THE JOINTS PROBLEM FOR VARIETIES

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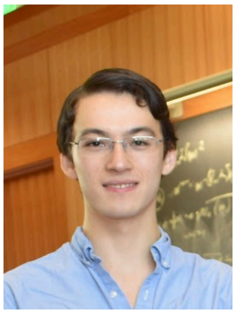
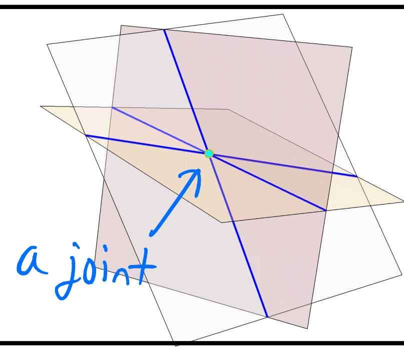
with



Jonathan Tidor & Hung-Hsun Hans Yu

Joints problem What's the max # of joints that N lines in $\mathbb{R}^3$ can make?

A **joint** is a point contained in 3 non-coplanar lines

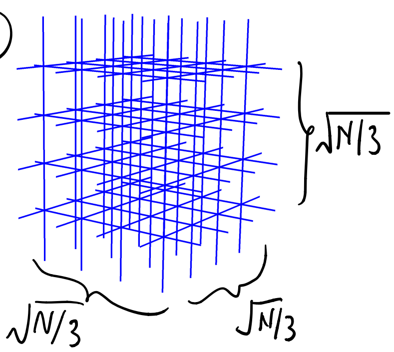


Examples

N lines

$\Theta(N^{3/2})$ joints

①

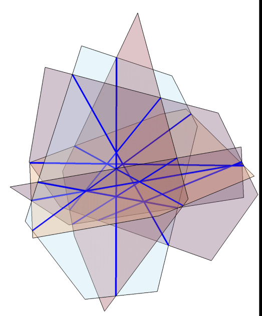


$\left(\frac{N}{3}\right)^{3/2}$ joints

② $k \sim \sqrt{2N}$ generic planes

$\rightsquigarrow$ pairwise form $\binom{k}{2} \sim N$ lines

& triplewise form $\binom{k}{3} \sim \frac{\sqrt{2}}{3} N^{3/2}$ joints



Introduced by Chazelle–Edelsbrunner–Guibas–Pollack–Seidel–Sharir–Snoeyink '92 $O(N^{7/4})$

$\boxed{\text{Guth–Katz (2010): } N \text{ lines in } \mathbb{R}^3 \text{ form } O(N^{3/2}) \text{ joints}}$

Subseq. generalized to arb dim & fields $(\mathbb{F}^d)$ Kaplan–Sharir–Shustin, Quilodrán

$\boxed{\text{Yu–Z. (2019+): optimal const, } \le \frac{\sqrt{2}}{3} N^{3/2} \text{ joints}}$

Connections

- Kakeya problem (Wolff)
- Finite field Kakeya problem (Dvir) ← polynomial method
- Multilinear Kakeya, "joints of tubes" (Bennett–Carbery–Tao, Guth)

$\boxed{\text{Joints of flats: max \# joints for } N \text{ planes in } \mathbb{F}^6\text{?}}$

a point contained in a triple of planes in spanning & indep directions

2-dim flats

<u>Construction</u> $\Theta(N^{3/2})$ joints: generic 4-flats

pairwise intersect → planes
triplewise intersect → joints

<u>Why I like this problem:</u>

- natural extension of the joints problem
- a key step in pf of joints thm fails badly
- need a new extension of the polynomial method

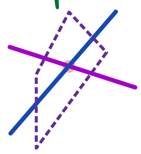
Incidence geometry for higher dimensional objects

Prior results on joints of higher dim objects

[Yang] N planes in $\mathbb{R}^6$ have $N^{\frac{3}{2}+o(1)}$ joints

Limitations
① Error term
② Only $\mathbb{R}$

[Yu-Z. / Carbery-Iliopoulou] N lines & M planes in $\mathbb{F}^4$ make $O(NM^{1/2})$ joints
(plane-line^2)



line-line-plane, in indep & spanning directions

Our results [Tidor-Yu-Z.]

Joints of flats	N planes in $\mathbb{F}^6$ have $O(N^{3/2})$ joints
Joints of varieties	A set of 2-dim varieties in $\mathbb{F}^6$ of total degree N has $O(N^{3/2})$ joints

$p \in V_1, V_2, V_3$ regular point
tangent planes at p spanning & indep directions

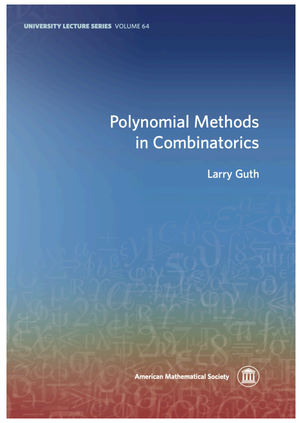
And more generally:

- arbitrary dimensions
- several sets of varieties (multijoints)
- counting joints with multiplicities

prev. for joints of lines
- conj. [Carbery]
- [Iliopoulou]
- [Zhang]

Review of the proof of: [Kaplan–Sharir–Shustin, Quilodrán]

N lines in $\mathbb{R}^3$ have $O(N^{3/2})$ joints



① Parameter counting:

using $\dim \mathbb{R}[x_1,\dots,x_d]_{\le n} = \binom{n+d}{d}$ deg $\le n$

deduce that $\exists$ non-zero poly g, $\deg \le C J^{1/3}$, vanishing on joints

Take g with min deg. $J = \#$joints

② Vanishing lemma: a single-variable polynomial cannot vanish more times than its degree

③ A joints-specific argument. If all lines have $> C J^{1/3}$ joints, then vanishing lemma $\Rightarrow$ g vanishes on all lines $\Rightarrow$ ∇g vanishes on all joints

$\Rightarrow$ one of $\partial_x g, \partial_y g, \partial_z g$ is nonzero, lower deg & vanish on all joints

So some line has $\le C J^{1/3}$ joints. Remove this line & repeat

$J \le C J^{1/3} N$ Thus $J = O(N^{3/2})$

How to generalize vanishing lemma to 2-var polynomials?

$\boxed{\text{Thm (Tidor–Yu–Z.) } N \text{ planes in } \mathbb{R}^6 \text{ have } O(N^{3/2}) \text{ joints}}$

Above proof would generalize if...

Attempt I

$g \in \mathbb{R}[x,y]_{\leq n}$ vanishing at $\binom{n+2}{2}$ distinct points $\overset{???}{\Longrightarrow}$ $g \equiv 0$

$\boxed{\text{NO}}$

Method of multiplicities:

Attempt II

$g \in \mathbb{R}[x,y]_{\leq n}$ vanishing to order $> n$ at a single point $\overset{???}{\Longrightarrow}$ $g \equiv 0$

ie. $\dfrac{\partial^{i+j} g}{\partial x^i \partial y^j}(p) = 0 \quad \forall\, i+j \leq n$

$\boxed{\text{YES, but how does it help?}}$

Attempt III

$\underbrace{g \in \mathbb{R}[x,y]_{\leq n}}_{\sim \frac{n^2}{2} \text{ dim}}$ vanishing $\underbrace{\text{to order } s \text{ at } \approx \frac{n^2}{s^2} \text{ points}}_{\sim \frac{n^2}{2} \text{ linear constraints}}$ $\overset{???}{\Longrightarrow}$ $g \equiv 0$

$\boxed{\text{NO}}$ e.g. $g(x,y) = y^s$

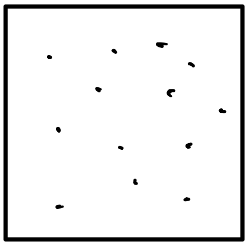
Linear dependencies among

vanishing conditions: linear constraints on $g \in \mathbb{R}[x,y]_{\leq n}$. e.g. $(\partial_{xx} - \partial_{yy})g(p) = 0$ for some fixed p

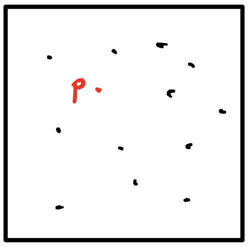
- viewed as both (derivative op, point) & linear functionals on $\mathbb{R}[x,y]_{\leq n}$

☆Key idea 1 Collecting linearly indep vanishing conditions

Restricting to a plane for now



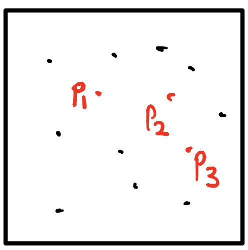
We will construct a set of $\dim \mathbb{R}[x,y]_{\leq n} = \binom{n+2}{2}$ linearly indep vanishing conditions on $\mathbb{R}[x,y]_{\leq n}$



Attached to each point p is a set of vanishing conditions for $g \in \mathbb{R}[x,y]_{\leq n}$:

$g(p)=0, \quad \partial_x g(p)=0, \quad \partial_y g(p)=0$

$\partial_{xx} g(p)=0, \quad \partial_{xy} g(p)=0, \quad \partial_{yy} g(p)=0, \quad \partial_{xxx} g(p)=0, \ldots$



The above vanishing conditions attached to several different points are lin. dep. as linear functionals on $\mathbb{R}[x,y]_{\leq n}$

We will select a basis of linear functionals on $\mathbb{R}[x,y]_{\leq n}$ via the following procedure.

First attempt

Cycle through the points on the plane

$p_1, p_2, p_3, \dots, p_1, p_2, p_3, \dots, p_1, p_2, p_3, \dots$

p_1 : add vanishing condition $g(p_1)=0$

p_2 : add vanishing condition $g(p_2)=0$ if nonredundant

⋮

p_1 : add a nonredundant subset of $\partial_x g(p_1)=0$, $\partial_y g(p_1)=0$

none implied by other added + prev. added

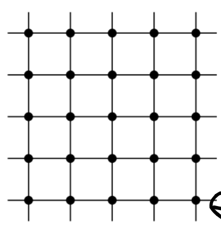
i.e. basis extension

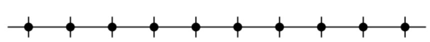
p_2 : add a nonredundant subset of $\partial_x g(p_2)=0$, $\partial_y g(p_2)=0$

and so on...

Can we control the # van. cond. attached to each pt?

Example





pts on grid get way more vanishing conditions than pts on the line

UNDESIRABLE

(This example also comes up for inverse Bézout; see Tao blog)

☆ Key idea 2 Let some points get a head start

e.g. (100 pts) $p_1, p_2, \dots, p_{50},\ p_1, \dots, p_{50},\ \dots,\ p_1, \dots, p_{50},\ p_1, \dots, p_{100},\ p_1, \dots, p_{100}, \dots$

Handicap $\vec{\alpha} \in \mathbb{Z}^J$ assigns an integer to each point

e.g.

points	a	b	c	d	e
handicap	0	1	3	0	-1

→ order: c c b c a b c d a b c d e a b c d e ...

Modify process of assigning vanishing conditions

c : add a nonredundant set of 0^{th} order derivative vanishing @ c

c ———————— 1^{st} ———————— c

b ———————— 0^{th} ———————— b

c ———————— 2^{nd} ———————— c

a ———————— 0^{th} ———————— a

Want a "good" choice of handicaps: treating all joints "fairly"

Handicap $\vec{\alpha} \in \mathbb{Z}^J$ $\longmapsto$ partition of $\binom{n+2}{2}$ among joints

$\binom{n+2}{2} = \dim \mathbb{R}[x,y]_{\leq n}$ (# vanish. cond. assigned)

Hard to compute!

☆ Key idea 3: Existence of good handicap via compactness / smoothing

① Monotonicity $\quad \alpha_p \nearrow \Rightarrow$ # van. cond at p cannot ↘

② Lipschitz continuity $\quad$ small Δ in handicap $\rightsquigarrow$ small Δ in # van cond

③ Bounded domain $\quad$ suffices to consider handicaps with bounded values (else some pt gets no van. cond.)

Putting different planes together

Handicap $\vec{\alpha} \in \mathbb{Z}^J$ assigns an integer to each joint

Separately for each plane F, apply above process to assign vanishing conditions ← (derivative op, point) restricted to F to joints on F

> **A new vanishing lemma** Given $0 \neq g \in \mathbb{R}[x_1, \cdots, x_6]_{\le n}$, $\exists$ joint p, contained in planes F_1, F_2, F_3 (indep & spanning directions) & derivative operator D_1 assigned to p on F_1 (& likewise D_2, D_3) s.t. $D_1 D_2 D_3\, g(p) \neq 0$.

Remark (a) We are assigning only a small # possible (D, p), else claim is trivial

(b) The proof relies on (D,p)'s coming from the procedure earlier

By parameter counting,

linear constraints

$$\sum_{\text{joints } p} \underbrace{(\#\, D_1 @ p \text{ on } F_1)(\#\, D_2 @ p \text{ on } F_2)(\#\, D_3 @ p \text{ on } F_3)}_{f(\vec{\alpha}, p)} \geqslant \dim \mathbb{R}[x_1, \dots, x_6]_{\le n} = \binom{n+6}{6}$$

By compactness/smoothing, considering the handicap $\vec{\alpha}$ that minimizes

$$\max_p f(\vec{\alpha}, p) - \min_p f(\vec{\alpha}, p)$$

we deduce that $\exists \vec{\alpha}$ s.t. $= o(n^6)$

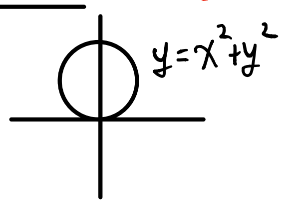
Total $\binom{n+2}{2}$ vanishing cond. assigned to each plane

Putting together + AM-GM $\Rightarrow$ joints of flats theorem $\square$

Joints of varieties

Flats : higher order directional directives along a flat

Varieties : derivatives in local coordinates

e.g. 

On the circle,

$$y = x^2+y^2$$
$$= x^2 + (x^2+y^2)^2$$
$$= x^2 + (x^2+(x^2+y^2)^2)^2 = \cdots$$
$$= x^2 + x^4 + 2x^6 + \cdots$$

Power series in local coord x

$\boxed{\text{completion}}$

2^{nd} order derivative operator at the origin is $\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial y}$ (not $\frac{\partial^2}{\partial x^2}$)
so that evaluations give linear functional on the space of regular functions

Extension to arbitrary fields $\mathbb{F}$

When differentiating, we only care about coeff extraction

Hasse derivatives (formal algebraic derivatives)

Question Other applications of this variant of polynomial method for higher dim objects?