Geometry and the Kato square root problem

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In the 1960’s, Kato considered the following abstract evolution equation

\[ \partial_t u(t) + A(t)u(t) = f(t), \quad t \in [0, T]. \]

on a Hilbert space \( \mathcal{H} \).
History of the problem

In the 1960’s, Kato considered the following abstract evolution equation

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on a Hilbert space \( \mathcal{H} \).

This has a unique strict solution \( u = u(t) \) if

\[ \mathcal{D}(A(t)^\alpha) = \text{const} \]

for some \( 0 < \alpha \leq 1 \) and \( A(t) \) and \( f(t) \) satisfy certain smoothness conditions.
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(ii) $J_t[u, u] \in S_{\omega+} = \{ \zeta \in \mathbb{C} : |\arg \zeta| \leq \omega \} \cup \{0\}$, and
(iii) $\mathcal{W}$ is complete under the norm

$$\|u\|_{\mathcal{W}}^2 = \|u\|^2 + \text{Re} \ J_t[u, u].$$
$T : \mathcal{D}(T) \rightarrow \mathcal{H}$ is called $\omega$-accretive if

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(iii) $\sigma(T) \subset S_{\omega^+}$.

A 0-accretive operator is non-negative and self-adjoint.
Let $A(t) : D(A(t)) \to \mathcal{H}$ be defined as the operator with largest domain such that

$$J_t[u, v] = \langle A(t)u, v \rangle \quad u \in D(A(t)), \ v \in \mathcal{W}.$$
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The theorem of Lax-Milgram guarantees that

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In 1962, Kato showed in [Kato] that for $0 \leq \alpha < 1/2$ and $0 \leq \omega \leq \pi/2$,

$$D(A(t)^\alpha) = D(A(t)^{\ast \alpha}) = D = \text{const}, \text{ and}$$

$$\|A(t)^\alpha u\| \simeq \|A(t)^{\ast \alpha} u\|, \quad u \in D. \quad (K_\alpha)$$
Let $A(t) : \mathcal{D}(A(t)) \to \mathcal{H}$ be defined as the operator with largest domain such that

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$$\mathcal{D}(A(t)^\alpha) = \mathcal{D}(A(t)^{\ast \alpha}) = \mathcal{D} = \text{const}, \quad \text{and}$$

$$\|A(t)^\alpha u\| \simeq \|A(t)^{\ast \alpha} u\|, \quad u \in \mathcal{D}. \tag{K_\alpha}$$

Counterexamples were known for $\alpha > 1/2$ and for $\alpha = 1/2$ when $\omega = \pi/2$. 

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Geometry and the Kato problem
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$$\| \partial_t \sqrt{A(t)}u \| \lesssim \| u \|$$

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In 1972, McIntosh provided a counter example in [Mc72] demonstrating that (K1) is false in such generality.

In 1982, McIntosh showed that (K2) also did not hold in general in [Mc82].
The *Kato square root problem* then became the following.

\[
J[u,v] = \langle A \nabla u, \nabla v \rangle, \quad u,v \in W^{1,2}(\mathbb{R}^n),
\]

where \(A \in L_\infty\) is a pointwise matrix multiplication operator satisfying

\[
\Re \langle J[u,u] \rangle \geq \kappa \| \nabla u \|,
\]

for some \(\kappa > 0\).

Under these conditions, is it true that

\[
D(\sqrt{\text{div} A \nabla}) = W^{1,2}(\mathbb{R}^n)
\]

\[
\| \sqrt{\text{div} A \nabla} u \| \approx \| \nabla u \|.
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(K1)

This was answered in the positive in 2002 by Pascal Auscher, Steve Hofmann, Michael Lacey, Alan McIntosh and Phillipe Tchamitchian in [AHLMcT].
The Kato square root problem then became the following. Set

\[ J[u, v] = \langle A \nabla u, \nabla v \rangle \quad u, v \in W^{1,2}(\mathbb{R}^n), \]

where \( A \in L_\infty \) is a pointwise matrix multiplication operator satisfying the following ellipticity condition:

\[ \text{Re} J[u, u] \geq \kappa \| \nabla u \|, \quad \text{for some } \kappa > 0. \]

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Let $\mathcal{M}$ be a smooth, complete Riemannian manifold with metric $g$, Levi-Civita connection $\nabla$, and volume measure $\mu_g$. 
Kato square root problem for functions and forms

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Write $\text{div} g = -\nabla^*$ in $L^2$. 
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Let $\Omega(\mathcal{M})$ denote the algebra of differential forms over $\mathcal{M}$. 
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Let $d$ be the exterior derivative as an operator on $L^2(\Omega(\mathcal{M}))$ and $d^*$ its adjoint, both of which are nilpotent operators.
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The Hodge-Dirac operator is then the self-adjoint operator $D = d + d^*$.
Kato square root problem for functions and forms

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Let $S = (I, \nabla)$. 
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Assume \( a \in L^\infty(\mathcal{M}) \) and
\[
A = (A_{ij}) \in L^\infty(\mathcal{M}, L(L^2(\mathcal{M}) \oplus L^2(T^*\mathcal{M}))).
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Consider the following second order differential operator \( L_A : D(L_A) \subset L^2(\mathcal{M}) \to L^2(\mathcal{M}) \) defined by:

\[
L_A u = aS^* ASu = -a \text{ div}(A_{11} \nabla u) - a \text{ div}(A_{10} u) + aA_{01} \nabla u + aA_{00} u.
\]
Let $S = (I, \nabla)$.

Assume $a \in L^\infty(\mathcal{M})$ and $A = (A_{ij}) \in L^\infty(\mathcal{M}, \mathcal{L}(L^2(\mathcal{M}) \oplus L^2(T^*\mathcal{M})))$.

Consider the following second order differential operator $L_A : \mathcal{D}(L_A) \subset L^2(\mathcal{M}) \to L^2(\mathcal{M})$ defined by:

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The Kato square root problem for functions is then to determine:

$\mathcal{D}(\sqrt{L_A}) = W^{1,2}(\mathcal{M})$ and $\|\sqrt{L_A}u\| \simeq \|\nabla u\| + \|u\| = \|u\|_{W^{1,2}}$ for all $u \in W^{1,2}(\mathcal{M})$. 
For an invertible $A \in L^\infty(\mathcal{L}(\Omega(M)))$, we consider perturbing $D$ to obtain the operator $D_A = d + A^{-1}d^*A$. 
For an invertible \( A \in L^\infty(\mathcal{L}(\Omega(\mathcal{M}))) \), we consider perturbing \( D \) to obtain the operator \( D_A = d + A^{-1}d^* A \).

The Kato square root problem for forms is then to determine the following whenever \( 0 \neq \beta \in \mathbb{C} \):

\[
\mathcal{D}(\sqrt{D_A^2 + |\beta|^2}) = \mathcal{D}(D_A) = \mathcal{D}(d) \cap \mathcal{D}(d^* A) \quad \text{and} \quad \| \sqrt{D_A^2 + |\beta|^2} u \| \simeq \| D_A u \| + \| u \|.
\]
Axelsson (Rosén)-Keith-McIntosh framework

\((H1)\) The operator \(\Gamma : \mathcal{D}(\Gamma) \subset \mathcal{H} \to \mathcal{H}\) is a closed, densely-defined and nilpotent operator, by which we mean \(\mathcal{R}(\Gamma) \subset \mathcal{N}(\Gamma)\),
Axelsson (Rosén)-Keith-McIntosh framework

(H1) The operator $\Gamma : \mathcal{D}(\Gamma) \subset \mathcal{H} \to \mathcal{H}$ is a closed, densely-defined and nilpotent operator, by which we mean $\mathcal{R}(\Gamma) \subset \mathcal{N}(\Gamma)$,

(H2) $B_1, B_2 \in \mathcal{L}(\mathcal{H})$ and there exist $\kappa_1, \kappa_2 > 0$ satisfying the accretivity conditions

\[ \Re \langle B_1 u, u \rangle \geq \kappa_1 \| u \|^2 \text{ and } \Re \langle B_2 v, v \rangle \geq \kappa_2 \| v \|^2, \]

for $u \in \mathcal{R}(\Gamma^*)$ and $v \in \mathcal{R}(\Gamma)$, and
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(H3) $B_1 B_2 \mathcal{R}(\Gamma) \subset \mathcal{N}(\Gamma)$ and $B_2 B_1 \mathcal{R}(\Gamma^*) \subset \mathcal{N}(\Gamma^*)$. 

Let us now define $\Pi = \Gamma + B_1 \Gamma^* B_2$ with domain $D(\Pi) = \mathcal{D}(\Gamma) \cap D(B_1 \Gamma^* B_2)$. 

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Axelsson (Rosén)-Keith-McIntosh framework

(H1) The operator $\Gamma : D(\Gamma) \subset \mathcal{H} \to \mathcal{H}$ is a closed, densely-defined and nilpotent operator, by which we mean $R(\Gamma) \subset N(\Gamma)$.

(H2) $B_1, B_2 \in \mathcal{L}(\mathcal{H})$ and there exist $\kappa_1, \kappa_2 > 0$ satisfying the accretivity conditions

$$\Re \langle B_1 u, u \rangle \geq \kappa_1 \|u\|^2 \text{ and } \Re \langle B_2 v, v \rangle \geq \kappa_2 \|v\|^2,$$

for $u \in R(\Gamma^*)$ and $v \in R(\Gamma)$, and

(H3) $B_1 B_2 R(\Gamma) \subset N(\Gamma)$ and $B_2 B_1 R(\Gamma^*) \subset N(\Gamma^*)$.

Let us now define $\Pi_B = \Gamma + B_1 \Gamma^* B_2$ with domain $D(\Pi_B) = D(\Gamma) \cap D(B_1 \Gamma^* B_2)$. 
Quadratic estimates

To say that $\Pi_B$ satisfies \textit{quadratic estimates} means that

$$
\int_0^\infty \| t\Pi_B (I + t^2 \Pi_B^2)^{-1} u \|^2 \, \frac{dt}{t} \simeq \| u \|^2, \quad (Q)
$$

for all $u \in \mathcal{R}(\Pi_B)$. 

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\]

for all $u \in \mathcal{R}(\Pi_B)$.

This implies that

\[
\mathcal{D}(\sqrt{\Pi_B^2}) = \mathcal{D}(\Pi_B) = \mathcal{D}(\Gamma) \cap \mathcal{D}(\Gamma^* B_2)
\]

\[
\| \sqrt{\Pi_B^2} u \| \simeq \| \Pi_B u \| \simeq \| \Gamma u \| + \| \Gamma^* B_2 u \|.
\]
The main theorem on manifolds

**Theorem (B.-Mc, 2012)**

Let $\mathcal{M}$ be a smooth, complete Riemannian manifold with $|\text{Ric}| \leq C$ and $\text{inj}(\mathcal{M}) \geq \kappa > 0$. Suppose the following ellipticity condition holds: there exists $\kappa_1, \kappa_2 > 0$ such that

\[
\Re \langle av, v \rangle \geq \kappa_1 \|v\|^2 \\
\Re \langle ASu, Su \rangle \geq \kappa_2 \|u\|^2_{W^{1,2}}
\]

for $v \in L^2(\mathcal{M})$ and $u \in W^{1,2}(\mathcal{M})$. Then,

$\mathcal{D}(\sqrt{L_A}) = \mathcal{D}(\nabla) = W^{1,2}(\mathcal{M})$ and

$\|\sqrt{L_A}u\| \simeq \|\nabla u\| + \|u\| = \|u\|_{W^{1,2}}$ for all $u \in W^{1,2}(\mathcal{M})$. 
Lipschitz estimates

Since we allow the coefficients $a$ and $A$ to be complex, we obtain the following stability result as a consequence:

**Theorem (B.-Mc, 2012)**

Let $\mathcal{M}$ be a smooth, complete Riemannian manifold with $|\text{Ric}| \leq C$ and $\text{inj}(\mathcal{M}) \geq \kappa > 0$. Suppose that there exist $\kappa_1, \kappa_2 > 0$ such that

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\text{Re} \langle av, v \rangle \geq \kappa_1 \|v\|^2 \quad \text{and} \quad \text{Re} \langle ASu, Su \rangle \geq \kappa_2 \|u\|_{W^{1,2}}^2
\]

for $v \in L^2(\mathcal{M})$ and $u \in W^{1,2}(\mathcal{M})$. Then for every $\eta_i < \kappa_i$, whenever $\|\tilde{a}\|_{\infty} \leq \eta_1$, $\|\tilde{A}\|_{\infty} \leq \eta_2$, the estimate

\[
\|\sqrt{L_A} u - \sqrt{L_{A+\tilde{A}}} u\| \lesssim (\|\tilde{a}\|_{\infty} + \|\tilde{A}\|_{\infty}) \|u\|_{W^{1,2}}
\]

holds for all $u \in W^{1,2}(\mathcal{M})$. The implicit constant depends in particular on $A$, $a$ and $\eta_i$. 
Let \( \{ \theta^i \} \) be an orthonormal frame at \( x \) for \( \Omega^1(M) = T^*M \).
Curvature endomorphism for forms

Let $\{\theta^i\}$ be an orthonormal frame at $x$ for $\Omega^1(M) = \mathsf{T}^{\ast}M$.

Denote the components of the curvature tensor in this frame by $R_{ijkl}$.

This can be seen as an extension of Ricci curvature for forms, since $g(R_{\omega},\eta) = \text{Ric}(\omega^{\flat},\eta^{\flat})$ whenever $\omega,\eta \in \Omega^1(M)$ and where $\flat: \mathsf{T}^{\ast}M \to \mathsf{T}M$ is the flat isomorphism through the metric $g$. 

The Weitzenb"{o}ck formula then asserts that $D^2 = \text{tr}_1 + \nabla^2 + R$. 


Curvature endomorphism for forms

Let \( \{ \theta^i \} \) be an orthonormal frame at \( x \) for \( \Omega^1(\mathcal{M}) = T^*\mathcal{M} \).

Denote the components of the curvature tensor in this frame by \( R_{mijkl} \). The curvature endomorphism is then the operator

\[
R \omega = R_{mijkl} \theta^i \wedge (\theta^j \wedge (\theta^k \wedge (\theta^l \wedge \omega)))
\]

for \( \omega \in \Omega_x(\mathcal{M}) \).
Curvature endomorphism for forms

Let \( \{ \theta^i \} \) be an orthonormal frame at \( x \) for \( \Omega^1(M) = T^*M \).

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for \( \omega \in \Omega^1_x(M) \).

This can be seen as an extension of Ricci curvature for forms, since

\[
g(R_\omega, \eta) = \text{Ric}(\omega^b, \eta^b)
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whenever \( \omega, \eta \in \Omega^1_x(M) \) and where \( b : T^*M \to TM \) is the flat isomorphism through the metric \( g \).
Curvature endomorphism for forms

Let \( \{ \theta^i \} \) be an orthonormal frame at \( x \) for \( \Omega^1(M) = \mathbb{T}^*M \).

Denote the components of the curvature tensor in this frame by \( R_{ijkl} \). The curvature endomorphism is then the operator

\[
R \omega = R_{ijkl} \theta^i \wedge (\theta^j \lrcorner (\theta^k \wedge (\theta^l \lrcorner \omega)))
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for \( \omega \in \Omega_x(M) \).

This can be seen as an extension of Ricci curvature for forms, since

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g(R \omega, \eta) = \text{Ric}(\omega^b, \eta^b)
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whenever \( \omega, \eta \in \Omega^1_x(M) \) and where \( b : \mathbb{T}^*M \to \mathbb{T}M \) is the flat isomorphism through the metric \( g \).

The Weitzenböck formula then asserts that

\[
D^2 = \text{tr}_{12} \nabla^2 + \mathbb{R}.
\]
Theorem (B., 2012)

Let $\mathcal{M}$ be a smooth, complete Riemannian manifold and let $\beta \in \mathbb{C} \setminus \{0\}$. Suppose there exist $\eta, \kappa > 0$ such that $|\text{Ric}| \leq \eta$ and $\text{inj}(\mathcal{M}) \geq \kappa$. Furthermore, suppose there is a $\zeta \in \mathbb{R}$ satisfying $g(\text{R} u, u) \geq \zeta |u|^2$, for $u \in \Omega_\times(\mathcal{M})$ and $A \in \mathcal{L}^\infty(\mathcal{L}(\Omega(\mathcal{M})))$ and $\kappa_1 > 0$ satisfying

$$\text{Re} \langle A u, u \rangle \geq \kappa_1 \|u\|^2.$$

Then, $\mathcal{D}(\sqrt{D_A^2 + |\beta|^2}) = \mathcal{D}(D_A) = \mathcal{D}(d) \cap \mathcal{D}(d^* A)$ and

$$\|\sqrt{D_A^2 + |\beta|^2} u\| \simeq \|D_A u\| + \|u\|.$$
The Kato problem for functions are captured in the AKM framework on letting \( \mathcal{H} = L^2(\mathcal{M}) \oplus (L^2(\mathcal{M}) \oplus L^2(T^*\mathcal{M})) \) and letting

\[
\Gamma = \begin{pmatrix} 0 & 0 \\ S & 0 \end{pmatrix}, \quad \Gamma^* = \begin{pmatrix} 0 & S^* \\ 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}.
\]
The Kato problem for functions are captured in the AKM framework on letting $\mathcal{H} = L^2(\mathcal{M}) \oplus (L^2(\mathcal{M}) \oplus L^2(T^*\mathcal{M}))$ and letting

$$\Gamma = \begin{pmatrix} 0 & 0 \\ S & 0 \end{pmatrix}, \quad \Gamma^* = \begin{pmatrix} 0 & S^* \\ 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}.$$

For the case of forms, the setup takes the form, $\mathcal{H} = L^2(\Omega(\mathcal{M})) \oplus L^2(\Omega(\mathcal{M}))$ and

$$\Gamma = \begin{pmatrix} d & 0 \\ \beta & -d \end{pmatrix}, \quad \Gamma^* = \begin{pmatrix} \delta & \bar{\beta} \\ 0 & -\delta \end{pmatrix}, \quad B_1 = \begin{pmatrix} A^{-1} & 0 \\ 0 & A^{-1} \end{pmatrix}, \quad B_2 = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}.$$
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Geometry enters the picture precisely in the harmonic analysis. We need to perform harmonic analysis on vector fields, not just functions. One can show that this is not artificial - the Kato problem on functions immediately provides a solution to the dual problem on vector fields.
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- A dyadic decomposition of the space
- A notion of averaging (in an integral sense)
- Poincaré inequality - on both functions and vector fields
- Control of $\nabla^2$ in terms of $\Delta$. 
Rough metrics

**Definition (Rough metric)**

Let $g$ be a $(2, 0)$ symmetric tensor field with measurable coefficients and that for each $x \in \mathcal{M}$, there is some chart $(U, \psi)$ near $x$ and a constant $C \geq 1$ such that

$$C^{-1} |u|_{\psi^* \delta(y)} \leq |u|_{g(y)} \leq C |u|_{\psi^* \delta(y)},$$

for almost-every $y \in U$ and where $\delta$ is the Euclidean metric in $\psi(U)$. Then we say that $g$ is a rough metric, and such a chart $(U, \psi)$ is said to satisfy the *local comparability condition*. 

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**Metric perturbations**

**Definition**

We say that two rough metrics $g$ and $\tilde{g}$ are $C$-close if

$$C^{-1} |u|_{\tilde{g}(x)} \leq |u|_{g(x)} \leq C |u|_{\tilde{g}(x)}$$

for almost-every $x \in M$ where $C \geq 1$. Two such metrics are said to be $C$-close everywhere if this inequality holds for every $x \in M$. 
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We also say that $g$ and $\tilde{g}$ are close if there exists some $C \geq 1$ for which they are $C$-close.

For two continuous metrics, $C$-close and $C$-close everywhere coincide.
Proposition

Let $g$ and $\tilde{g}$ be two rough metrics that are $C$-close. Then, there exists $B \in \Gamma(T^*\mathcal{M} \otimes T\mathcal{M})$ such that it is symmetric, almost-everywhere positive and invertible, and

$$\tilde{g}_x(B(x)u, v) = g_x(u, v)$$

for almost-every $x \in \mathcal{M}$. Furthermore, for almost-every $x \in \mathcal{M}$,

$$C^{-2} |u|_{\tilde{g}(x)} \leq |B(x)u|_{\tilde{g}(x)} \leq C^2 |u|_{\tilde{g}(x)},$$

and the same inequality with $\tilde{g}$ and $g$ interchanged. If $\tilde{g} \in C^k$ and $g \in C^l$ (with $k, l \geq 0$), then the properties of $B$ are valid for all $x \in \mathcal{M}$ and $B \in C^{\min\{k,l\}}(T^*\mathcal{M} \otimes T\mathcal{M})$. 
The measure $\mu_g(x) = \theta(x) \, d\mu_\tilde{g}(x)$, where $\theta(x) = \sqrt{\det B(x)}$. 
The measure $\mu_g(x) = \theta(x) \, d\mu_{\tilde{g}}(x)$, where $\theta(x) = \sqrt{\det B(x)}$. Consequently,

(i) whenever $p \in [1, \infty)$, $L^p(\mathcal{T}^{(r,s)} \mathcal{M}, g) = L^p(\mathcal{T}^{(r,s)} \mathcal{M}, \tilde{g})$ with

$$C^{-\left(r+s+\frac{n}{2p}\right)} \|u\|_{p,\tilde{g}} \leq \|u\|_{p,g} \leq C^{r+s+\frac{n}{2p}} \|u\|_{p,\tilde{g}},$$
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(ii) for $p = \infty$, $L^\infty(\mathcal{T}^{(r,s)} M, g) = L^\infty(\mathcal{T}^{(r,s)} M, \tilde{g})$ with

$$C^{-\left(r+s\right)} \|u\|_{\infty,\tilde{g}} \leq \|u\|_{\infty,g} \leq C^{r+s} \|u\|_{\infty,\tilde{g}},$$
(iii) the Sobolev spaces $W^{1,p}(\mathcal{M}, g) = W^{1,p}(\mathcal{M}, \tilde{g})$ and $W^{1,p}_0(\mathcal{M}, g) = W^{1,p}_0(\mathcal{M}, \tilde{g})$ with

$$C^{-\left(1+\frac{n}{2p}\right)} \|u\|_{W^{1,p}, \tilde{g}} \leq \|u\|_{W^{1,p}, g} \leq C^{1+\frac{n}{2p}} \|u\|_{W^{1,p}, \tilde{g}},$$
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(v) the divergence operators satisfy $\text{div}_{D,g} = \theta^{-1} \text{div}_{D,\tilde{g}} \theta B$ and $\text{div}_{N,g} = \theta^{-1} \text{div}_{N,\tilde{g}} \theta B$. 
Case of functions

Theorem (B, 2014)

Let \( \tilde{g} \) be a smooth, complete metric and suppose that there exists \( \kappa > 0 \) and \( \eta > 0 \) such that

(i) \( \text{inj}(\mathcal{M}, \tilde{g}) \geq \kappa \) and,

(ii) \( |\text{Ric}(\tilde{g})| \leq \eta \).

Then, for any rough metric \( g \) that is close, the Kato square root problem for functions has a solution on \( (\mathcal{M}, g) \).
Case of forms

**Theorem (B, 2014)**

Let $\mathfrak{g}$ be a rough metric close to $\tilde{g}$, a smooth, complete metric, and suppose that:

(i) there exists $\kappa > 0$ such that $\text{inj} (M, \tilde{g}) \geq \kappa$,
(ii) there exists $\eta > 0$ such that $|\text{Ric}(\tilde{g})| \leq \eta$, and
(iii) there exists $\zeta \in \mathbb{R}$ such that $\tilde{g}(R\omega, \omega) \geq \zeta |\omega|_{\tilde{g}}^2$.

Then, the Kato square root problem for forms has a solution on $(M, \mathfrak{g})$. 
Compact manifolds with rough metrics

Theorem (B, 2014)

Let $M$ be a smooth, compact manifold and $g$ a rough metric. Then, the Kato square root problem (on functions and forms, respectively) has a solution.
Cones and induced metrics

Let $C_{r,h}^n$ be the $n$-cone of height $h > 0$ and radius $r > 0$. 
Cones and induced metrics

Let $C^n_{r,h}$ be the $n$-cone of height $h > 0$ and radius $r > 0$. The cone can be realised as the image of the graph function

$$F_{r,h}(x) = \left(x, h \left(1 - \frac{|x|}{r}\right)\right).$$
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$$F_{r,h}(x) = \left( x, h \left( 1 - \frac{|x|}{r} \right) \right).$$

Let $U$ be an open set in $\mathbb{R}^n$ such that $B_r(0) \subset U$. Then, define $G_{r,h} : U \to \mathbb{R}^{n+1}$ as the map $F_{r,h}$ whenever $x \in B_r(0)$ and $(x,0)$ otherwise.
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Then we obtain that the map $G_{r,h}$ satisfies

$$|x - y| \leq |G_{r,h}(x) - G_{r,h}(y)| \leq \sqrt{1 + (hr^{-1})^2} |x - y|.$$
Let $\gamma : I \to U$ be a smooth curve such that $\gamma(0) \notin \{0\} \cup \partial B_r(0)$. Then,

$$|\gamma'(0)| \leq |(G_{r,h} \circ \gamma)'(0)| \leq \sqrt{1 + \frac{h^2}{r^2}} |\gamma'(0)|.$$ 

Moreover, for $u \in T_x U$, $x \notin \{0\} \cup \partial B_r(0)$ (and in particular for almost-every $x$),

$$|u|_\delta \leq |u|_g \leq \sqrt{1 + \frac{h^2}{r^2}} |u|_\delta,$$

where $\delta$ is the usual inner product on $U$ induced by $\mathbb{R}^n$. 

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Proposition

Let $\gamma : I \to U$ be a smooth curve such that $\gamma(0) \not\in \{0\} \cup \partial B_r(0)$. Then,

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where $\delta$ is the usual inner product on $U$ induced by $\mathbb{R}^n$.

A particular consequence is that the metrics $g = G_{r,h}^* \delta_{\mathbb{R}^{n+1}}$ and $\delta_{\mathbb{R}^n}$ are $\sqrt{1 + (hr^{-1})^2}$-close on $U$. 
Lemma

Given \( \varepsilon > 0 \), there exists two points \( x, x' \) and distinct minimising smooth geodesics \( \gamma_{1,\varepsilon} \) and \( \gamma_{2,\varepsilon} \) between \( x \) and \( x' \) of length \( \varepsilon \)

Furthermore, there are two constants \( C_{1,r,h,\varepsilon}, C_{2,r,h,\varepsilon} > 0 \) depending on \( h, r \) and \( \varepsilon \) such that the geodesics \( \gamma_{1,\varepsilon} \) and \( \gamma_{2,\varepsilon} \) are contained in \( G_{r,h}(A_{\varepsilon}) \) where \( A_{\varepsilon} \) is the Euclidean annulus

\[
\{ x \in B_r(0) : C_{1,r,h,\varepsilon} < |x| < C_{2,r,h,\varepsilon} \}.
\]
Theorem (B., 2014)

For any $C > 1$, there exists a smooth metric $g$ which is $C$-close to the Euclidean metric $\delta$ for which $\text{inj}(\mathbb{R}^2, g) = 0$. Furthermore, the Kato square root problem for functions can be solved for $(\mathbb{R}^2, g)$ under the.
In higher dimensions, we obtain a similar result since the 2-dimensional cone can be realised as a totally geodesic submanifold.
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**Theorem (B., 2014)**

Let $\mathcal{M}$ be a smooth manifold of dimension at least 2 and $g$ a continuous metric. Given $C > 1$, and a point $x_0 \in \mathcal{M}$, there exists a rough metric $h$ such that:

(i) it induces a length structure and the metric $d_g$ preserves the topology of $\mathcal{M}$,
(ii) it is smooth everywhere except $x_0$,
(iii) the geodesics through $x_0$ are Lipschitz,
(iv) it is $C$-close to $g$,
(v) $\text{inj}(\mathcal{M} \setminus \{x_0\}, h) = 0$. 

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