac{1}{\delta}\int_0^\delta \mathbf{m}(\cdot, z)dz$

$$E_s(\overline{\mathbf{m}}) = \frac{1}{\delta}\int_{\mathbb{T}_\ell\times(0,\delta)} |m_\parallel|^2 d^3r - \frac{\delta}{8\pi}\int_{\mathbb{T}_\ell}\int_{\mathbb{R}^2} \frac{(\overline{m}_\parallel(\mathbf{r}) - \overline{m}_\parallel(\mathbf{r}'))^2}{|\mathbf{r}-\mathbf{r}'|^3} d^2r\, d^2r' + \frac{\delta}{4\pi}\int_{\mathbb{T}_\ell}\int_{\mathbb{R}^2} \frac{\nabla\cdot\overline{\mathbf{m}}_\perp(\mathbf{r})\nabla\cdot\overline{\mathbf{m}}_\perp(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^2r\, d^2r'$$

Theorem *There exists a universal constant $C > 0$ such that*

$$|\mathcal{E}_s(\mathbf{m}) - E_s(\overline{\mathbf{m}})| \leq C\delta \int_{\mathbb{T}_\ell\times(0,\delta)} |\nabla\mathbf{m}|^2 d^3r$$

the difference between the energies is <u>lower order</u> in $\delta \ll 1$ $\sim \mathcal{E}(\mathbf{m})\delta^2$

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Knüpfer, M, Nolte, 2019

Reformulation

$$F_{\text{vol}}(f) := \frac{1}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{\nabla\cdot f(x)\nabla\cdot f(\tilde{x})}{|x-\tilde{x}|}\,d\tilde{x}\,dx,$$

$$F_{\text{surf}}(\tilde{f}) := \frac{1}{8\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{(\tilde{f}(x)-\tilde{f}(\tilde{x}))^2}{|x-\tilde{x}|^3}\,d\tilde{x}\,dx,$$

rescale

$$\bar{x} := \frac{Q-1}{\kappa+\delta}x \quad \text{and} \quad \bar{m}(\bar{x}) := m\left(\frac{\kappa+\delta}{Q-1}\bar{x}\right)$$

<u>*notation*</u>:

$$m = (m', m_3)$$
$$m' = (m_1, m_2)$$

<u>*energy*</u>:

$$E_{\sigma,\lambda}(m) = \int_{\mathbb{R}^2}|\nabla m|^2\,dx + \sigma^2\Bigg(\int_{\mathbb{R}^2}|m'|^2\,dx - 2\lambda\int_{\mathbb{R}^2}m'\cdot\nabla m_3\,dx + (1-\lambda)\left(F_{\text{vol}}(m') - F_{\text{surf}}(m_3)\right)\Bigg)$$

<u>*parameters*</u>:

$$\sigma := \frac{\kappa+\delta}{\sqrt{Q-1}} \quad \text{and} \quad \lambda := \frac{\kappa}{\kappa+\delta}$$

ultimately, we wish to consider the asymptotic limit $\sigma \to 0$ with λ fixed

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conformal limit

Skyrmions

- topologically nontrivial configurations of nonlinear field theories

- introduced by Tony Skyrme in the early 1960s to empirically describe the low-energy properties of baryons

- received attention in the mathematical literature from the 1980s onward

Esteban, 1986; Esteban, 1992; Faddeev and Niemi, 1997; Esteban, 2004; Lin and Yang, 2004

- relevant example: *baby skyrmions*

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^2} \left\{ |\nabla u|^2 + \frac{\lambda}{2} |\partial_1 u \times \partial_2 u|^2 + \frac{\mu}{2} (1 - \mathbf{n} \cdot u)^2 \right\} \mathrm{d}x$$

- existence of minimizers of

$$E_k = \inf\{E(u) : E(u) < \infty, \deg(u) = k\}$$

$$\deg(u) = \frac{1}{4\pi} \int_{\mathbb{R}^2} u \cdot (\partial_1 u \times \partial_2 u) \mathrm{d}x$$

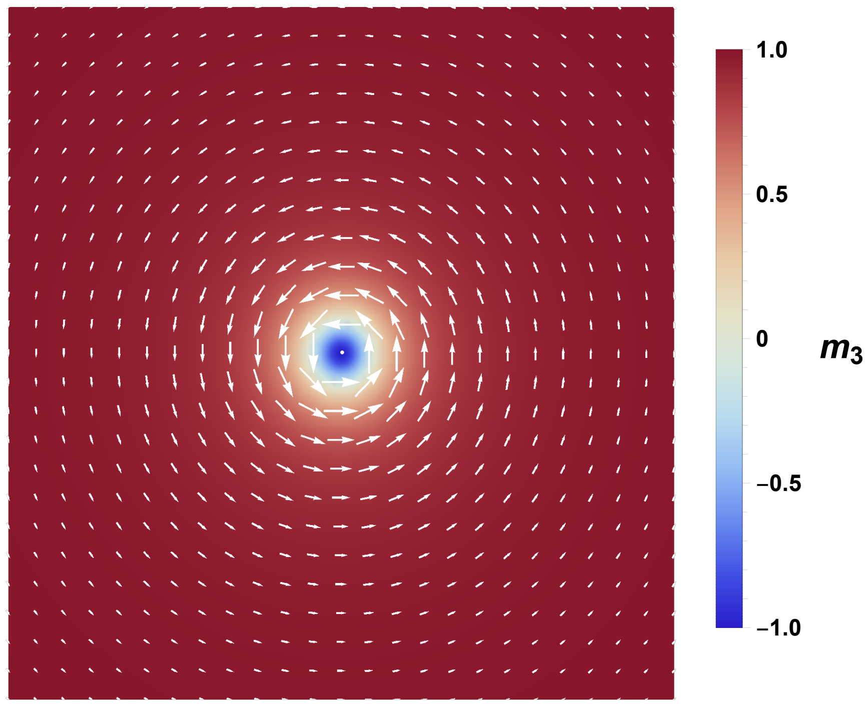
Lin and Yang, 2004; Li and Zhu, 2011

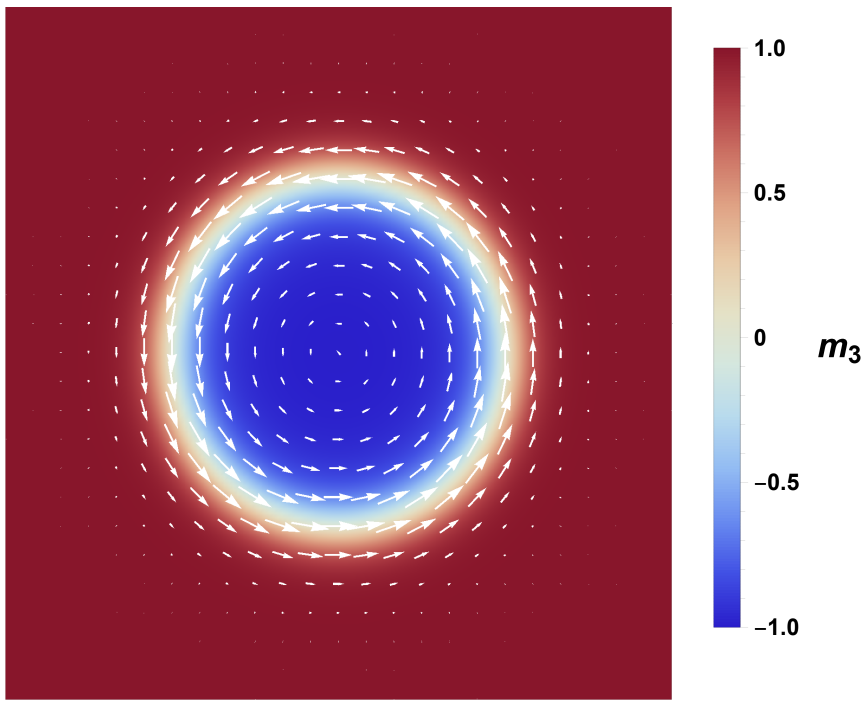
Admissible class

$$\mathcal{N}(m) := \frac{1}{4\pi} \int_{\mathbb{R}^2} m \cdot (\partial_1 m \times \partial_2 m)\, \mathrm{d}x$$

topological degree

compact skyrmion vs. skyrmionic bubble





for bubble skyrmion, the stray field energy *diverges* with radius:

$$F_{\mathrm{surf}}(m_{R,3}) \sim R \ln R$$

M, Simon, 2019

hence $\quad m : \mathbb{R}^2 \to \mathbb{S}^2,\ \nabla m \in L^2,\ m' \in L^2 \quad \not\Rightarrow \quad E_{\sigma,\lambda}(m) > -\infty$

no hope to construct solutions as absolute minimizers with prescribed degree

contrast this with the local case

Bogdanov, Yablonskii, 1989
Bogdanov, Kudinov, Yablonskii, 1989
Melcher, 2014; Li, Melcher, 2018

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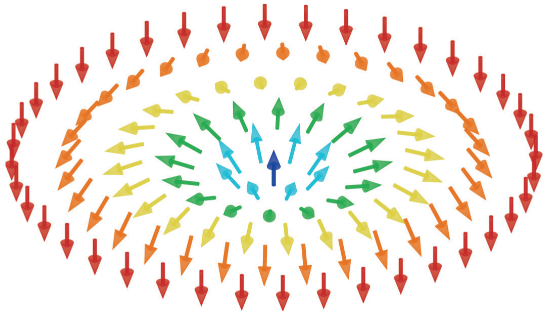
Compact skyrmions as local minimizers



introduce:

$$\mathcal{N}(m) := \frac{1}{4\pi} \int_{\mathbb{R}^2} m \cdot (\partial_1 m \times \partial_2 m) \, \mathrm{d}x$$

$$\mathcal{A} := \left\{ m \in \mathring{H}^1(\mathbb{R}^2; \mathbb{S}^2) : \int_{\mathbb{R}^2} |\nabla m|^2 \, \mathrm{d}x < 16\pi, \; m + e_3 \in L^2(\mathbb{R}^2), \; \mathcal{N}(m) = 1 \right\}$$

why 16π? Topological lower bound: $m \in \mathring{H}^1(\mathbb{R}^2; \mathbb{S}^2)$

$$\int_{\mathbb{R}^2} |\nabla m|^2 \, \mathrm{d}x \geq 8\pi \, |\mathcal{N}(m)|$$

$$|\nabla m|^2 \pm 2m \cdot (\partial_1 m \times \partial_2 m) = |\partial_1 m \mp m \times \partial_2 m|^2$$

allows to exclude splitting in the concentration compactness arguments

Theorem 1. *Let $\sigma > 0$ and $\lambda \in [0, 1]$ be such that $\sigma^2(1 + \lambda)^2 \leq 2$. Then there exists $m_{\sigma,\lambda} \in \mathcal{A}$ such that*

$$E_{\sigma,\lambda}(m_{\sigma,\lambda}) = \inf_{\widetilde{m} \in \mathcal{A}} E_{\sigma,\lambda}(\widetilde{m}).$$

adapting arguments of Melcher, 2014, and Döring and Melcher, 2017
see also Greco, 2019

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main point: $m_n + e_3 \to m_\sigma + e_3$ in $L^2(\mathbb{R}^2; \mathbb{R}^3)$

Conformal limit

$$E_{\sigma,\lambda}(m) = \int_{\mathbb{R}^2} |\nabla m|^2 \, \mathrm{d}x + \sigma^2 \bigg(\int_{\mathbb{R}^2} |m'|^2 \, \mathrm{d}x - 2\lambda \int_{\mathbb{R}^2} m' \cdot \nabla m_3 \, \mathrm{d}x + (1-\lambda) \left(F_{\mathrm{vol}}(m') - F_{\mathrm{surf}}(m_3) \right) \bigg)$$

setting $\sigma = 0$ leads to harmonic maps from $\mathbb{R}^2$ to $\mathbb{S}^2$ with prescribed degree

complete solution formally obtained by Belavin and Polyakov, 1975

degree 1 minimizers of $F(m) := \int_{\mathbb{R}^2} |\nabla m|^2 \, \mathrm{d}x$ over $m \in \mathring{H}^1(\mathbb{R}^2; \mathbb{S}^2)$ belong to:

$$\mathcal{B} := \left\{ S\Phi(\rho^{-1}(\bullet - x)) : S \in \mathrm{SO}(3), \, \rho > 0, \, x \in \mathbb{R}^2 \right\}$$

i.e., *dilations, rotations and translations* of:

$$\Phi(x) := \left(-\frac{2x}{1+|x|^2}, \frac{1-|x|^2}{1+|x|^2} \right)$$

see also:
Eells and Sampson, 1964
Lemaire, 1978
Wood, 1974
Brezis, Coron, 1985

furthermore, if $\phi \in \mathcal{B}$ then $\int_{\mathbb{R}^2} |\nabla \phi|^2 \, \mathrm{d}x = 8\pi$ and

$$\Delta\phi + |\nabla\phi|^2 \phi = 0 \qquad \mathcal{N}(\phi) = 1$$

and vice versa

minimizers of $E_{\sigma,\lambda}$ as $\sigma \to 0$ are almost minimizers of F

=> minimizers are close to $\mathcal{B}$ — Lin, 1999

Rigidity estimate for degree ±1 harmonic maps

define the class of degree 1 Sobolev maps from $\mathbb{R}^2$ to $\mathbb{S}^2$

$$\mathcal{C} := \left\{ \tilde{m} \in \mathring{H}^1(\mathbb{R}^2; \mathbb{S}^2) : \mathcal{N}(\tilde{m}) = 1 \right\}$$

and the *Dirichlet distance* to degree 1 Belavin-Polyakov profiles:

$$D(m; \mathcal{B}) := \inf_{\widetilde{\phi} \in \mathcal{B}} \left(\int_{\mathbb{R}^2} \left| \nabla \left(m - \widetilde{\phi} \right) \right|^2 \mathrm{d}x \right)^{\frac{1}{2}}$$

Theorem 2. *For every $m \in \mathcal{C}$ there exists $\phi \in \mathcal{B}$ that achieves the infimum in the Dirichlet distance $D(m; \mathcal{B})$. Furthermore, there exists a universal constant $\eta > 0$ such that*

$$\eta D^2(m; \mathcal{B}) \leq F(m) - 8\pi.$$

Gustafson, Kang, Tsai, 2007

- conformal invariance of the harmonic maps => switch to maps from $\mathbb{S}^2$ to $\mathbb{S}^2$
- reduce the problem to that of stability of the identity map on $\mathbb{S}^2$
- spectral gap for the linearized problem via vectorial spherical harmonics

Smith, 1975; Mazet, 1973; Davila, del Pino, Wei, 2019; Chen, Liu, Wei, 2020
Di Fratta, Slastikov, Zarnescu, 2019; Luckhaus, Zemas, 2019

Reduction to maps between spheres

given a map $m \in \mathcal{C}$ that is close to $\phi \in \mathcal{B}$ in $\mathring{H}^1(\mathbb{R}^2;\mathbb{R}^3)$, the map $m \circ \phi^{-1}$ is close to $\mathrm{id}_{\mathbb{S}^2}$ in $H^1(\mathbb{S}^2;\mathbb{R}^3)$.

<u>Hessian</u> of the Dirichlet energy at the identity map: $\zeta, \xi \in H^1(\mathbb{S}^2; T\mathbb{S}^2)$

$$\mathfrak{H}(\zeta,\xi) := \int_{\mathbb{S}^2} (\nabla\zeta : \nabla\xi - 2\zeta\cdot\xi)\,\mathrm{d}\mathcal{H}^2$$

define the *Jacobi fields*: $\dim J \geq 6$

$$J := \left\{\zeta \in H^1\left(\mathbb{S}^2; T\mathbb{S}^2\right) : \mathfrak{H}(\zeta,\zeta) = 0\right\}$$

vector spherical harmonics:

$$\mathcal{Y}^{(1)}_{n,j}(y) := Y_{n,j}(y)\,y, \qquad \mathcal{Y}^{(1)}_{0,0}(y) := \frac{1}{\sqrt{4\pi}}y,$$
$$\mathcal{Y}^{(2)}_{n,j}(y) := \frac{1}{\sqrt{n(n+1)}}\nabla Y_{n,j}(y),$$
$$\mathcal{Y}^{(3)}_{n,j}(y) := \frac{1}{\sqrt{n(n+1)}}\,y \times \nabla Y_{n,j}(y).$$

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Freeden and Schreiner, *Spherical Functions of Mathematical Geosciences, 2009*

Jacobi fields and a spectral gap

Proposition *We have* $J = \operatorname{span}\left\{\mathcal{Y}_{1,j}^{(2)}, \mathcal{Y}_{1,j}^{(3)};\ j = -1, 0, 1\right\}$. *In particular, all Jacobi fields are smooth and it holds that* $\dim J = 6$. *Furthermore, we have the spectral gap property*

$$\mathfrak{H}(\xi, \xi) \geq \frac{2}{3} \int_{\mathbb{S}^2} |\nabla \xi|^2 \, \mathrm{d}\mathcal{H}^2$$

for all $\xi \in \mathbb{H}^1$. *Finally, the* L^2*-orthogonal projection* $\pi_J : L^2(\mathbb{S}^2; T\mathbb{S}^2) \to L^2(\mathbb{S}^2; T\mathbb{S}^2)$ *onto* J *is well-defined and orthogonal with respect to the inner product in* $\mathring{H}^1(\mathbb{S}^2)$.

here:

$$\mathbb{H}^1 := \left\{\xi \in H^1(\mathbb{S}^2; T\mathbb{S}^2) : \int_{\mathbb{S}^2} (\nabla \xi : \nabla \zeta) \, \mathrm{d}\mathcal{H}^2 = 0 \text{ for all } \zeta \in J\right\}$$

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From linear stability to rigidity

find $\phi \in \mathcal{B}$ that best approximates $m \in \mathcal{C}$ in Dirichlet distance $D(m;\mathcal{B})$

Lemma *For any $m \in \mathcal{C}$ there exists $\phi \in \mathcal{B}$ such that*

$$D(m;\mathcal{B}) = \left(\int_{\mathbb{R}^2} |\nabla(m-\phi)|^2 \,\mathrm{d}x \right)^{\frac{1}{2}}.$$

let $\phi_n := R_n \Phi\left(\rho_n^{-1}(\bullet - x_n)\right) \in \mathcal{B}$ be a minimizing sequence (wlog $R_n = R$)

arguing by contradiction, we have

$$\lim_{n\to\infty} \rho_n = 0, \ \lim_{n\to\infty} \rho_n = \infty, \text{ or } \lim_{n\to\infty} x_n = \infty.$$

=> either $\nabla \phi_n \rightharpoonup 0 \text{ in } L^2$ or $\nabla m_n \rightharpoonup 0 \text{ in } L^2$ (after rescaling) =>

$$\int_{\mathbb{R}^2} |\nabla(m-\phi)|^2 \,\mathrm{d}x > \int_{\mathbb{R}^2} |\nabla m|^2 \,\mathrm{d}x + 8\pi \iff \int_{\mathbb{R}^2} \nabla m : \nabla \phi \,\mathrm{d}x < 0 \quad \text{for all } \phi \in \mathcal{B}.$$

conclude by testing against Belavin-Polyakov profiles

with permuted and reflected components

From linear stability to rigidity (cont.)

Lemma 3. *There exists a universal constant* $\tilde\eta > 0$ *such that the following holds: Let* $p \in [1,\infty)$. *Then there exists a constant* $C_p > 0$ *such that if* $m \in H^1(\mathbb{S}^2;\mathbb{S}^2)$ *satisfies* $\int_{\mathbb{S}^2} |\nabla(m - \mathrm{id}_{\mathbb{S}^2})|^2 \,\mathrm{d}\mathcal{H}^2 \le \tilde\eta$, *then we have the estimate*

$$\left(\int_{\mathbb{S}^2} |m - \mathrm{id}_{\mathbb{S}^2}|^p \,\mathrm{d}\mathcal{H}^2\right)^{\frac{1}{p}} \le C_p \left(\int_{\mathbb{S}^2} |\nabla(m - \mathrm{id}_{\mathbb{S}^2})|^2 \,\mathrm{d}\mathcal{H}^2\right)^{\frac{1}{2}}.$$

Furthermore, there exists a universal constant $C > 0$ *such that the Moser-Trudinger type inequality*

$$\int_{\mathbb{S}^2} e^{\frac{2\pi}{3}\frac{|m-\mathrm{id}_{\mathbb{S}^2}|^2}{\|\nabla(m-\mathrm{id}_{\mathbb{S}^2})\|_2^2}} \,\mathrm{d}\mathcal{H}^2 \le C$$

holds.

Lemma 4. *Let* $\tilde\eta > 0$ *be as in Lemma 3. For* $m \in \mathcal{C}$ *with* $D^2(m;\mathcal{B}) < \tilde\eta$ *we have*

$$\left(\frac{2}{3} - \frac{2}{3}C_4^2 D(m;\mathcal{B}) - \frac{19}{12}C_4^4 D^2(m;\mathcal{B})\right) D^2(m;\mathcal{B}) \le F(m) - 8\pi,$$

where C_4 *is the constant from Lemma 3.*

Back to the conformal limit: an Ansatz

every ansatz-free analysis requires a good ansatz 🙂

$$f(r) := \frac{2r}{1+r^2}$$

problem: $\phi \in \mathcal{B} \;\Rightarrow\; E_{\sigma,\lambda}(\phi) = \infty$!

anisotropy energy blows up logarithmically => need a suitable cutoff at infinity

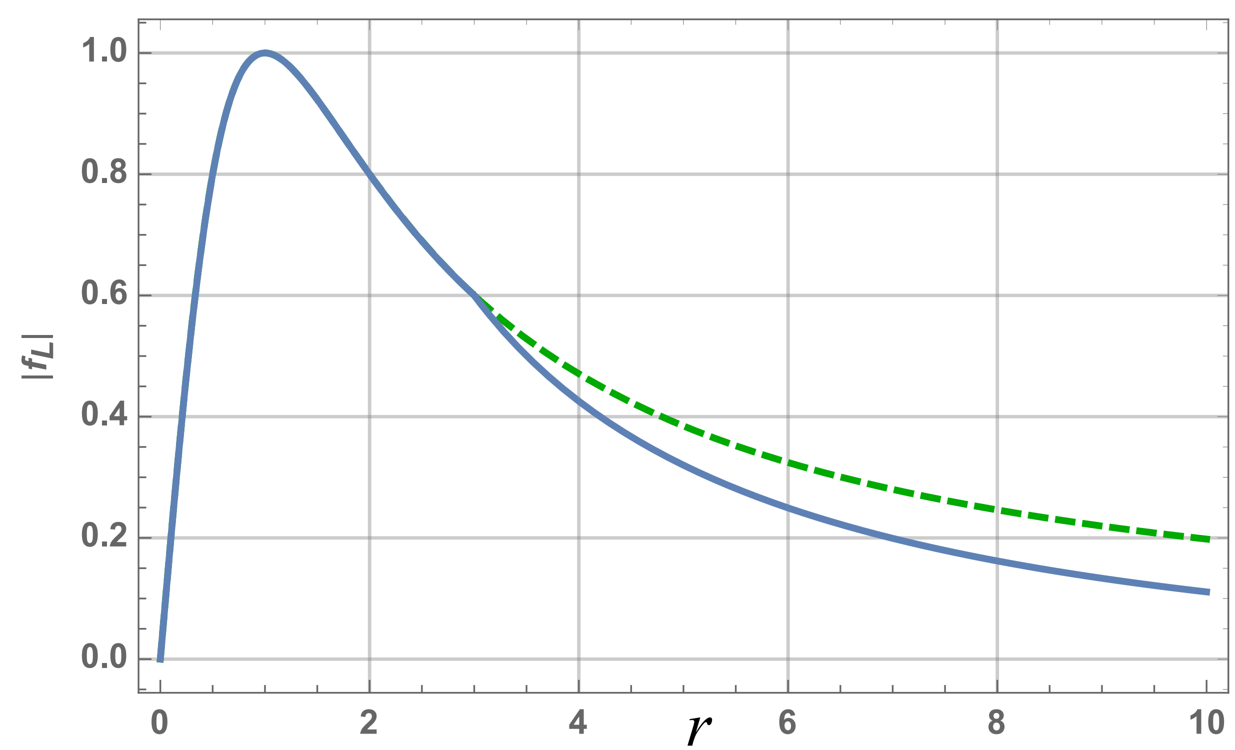
for $L > 1$, introduce

$$\Phi_L(x) := \left(-f_L(|x|)\frac{x}{|x|}, \operatorname{sign}(1-|x|)\sqrt{1-f_L^2(|x|)} \right)$$

$$f_L(r) := \begin{cases} f(r) & \text{if } r \le L^{\frac{1}{2}}, \\ f\left(L^{\frac{1}{2}}\right) \dfrac{K_1\left(rL^{-1}\right)}{K_1\left(L^{-\frac{1}{2}}\right)} & \text{if } r > L^{\frac{1}{2}}. \end{cases}$$

K_1 is the modified Bessel function of the second kind of order 1

- decays exponentially
- arises from the exact minimizers of the exchange + anisotropy at infinity



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Upper bound on energy

$$S_\theta := \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

fix $\rho > 0,\ \theta \in [-\pi, \pi),\ L > 1$ and $S_\theta \in \mathrm{SO}(3)$ as above

define a test profile

$$\phi_{\rho,\theta,L}(x) := S_\theta \Phi_L(\rho^{-1}x)$$

then

$$E_{\sigma,\lambda}(\phi_{\rho,\theta,L}) \simeq 8\pi + \frac{4\pi}{L^2} + 4\pi\sigma^2\rho^2 \log\left(\frac{4L^2}{e^{2(1+\gamma)}}\right)$$
$$- 8\pi\sigma^2\lambda\rho\cos\theta + \sigma^2(1-\lambda)\frac{\pi^3\rho}{8}\left(3\cos^2\theta - 1\right)$$

minimized by

$$\rho_0 \simeq \frac{\bar{g}(\lambda)}{16\pi}\frac{1}{|\log\sigma|}, \quad L_0 \simeq \frac{16\pi}{\bar{g}(\lambda)}\frac{|\log\sigma|}{\sigma}, \quad \theta_0^\pm := \begin{cases} 0 & \text{if } \lambda \geq \lambda_c, \\ \pm\arccos\left(\frac{32\lambda}{3\pi^2(1-\lambda)}\right) & \text{else.} \end{cases}$$

$$\lambda_c := \frac{3\pi^2}{32+3\pi^2}, \qquad \bar{g}(\lambda) := \begin{cases} (8+\frac{\pi^2}{4})\pi\,\lambda - \frac{\pi^3}{4} & \text{if } \lambda \geq \lambda_c \\ \frac{128\lambda^2}{3\pi(1-\lambda)} + \frac{\pi^3}{8}(1-\lambda) & \text{else} \end{cases}$$

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see also Komineas, Melcher, Venakides, 2020

Main result

Theorem 5. *Let* $\lambda \in [0,1]$. *Let* m_σ *be a minimizer of* $E_{\sigma,\lambda}$ *over* $\mathcal{A}$. *Then there exist* $x_\sigma \in \mathbb{R}^2$, $\rho_\sigma > 0$ *and* $\theta_\sigma \in [-\pi, \pi)$ *such that* $m_\sigma - S_{\theta_\sigma}\Phi(\rho_\sigma^{-1}(\bullet - x_\sigma)) \to 0$ *in* $\mathring{H}^1(\mathbb{R}^2;\mathbb{R}^3)$ *as* $\sigma \to 0$, *and*

$$\lim_{\sigma\to 0} |\log\sigma|\rho_\sigma = \frac{\bar{g}(\lambda)}{16\pi}, \qquad \lim_{\sigma\to 0} |\theta_\sigma| = \theta_0^+,$$

as well as

$$\lim_{\sigma\to 0} \frac{|\log\sigma|^2}{\sigma^2 \log|\log\sigma|} \left| E_{\sigma,\lambda}(m_\sigma) - 8\pi + \frac{\sigma^2}{|\log\sigma|}\left(\frac{\bar{g}^2(\lambda)}{32\pi} - \frac{\bar{g}^2(\lambda)}{32\pi} \frac{\log|\log\sigma|}{|\log\sigma|} \right) \right| = 0.$$

$$\lambda_c := \frac{3\pi^2}{32 + 3\pi^2}, \qquad \bar{g}(\lambda) := \begin{cases} (8 + \frac{\pi^2}{4})\pi\,\lambda - \frac{\pi^3}{4} & \text{if } \lambda \geq \lambda_c \\ \frac{128\lambda^2}{3\pi(1-\lambda)} + \frac{\pi^3}{8}(1-\lambda) & \text{else} \end{cases}$$

Remark: quantitative estimate of the closeness to the BP profile available

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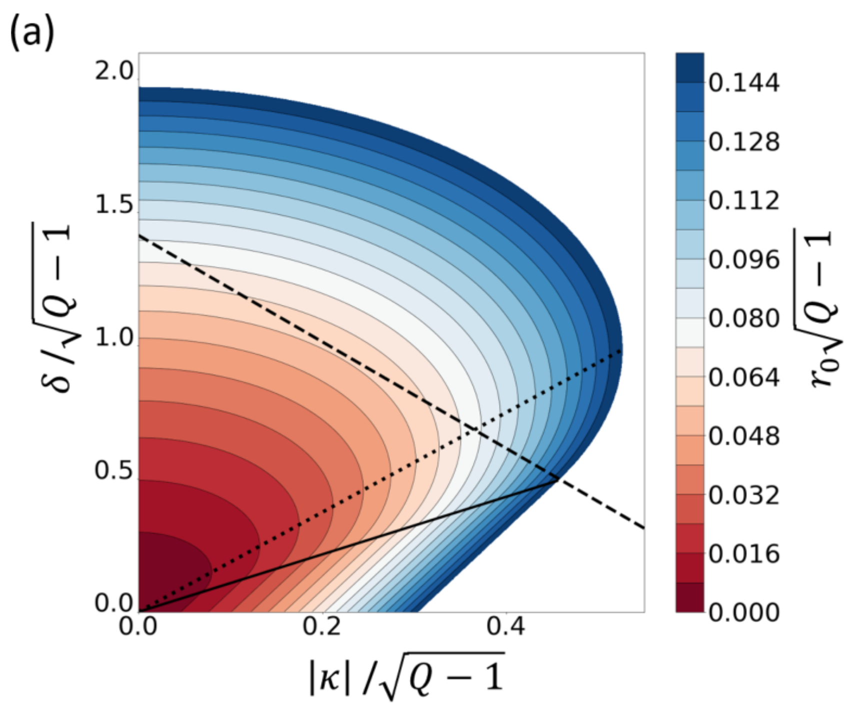
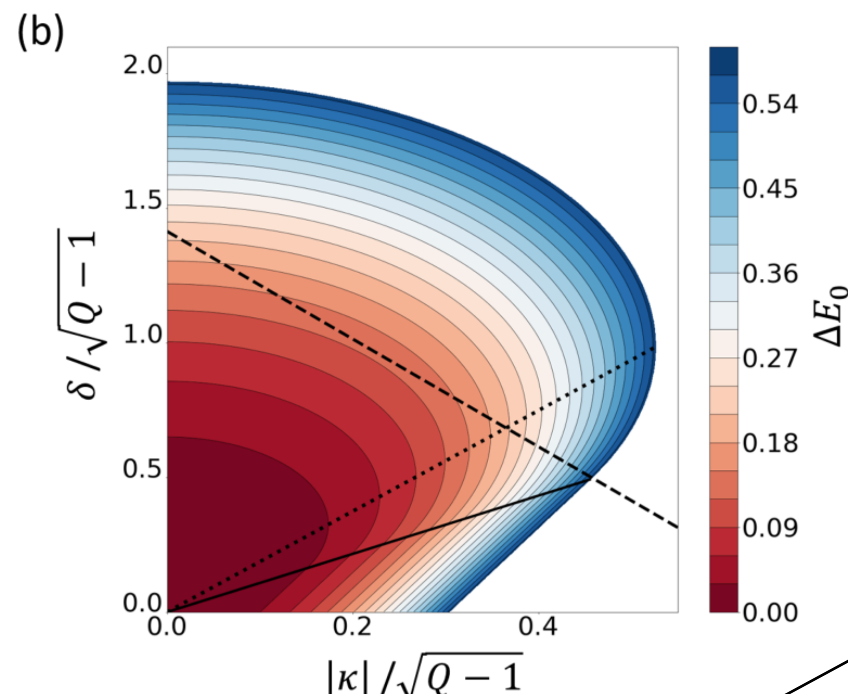
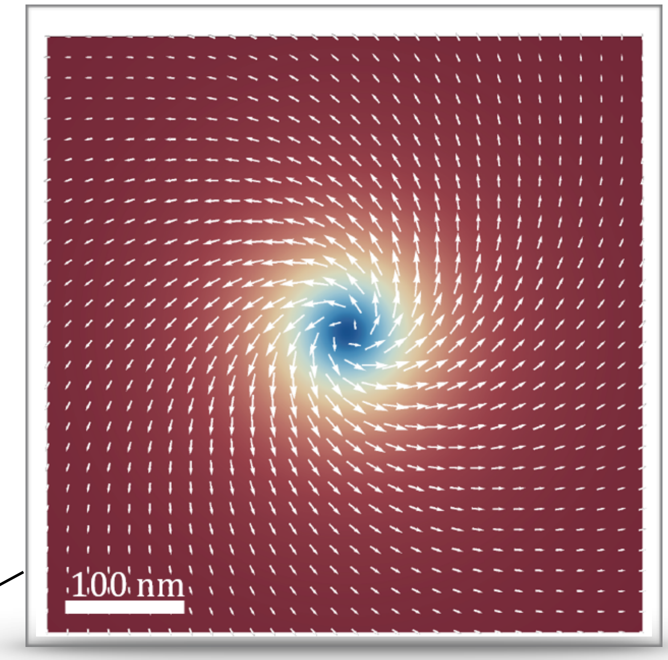
Bernand-Mantel, M, Simon, 2020

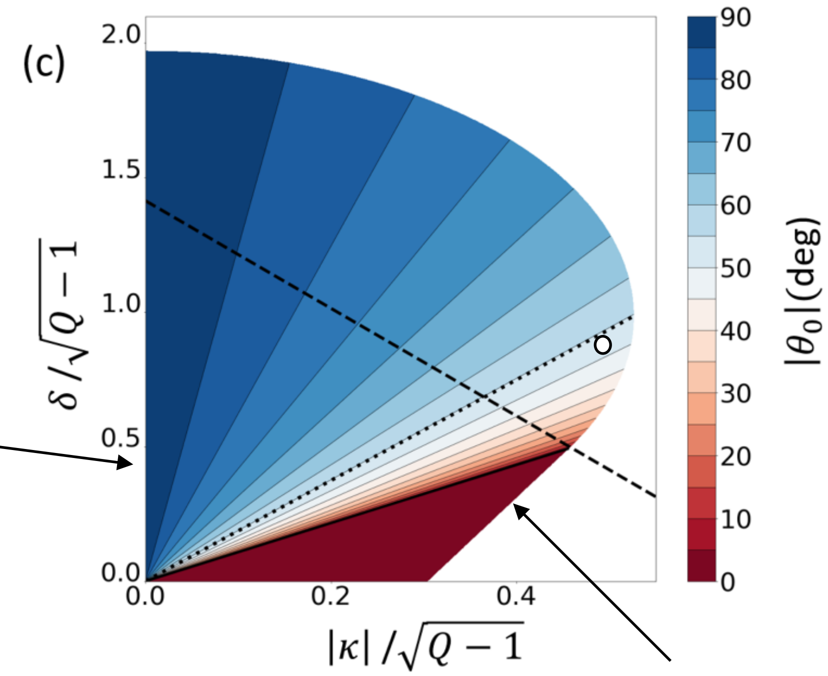
Interpretation

$$E(\mathbf{m}) \simeq \int_{\mathbb{R}^2} \left\{ |\nabla \mathbf{m}|^2 + (Q-1)|\mathbf{m}_\perp|^2 - 2\kappa \mathbf{m}_\perp \cdot \nabla m_\parallel \right\} d^2r + \frac{\delta}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\nabla \cdot \mathbf{m}_\perp(\mathbf{r}) \nabla \cdot \mathbf{m}_\perp(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^2r \, d^2r' - \frac{\delta}{8\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(m_\parallel(\mathbf{r}) - m_\parallel(\mathbf{r}'))^2}{|\mathbf{r}-\mathbf{r}'|^3} d^2r \, d^2r'$$

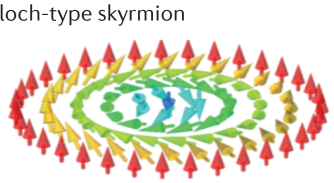








Bloch-type skyrmion



$\theta_0^{\text{opt}} = \pm \arccos(\text{sgn}(\kappa)/\sqrt{3})$, i.e., $\theta_0^{\text{opt}} \approx \pm 54.74^\circ$ for $\kappa > 0$ or $\theta_0^{\text{opt}} \approx \pm 125.3^\circ$ for $\kappa < 0$.

Néel-type skyrmion

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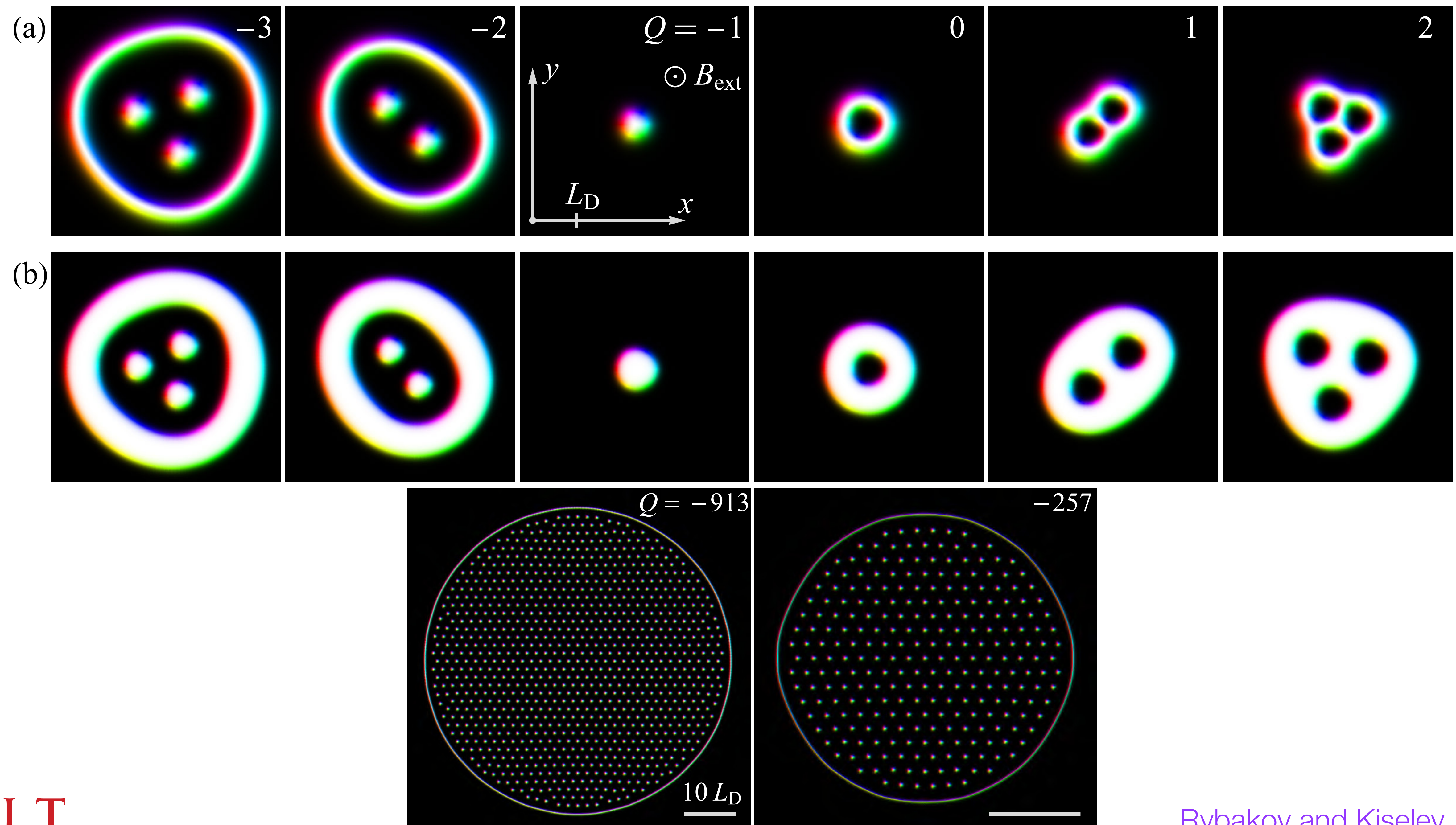
Bernand-Mantel, M, Simon, 2020

Outline of proof

- matching upper and lower bounds in terms of energies of truncated BP profiles in the spirit of Γ-equivalence
- use the established rigidity of degree 1 harmonic maps to estimate the remaining terms in the energy
- the main difficulty is that the limiting BP profile may not satisfy $\lim_{|x|\to\infty} \phi(x) = -e_3$
- estimate the anisotropy energy penalty for deviations of $\nu := \lim\limits_{|x|\to\infty} \phi(x)$ from -e_3, using our version of Moser-Trudinger inequality
- relate the difference between ν and $-e_3$ to the Dirichlet excess via relaxing the unit length constraint and minimizing the exchange + anisotropy
- conclude by utilizing the rigidity of the finite-dimensional energy of BP profiles

Skyrmion bags

many more solutions in the homotopy classes (even w/o stray field):



Rybakov and Kiselev, 2019

Questions?

Cosmonauts A.Balandin and G.Strekalov with the Banner of Peace. Mir Space Station.