CHAPTER 1

Some basic elements of complex analysis

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1. Complex notation and holomorphic functions

Let x and y be the standard coordinates in $\mathbb{C} \simeq \mathbb{R}^2$ and consider the complex-valued functions $z = x + iy$ and $\bar{z} = x - iy$. The exterior derivative d extends to a complex-linear operator on complex-valued functions and forms. In particular, $dz = dx + idy$ and $d\bar{z} = dx - idy$, and an easy computation reveals that

$$\frac{i}{2} dz \wedge d\bar{z} = dx \wedge dy = dV(z),$$

the area form on $\mathbb{C}$. In particular, dz and $d\bar{z}$ are linearly independent so any 1-form ξ can be written $\xi = \alpha dz + \beta d\bar{z}$. In particular, if f is a differentiable function (at a given point), then we have

$$df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z} = \partial f + \bar{\partial} f,$$

with

$$\frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big).$$

It is easily checked that the limit

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists if and only if f is differentiable at a and $(\partial f / \partial \bar{z})(a) = 0$. In that case, then $f'(a) = (\partial f / \partial z)(a)$. A function $f \in C^1(\Omega)$ is said to be holomorphic, $f \in \mathcal{O}(\Omega)$, if $\bar{\partial} f = 0$.

In the same way, in $\mathbb{C}^n$ we have the standard coordinates $z_k = x_k + iy_k$, and

$$\left(\frac{i}{2}\right)^n dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge dz_n \wedge d\bar{z}_n = dx_1 \wedge dy_1 \wedge \ldots \wedge dx_n \wedge dy_n = dV(z) \tag{1.1}$$

is the volume form in $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. Therefore, cf., Exercise 1, the forms $dz_j, d\bar{z}_k$ form a basis for the cotangent space at each point, and hence any complex-valued form ξ can be written

$$\xi = \sum_{I,K}{}' \xi_{I,K} dz_I \wedge d\bar{z}_K,$$

with the usual multi-index notation, so that $dz_I = dz_{I_1} \wedge dz_{I_2} \wedge \ldots \wedge dz_{I_{|I|}}$, etc, and the prime indicates that the summation is performed over increasing multiindices. We say that ξ has *bidegree* (p, q) if it can be written with $|I| = p$ and $|K| = q$. If f is a function, we have

$$df = \sum_k \frac{\partial f}{\partial z_k} dz_k + \sum_k \frac{\partial f}{\partial \bar{z}_k} d\bar{z}_k = \partial f + \bar\partial f,$$

and in general

$$d\xi = \sum_{I,K}{}' d\xi_{I,K} \wedge dz_I \wedge d\bar{z}_K = \sum_{I,K} \partial \xi_{I,K} \wedge dz_I \wedge d\bar{z}_K + \sum_{I,K} \bar\partial \xi_{I,K} \wedge dz_I \wedge d\bar{z}_K = \partial\xi + \bar\partial\xi.$$

Notice that since $0 = d^2 = (\partial + \bar\partial)^2 = \partial^2 + \bar\partial\partial + \partial\bar\partial + \bar\partial^2$, for degree reasons,

$$\partial^2 = 0, \quad \bar\partial^2 = 0, \quad \bar\partial\partial + \partial\bar\partial = 0.$$

As in the case $n = 1$ we say that a function $f \in C^1(\Omega)$, $\Omega \subset \mathbb{C}^n$, is *holomorphic*, $f \in \mathcal{O}(\Omega)$, if $\bar\partial f = 0$. Since $\bar\partial$ is linear, clearly $\mathcal{O}(\Omega)$ is a linear space. Moreover, for functions f, g,

$$\partial(fg) = f\partial g + g\partial f, \quad \bar\partial(fg) = f\bar\partial g + g\bar\partial f,$$

so $\mathcal{O}(\Omega)$ is also a ring, i.e., closed under multiplication. If $H \subset \mathbb{C}^n$ is any set, then $f \in \mathcal{O}(H)$ means that f is holomorphic in some open neighborhood of H.

In the same way, a smooth $(p, 0)$-form f is *holomorphic* if $\bar\partial f = 0$. If $f = \sum'_{|I|=p} f_I dz_I$ this just means that each f_I is holomorphic.

We say that $f \colon \Omega \to \mathbb{C}^m$ is holomorphic, $f \in \mathcal{O}(\Omega, \mathbb{C}^m)$, if each of its coordinate functions are holomorphic. If this holds and if w

are coordinates on $\mathbb{C}^m$, then $dw_j = df_j = \partial f_j$ and $d\bar{w}_j = d\bar{f}_j = \bar{\partial}\bar{f}_j$. Therefore, if ξ is a (p,q)-form in a neighborhood of the image of f, then the pull-back $f^*\xi$ is again a (p,q)-form in Ω. In particular, if g is holomorphic, then $d(g\circ f) = d(f^*g) = f^*(dg) = f^*(\partial g)$, which shows that $d(g\circ f)$ is a $(1,0)$-form and thus $g\circ f$ is holomorphic.

Now suppose that $\Omega \subset \mathbb{C}^n$ and $f \in \mathcal{O}(\Omega, \mathbb{C}^n)$. If $w = f(z)$, then

$$dw_1 \wedge \ldots \wedge dw_n = \det \frac{\partial f}{\partial z} dz_1 \wedge \ldots \wedge dz_n,$$

where

$$\frac{\partial f}{\partial z} = \Big[\frac{\partial f_j}{\partial z_k}\Big]_{jk}.$$

Let Df be the real-linear derivative of f (at some fixed point). In view of (1.1) we then have

$$dV(w) = \Big(\frac{i}{2}\Big)^n dw_1 \wedge d\bar{w}_1 \wedge \ldots \wedge dw_n \wedge d\bar{w}_n =$$
$$\Big|\det \frac{\partial f}{\partial z}\Big|^2 \Big(\frac{i}{2}\Big)^n dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge dz_n \wedge d\bar{z}_n = \Big|\det \frac{\partial f}{\partial z}\Big|^2 dV(z),$$

which means that $\det Df = |\det(\partial f/\partial z)|^2$; in particular $\det Df$ is non-negative. By the inverse function theorem, f is locally a diffeomorphism if (and only if) $df_1 \wedge \ldots \wedge df_n \neq 0$. In that case dz_j are linear combinations of dw_k which implies that the inverse is holomorphic as well. We say that f is (locally) biholomorphic; since then $\det Df > 0$, it preserves orientation.

For an open set Ω we make $\mathcal{O}(\Omega)$ into a Frechet space with the seminorms $\|f\|_K = \sup_K |f|$ for compact subsets K, so that a sequence f_j tends to 0 if and only if $f_j \to 0$ uniformly on each K. As usual $\mathcal{E}(\Omega)$, the Frechet space of smooth functions in Ω, has the topology so that $f_k \to 0$ if and only if on each K all derivatives of f_k tend to 0 uniformly. We will see later, in Section 3, that in fact $\mathcal{O}(\Omega)$ is a closed subspace of $\mathcal{E}(\Omega)$ with the induced topology. If K is a compact set we let $\mathcal{O}(K)$ be the space of functions that are holomorphic in some open neighborhood of K, and $f_k \to 0$ if(f) all f_k are holomorphic in a fixed open neighborhood and tends to 0 there.

We say that a differentiable manifold X is a *complex manifold* if it is covered by coordinate neighborhoods (z, U) such that U via z is diffeomorphic to an open set in $\mathbb{C}^n$, and all the induced diffeomorphisms are biholomorphic.

Locally, thus X has the complex structure of $\mathbb{C}^n$ and hence all notions like holomorphic function, (p,q)-form etc are well-defined. Moreover, we have a natural orientation, inherited from the standard orientation on $\mathbb{C}^n$.

A closed subset Y is a *complex submanifold* of dimension p if for each point $z^0 \in Y$ there is a neighborhood U and local holomorphic coordinates z, such that $Y \cap U = \{z \in U;\ z_{p+1} = \cdots = z_n = 0\}$. In that case clearly $z_1, \ldots, z_p$ is a local coordinate system on Y, so Y is a complex manifold itself.

2. The Cauchy formula and some consequences

For fixed $z \in \mathbb{C}$,

$$\omega_{\zeta - z} = \frac{1}{2\pi i}\frac{d\zeta}{\zeta - z}$$

is the Cauchy kernel with pole at z. It is holomorphic in $\mathbb{C} \setminus \{z\}$ and locally integrable in $\mathbb{C}$. We first study the local nature of the Cauchy kernel at the origin.

LEMMA 2.1. *For each C^1-function ϕ in $\mathbb{C}$ with compact support we have*

$$-\int \bar\partial\phi \wedge \frac{d\zeta}{2\pi i\zeta} = \phi(0). \tag{2.1}$$

PROOF. Notice that outside the origin $d(\phi\omega_\zeta) = \bar\partial\phi\wedge\omega_\zeta$. By Stokes' theorem we therefore have

$$-\int_{|\zeta|>\epsilon} \bar\partial\phi \wedge \omega_\zeta = \int_{|\zeta|=\epsilon} \phi\omega_\zeta = \frac{1}{2\pi i}\int_{|\zeta|=\epsilon} f(\zeta)\frac{d\zeta}{\zeta} =$$
$$\frac{1}{2\pi i\epsilon^2}\int_{|\zeta|=\epsilon} f(\zeta)\bar\zeta d\zeta = \frac{1}{2\pi i\epsilon^2}\int_{|\zeta|<\epsilon} [f(\zeta) + \mathcal{O}(|\zeta|)]d\bar\zeta \wedge d\zeta =$$
$$\frac{1}{\pi\epsilon^2}\int_{|\zeta|<\epsilon} (f(\zeta) + \mathcal{O}(|\zeta|))dV(\zeta)$$

which tends to $f(0)$ when $\epsilon \to 0$, $\square$

PROPOSITION 2.2 (Cauchy-Green's formula). *If f is C^1 in Ω and $D \subset \Omega$ is bounded and has smooth boundary (or at least some reasonable regularity, like piecewise C^1) then*

$$f(z) = \int_{\partial D} \omega_{\zeta - z} f + \int_D \omega_{\zeta - z} \wedge \bar\partial f, \quad z \in D. \tag{2.2}$$

As an immediate corollary we have, for a holomorphic function f, the Cauchy formula,

$$f(z) = \int_{\partial D} f(\zeta)\omega_{\zeta-z}, \quad z \in D.$$

This formula is the corner stone in the theory of one complex variable and probably one of the most remarkable formulas in analysis.

Notice that

$$\omega_{\zeta-z} \wedge \bar\partial f = \frac{1}{2\pi i}\frac{\partial f}{\partial \bar z}\frac{dz \wedge d\bar z}{\zeta - z} = -\frac{1}{\pi}\frac{\partial f}{\partial \bar z}\frac{dV(z)}{\zeta - z}.$$

Proof. Notice that (2.2) is just (2.1) if f has compact support in D. Now suppose that f vanishes identically in a neighborhood of the point z. Then $d(\omega_{\zeta-z}f) = -\omega_{\zeta-z} \wedge \bar\partial f$ and hence (2.2) follows from Stokes' theorem. For the general case, let χ be a smooth cutoff function that has compact support in D and is identically 1 in a neighborhood of the point z. Then $f = \chi f + (1-\chi)f = f_1 + f_2$ where f_1 has compact support in D and f_2 vanishes in a neighborhood of z. Since (2.2) is linear in f, the general case now follows. $\square$

See Exercise 7 for an alternative way to obtain (2.2) from (2.1).

Now let f be holomorphic in a neighborhood of the origin in $\mathbb{C}^n$, say in the so-called polydisk $\{z \in \mathbb{C}^n;\ |z_j| \le r_j\}$, and take $\epsilon_j < r_j$. By iterated use of the Cauchy formula we get

$$f(z_1, \dots, z_n) = \int_{|\zeta_1|=\epsilon_1} \cdots \int_{|\zeta_n|=\epsilon_n} f(\zeta)\omega_{\zeta_1-z_1} \wedge \dots \wedge \omega_{\zeta_n-z_n}, \tag{2.3}$$

for z with $|z_j| < \epsilon_j$. If we expand the Cauchy kernels in geometric series, we get the power series representation

$$f(z) = \sum_{|\alpha|\ge 0} c_\alpha z^\alpha, \quad |z_j| < \epsilon_j,$$

where $z^\alpha = z_1^{\alpha_1}\cdots z_n^{\alpha_n}$, and ($\alpha! = \alpha_1!\cdots\alpha_n!$)

$$c_\alpha = \frac{\partial^\alpha f}{\partial z^\alpha}(0)/\alpha!.$$

In particular, we see that if all derivatives of f vanish at a point, then f is identically zero in a neighborhood. Therefore, if $f \in \mathcal{O}(\Omega)$ and A is the set of points in Ω at which all derivatives of f vanish, then A is both open and closed. We therefore have

Proposition 2.3 (Uniqueness theorem). *Assume that $f \in \mathcal{O}(\Omega)$ and Ω is connected. If all derivatives of f vanish at some point in Ω, then $f \equiv 0$ in Ω.*

Since we can differentiate (2.3) with respect to z any number of times at the origin, which is an "arbitrary" point, we find that each holomorphic function is of class C^∞.

We leave it to the reader to prove the maximum principle (it can be reduced to the one-variable case):

Assume that $f \in \mathcal{O}(\Omega)$ and Ω is connected. Then for each interior point a, $|f(a)| < \sup_\Omega |f|$ unless f is constant.

Let f be a (p,q)-form and consider the equation $\bar\partial u = f$. Since $\bar\partial^2 = 0$ a necessary condition for solvability is that $\bar\partial f = 0$. We will see later on that locally this is also sufficient. However, we start with :

PROPOSITION 2.4. *Assume $f = f_1 d\bar z$ is C^k, $k \geq 1$, has compact support in $\mathbb{C}$. Then*

$$u(z) = \int_\zeta \omega_{\zeta - z} \wedge f(\zeta)$$

is in C^k and solves $\bar\partial u = f$.

PROOF. By a linear change of variables,

$$u(z) = \int_\zeta f_1(\zeta + z) d\bar\zeta \wedge \omega_\zeta$$

and it is now clear that we can differentiate the integral k times, so u is (at least) C^k. Moreover,

$$\partial u/\partial \bar z = \int_\zeta (\partial f_1/\partial\bar\zeta)(\zeta + z) d\bar\zeta \wedge \omega_\zeta = \int_\zeta (\partial f(\zeta)/\partial\bar\zeta) d\bar\zeta \wedge \omega_{\zeta - z}$$

which is equal to $f_1(z)$ by (2.1). □

In general, the solution will not have compact support. In fact, if it has, then

$$\int f \wedge dz = \int \bar\partial u \wedge dz = \int d(u dz) = 0$$

by Stokes' theorem, so for a counterexample just take any $f = f_1 d\bar z$ such that f_1 has non-vanishing integral. However, if $n > 1$ we have

PROPOSITION 2.5. *Assume that f is a smooth $\bar\partial$-closed $(0,1)$-form in $\mathbb{C}^n$, $n > 1$, with compact support. Then there is a smooth solution to $\bar\partial u = f$ with compact support.*

Since u is holomorphic in the (interior of the) set where $f = 0$, by the uniqueness theorem, u is identically zero in the unbounded component of the complement of the support of f.

PROOF. For $f = \sum_k f_k d\bar{z}_k$, the condition $\bar\partial f = 0$ means that

$$\frac{\partial f_k}{\partial \bar{z}_\ell} = \frac{\partial f_\ell}{\partial \bar{z}_k}. \tag{2.4}$$

For simplicity we assume $n = 2$. Let

$$u(z_1, z_2) = \int_{\zeta_1} f_1(\zeta_1, z_2)\omega_{\zeta_z - z_1}. \tag{2.5}$$

Then, by Proposition 2.4, $\partial u/\partial \bar{z}_1 = f_1$. However, we also have that

$$\frac{\partial u}{\partial \bar{z}_2} = \int_{\zeta_1} \frac{\partial f_1}{\partial \bar{z}_2}(\zeta_1, z_2)\omega_{\zeta_z - z_1} = \int_{\zeta_1} \frac{\partial f_2}{\partial \bar{\zeta}_1}(\zeta_1, z_2)\omega_{\zeta_z - z_1} = f_2(z_1, z_2),$$

where we have used (2.4) for the second equality and Cauchy-Green's formula for the last equality. Therefore $\bar\partial u = f$. From (2.5) it is obvious that u vanishes if $|z_2|$ is large. Since moreover $\bar\partial u$ vanishes outside some big ball, u is holomorphic outside this ball, and by the uniqueness it follows that u vanishes identically there. $\square$

One of the first striking phenomenon that was noticed in the several-dimensional case is the followin. See Exercise refharex for a real analogue.

PROPOSITION 2.6 (Hartogs' phenomenon). *Let Ω be an open set in $\mathbb{C}^n$, $n > 1$, and let K be a compact subset such that $\Omega \setminus K$ is connected. For each $f \in \mathcal{O}(\Omega \setminus K)$ there is a function $F \in \mathcal{O}(\Omega)$ such that $F = f$ in $\Omega \setminus K$.*

If Ω is connected and $\Omega \setminus K$ is not connected, then it means that K has some "holes"; so extending K to the compact set H by filling out theses holes, we get that $\Omega \setminus H$ is connected.

PROOF. Take a cutoff function χ in Ω that is identically 1 in a neigborhood of K. Then $(1 - \chi)f$ is smooth in Ω and moreover, $g = \bar\partial(1 - \chi)f = -f\bar\partial\chi$ has compact support in $\mathbb{C}^n$ and is $\bar\partial$-closed. Therefore, by (the remark after) Proposition 2.5 we can find a solution to $\bar\partial v = g$ that vanishes in the unbounded component of the complement of the support of χ. However, the boundary of this set belongs to $\Omega \setminus K$ so $F = (1-\chi)f - v$ is a holomorphic function in Ω that coincides with f on some open subset of $\Omega \setminus K$. Since $\Omega \setminus K$ is connected, it follows from the uniqueness theorem that $F = f$ in this whole set. $\square$

In particular it follow that the set where $f \in \mathcal{O}(\Omega)$ vanishes cannot be compact in Ω unless $n = 1$. In one variable, on the other hand, holomorphic functions of course may have isolated zeros and singularities.

PROPOSITION 2.7. *Suppose that f is holomorphic in a neighborhood of $0 \in \mathbb{C}^n$, $f(0) = 0$ but f is not vanishing identically when $z_n = 0$. Then $f = aW$ where $a(0) \neq 0$ and*

$$W(z) = z_n^r + \alpha_{r-1}(z')z_n^{r-1} + \cdots + \alpha_1(z')z_n + \alpha_0,$$

where $z' = (z_1, \ldots, z_{n-1})$ and $\alpha_j(0') = 0$.

PROOF. Se tex GH!! ????????? □

3. The Bochner-Martinelli and Cauchy-Fantappié-Leray formulas

If $s = \sum_1^n s_j d\zeta_j$ is a $(1,0)$-form and ζ is a point, we let

$$s \cdot \zeta = \langle s, \zeta \rangle = \sum_1^n s_j \zeta_j.$$

Let $s = \sum_1^n s_j d\zeta_j$ be a smooth $(1,0)$-form such that $2\pi i s \cdot \zeta = 1$. Such a form always exists outside 0; for instance one can take

$$b = \frac{\partial |\zeta|^2}{2\pi i |\zeta|^2} = \frac{\sum_1^n \bar{\zeta}_j d\zeta_j}{2\pi i |\zeta|^2}.$$

Then

$$0 = \bar{\partial} 1 = \sum_1^n \zeta_j \bar{\partial} s_j,$$

so the 1-forms $\bar{\partial} s_1, \ldots, \bar{\partial} s_n$ are linearly dependent and thus $\bar{\partial} s_1 \wedge \bar{\partial} s_2 \wedge \ldots \bar{\partial} s_n = 0$. Since $s \wedge (\bar{\partial} s)^{n-1}$ has bidegree $(n, n-1)$, therefore

$$d(s \wedge (\bar{\partial} s)^{n-1}) = \bar{\partial}(s \wedge (\bar{\partial} s)^{n-1}) = (\bar{\partial} s)^n = (\sum_1^n \bar{\partial} s_j \wedge d\zeta_j)^n = n! \bar{\partial} s_1 \wedge \ldots \wedge \bar{\partial} s_n \wedge d\zeta_n \wedge \ldots \wedge d\zeta_1 = 0. \tag{3.1}$$

The form $B = b \wedge (\bar{\partial} b)^{n-1}$ is called the Bochner-Martinelli kernel. Notice that

$$B = \frac{1}{(2\pi i)^n} \frac{\partial |\zeta|^2}{|\zeta|^2} \wedge \Big(\bar{\partial} \frac{\partial |\zeta|^2}{|\zeta|^2}\Big)^{n-1} = \frac{1}{(2\pi i)^n} \frac{\partial |\zeta|^2 \wedge (\bar{\partial}\partial |\zeta|^2)^{n-1}}{|\zeta|^{2n}}; \tag{3.2}$$

here we use the fact that $\partial |\zeta|^2 \wedge \partial |\zeta|^2 = 0$. Therefore, $B = \mathcal{O}(|\zeta|^{-2n+1})$ and thus locally integrable. We have the following multivariable analog of (2.1).

LEMMA 3.1. *If ϕ is a C^1-function in $\mathbb{C}^n$ with compact support, then*

$$-\int \bar{\partial} \phi \wedge B = \phi(0). \tag{3.3}$$

Proof. Outside the origin $d(\phi B) = \bar\partial\phi \wedge B$. By Stokes' formula and (6.10),

$$-\int_{|\zeta|>\epsilon} \bar\partial\phi \wedge B = \int_{|\zeta|=\epsilon} \phi \wedge B =$$

$$\frac{1}{(2\pi i)^n}\frac{1}{\epsilon^{2n}}\int_{|\zeta|=\epsilon} \phi\partial|\zeta|^2(\bar\partial\partial|\zeta|^2)^{n-1} = \frac{1}{(2\pi i)^n}\frac{1}{\epsilon^{2n}}\int_{|\zeta|<\epsilon} \big[\phi(\bar\partial\partial|\zeta|^2)^n + \mathcal{O}(|\zeta|)\big].$$

The right hand side tends to $\phi(0)$ when $\epsilon \to 0$ since

$$(\frac{i}{2}\partial\bar\partial|\zeta|^2)^n = \Big(\frac{i}{2}\sum_1^n d\zeta_j \wedge d\bar\zeta_j\Big)^n =$$

$$n!\Big(\frac{i}{2}\Big)^n d\zeta_1 \wedge d\bar\zeta_1 \wedge \ldots \wedge d\zeta_n \wedge d\bar\zeta_n = n! dV(\zeta)$$

and the volume of the unit ball is $\pi^n/n!$, see Exercise 5. □

We have the following global formula.

Proposition 3.2 (The Bochner-Martinelli formula). *Let a be a point in $D \subset\subset \Omega$ and let s be a smooth $(1,0)$-form in Ω such that $2\pi i \langle s, \zeta - a\rangle = 1$ outside a and equal to $b(\zeta - a)$ in a neighborhood of a. Let $K = s \wedge (\bar\partial s)^{n-1}$. Then for any C^1-function f we have*

$$f(a) = \int_{\partial D} Kf + \int_D K \wedge \bar\partial f. \tag{3.4}$$

Notice that if $n = 1$ then K is just the Cauchy kernel so we get back the Cauchy-Green formula. In higher dimensions, however, there are many choices. Later on we shall see that it is enough that $s(\zeta)$ "behaves like" $b(\zeta - a)$ close to a.

Proof. Notice that $K = B$ in a neighborhood of a. By (3.1) we have that $d(Kf) = -K\bar\partial f$ if f vanishes in a neighborhood of a. In view of (3.3) we now obtain (3.4) in the same way as (2.2). □

Corollary 3.3 (The Cauchy-Fantappiè-Leray formula). *Assume that f is holomorphic in a neighborhood Ω of $\overline{D}$, and that σ is a smooth $(1,0)$-form on ∂D such that $\sigma \cdot (\zeta - a) = 1$, $a \in D$. Then for any holomorphic function f we have*

$$f(a) = \int_{\partial D} \sigma \wedge (\bar\partial\sigma)^{n-1} f.$$

To interprete the integral, let σ denote any smooth extension to a neighborhood of ∂D. Since $\sigma \wedge (\bar\partial\sigma)^{n-1} = \sigma \wedge (d\sigma)^{n-1}$ for degree reasons, the pull-back to ∂D is an intrinsically defined form.

PROOF. With no loss of generality we may assume that $a = 0$. Let σ be any extension to a neighborhood of ∂D. By continuity $\sigma \cdot \zeta \neq 0$ close to ∂D and if χ is an appropriately chosen cutoff function

$$s = (1-\chi)\frac{\sigma}{2\pi i \sigma \cdot \zeta} + \chi b$$

satisfies the assumption in Proposition 3.2. Therefore, the corollary follows, noting that $s \wedge (\bar\partial s)^{n-1} = \sigma \wedge (\bar\partial \sigma)^{n-1}$ on ∂D. □

For any domain D and $a \in D$ we can use $\sigma(\zeta) = b(\zeta - a)$ and thus obtain a representation formula for holomorphic functions, generalizing the Cauchy formula for $n = 1$. When $n > 1$ unfortunately it will not depend holomorphically on the variable a. However, in certain cases one can find a formula with holomorphic dependence of a.

REMARK 3.1. A necessary condition for the existence of a form $s(\zeta, z)$ for $\zeta \in \partial D$ depending holomorphically on z such that $\langle s(\zeta, z), \zeta - z\rangle = 1$, such an s is called a holomorphic support function, is that D is pseudoconvex. In general. though, it is not sufficient, even if we assume that D has smooth boundary. However, if D is strictly pseudoconvex one can always find a holomorphic support function, see Ch 2. It is also true for a large class of weakly pseudoconvex domains, e.g., all convex domains, see Example 3.2 below. □

EXAMPLE 3.1. Let $\mathbb{B} = \{\zeta;\ |\zeta| < 1\}$ be the unit ball. Then for $a \in \mathbb{B}$ we can take

$$\sigma = \frac{\partial|\zeta|^2}{2\pi i(1 - \bar\zeta \cdot a)}.$$

We then get, by a similar argument as above,

$$f(a) = \int_{|\zeta|=1} \frac{f(\zeta)\partial|\zeta|^2 \wedge (\bar\partial\partial|\zeta|^2)^{n-1}}{(2\pi i)^n(1-\bar\zeta\cdot a)^n} = \int_{|\zeta|=1}\frac{f(\zeta)d\nu(\zeta)}{(2\pi i)^n(1-\bar\zeta\cdot a)^n}.$$

We leave it as an exercise to the reader to check that

$$d\nu(\zeta) = \frac{1}{(2\pi i)^n}\partial|\zeta|^2 \wedge (\bar\partial\partial|\zeta|^2)^{n-1}$$

is the normalized surface measure on $\partial\mathbb{B}$, cf., Exercise 12. This representation formula is called the Szegö integral. □

EXAMPLE 3.2. More generally, let $D = \{\rho < 0\}$ be a convex domain in $\mathbb{C}^n$ with defining function ρ, i.e, $d\rho \neq 0$ on ∂D; it is not necessary to assume that ρ is a convex function. Then for any $a \in D$,

$$2\mathrm{Re}\,\langle \partial\rho(\zeta), \zeta - a\rangle > 0, \quad \zeta \in \partial D, \tag{3.5}$$

see Exercise 8. Taking this for granted we can use

$$s(\zeta, z) = \partial\rho(\zeta)/\left\langle \partial\rho(\zeta), \zeta - a\right\rangle$$

and get the classical representation formula

$$f(a) = \frac{1}{(2\pi i)^n}\int_{\partial D}\frac{f(\zeta)\partial\rho\wedge(\bar\partial\partial\rho)^{n-1}}{\langle\partial\rho(\zeta), \zeta - a\rangle^n}.$$

□

Example 3.3 (Cauchy estimates). Sometimes it is useful to have a representation of holomorphic functions where the integration is performed over something thicker than just a boundary ∂D. Let H be any compact subset of Ω and let χ be a cutoff function in Ω that is identically 1 in a neighborhood of H. Using $s(\zeta) = b(\zeta - a)$ for $a \in H$, and applying (3.4) to $f\chi$ we get the representation

$$f(a) = -\int f\bar\partial\chi\wedge K, \quad f\in\mathcal{O}(\Omega). \tag{3.6}$$

Since $\bar\partial\chi\wedge K$ is smooth for a in a neighborhood of H we can differentiate an arbitrary number of times, and hence we get constants C_M such that

$$\sum_{|\alpha|\le M}\sup_K|\partial^\alpha f/\partial z^\alpha| \le C_M\int_\Omega|f|dV, \quad f\in\mathcal{O}(\Omega).$$

Thus all derivatives of a holomorphic function can be estimated uniformly on K by the L^1-norm of f over a slightly larger open set, in particular by the supremum of the function itself over the larger set. Estimates of this kind are called *Cauchy estimates*.

It follows that if $\mathcal{E}(\Omega)$ is equipped with the usual topology, then $\mathcal{O}(\Omega)$ has the induced topology as a closed subspace of $\mathcal{E}(\Omega)$. □

Example 3.4. In (3.6) the kernel does not depend holomorphically on a unless $n = 1$. However as before one can obtain holomorphic dependence in special cases like the ball. Let χ be a cutoff function in the ball $\mathbb{B}$ that is identically 1 in a neighborhood of the closure of $r\mathbb{B}$ for some $r < 1$. Moreover, for $a \in K$, let

$$s(\zeta) = \chi(\zeta)b(\zeta - a) + (1-\chi(\zeta))\frac{\partial|\zeta|^2}{2\pi i(|\zeta|^2 - \bar\zeta\cdot a)}.$$

For holomorphic f, applying (3.4) to χf, we then get the representation

$$f(a) = -\int f(\zeta)\bar\partial\chi(\zeta)\wedge K, \tag{3.7}$$

where the kernel

$$K = \frac{1}{(2\pi i)^n} \frac{\partial|\zeta|^2 \wedge (\bar\partial\partial|\zeta|^2)^{n-1}}{(|\zeta|^2 - \bar\zeta \cdot a)^n}$$

is holomorphic in a for $a \in r\mathbb{B}$ and ζ on the support of $\bar\partial\chi$. $\square$

4. Koppelman's formula

If $f = f_1 d\zeta$ is a smooth $(0,1)$-form in $D \subset \mathbb{C}$ such that

$$\int_D |f_1| dV < \infty$$

then

$$u(z) = \int_D \omega_{\zeta - z} \wedge f$$

is a smooth solution to $\bar\partial u = f$ in D; this follows from Proposition 2.4 by writing $f = \chi f + (1-\chi) f$. We shall now consider multivariable analogues.

Let $\alpha(\zeta, z)$ be any form of on $\mathbb{C}^n \times \mathbb{C}^n$ with compact support. We then define

$$\int_\zeta \alpha(\zeta, z) \tag{4.1}$$

as the form in z such that

$$\int_z \phi(z) \wedge \int_\zeta \alpha(\zeta, z) = \int\int_{z,\zeta} \phi(z) \wedge \alpha(\zeta, z)$$

for all forms ϕ. The right hand side is well-defined since $\mathbb{C}^n$ has even real dimension so the orientation (volume form) on $\mathbb{C}^n \times \mathbb{C}^n$ is unambiguously defined. A moment of thought reveals that the definition practically means that one first moves all differentials of ζ to the right (or to the left) and then perform the integration with respect to ζ. For instance, if $\psi(\zeta, z)$ is a function, then

$$\int_\zeta \psi(\zeta, z) d\zeta \wedge dz \wedge d\bar\zeta = -\Big[\int_\zeta \psi(\zeta, z) d\zeta \wedge d\bar\zeta\Big] dz.$$

Clearly, only components of α that have bidegree (n, n) in ζ can give any contribution in (4.1). We have the Fubini theorem

$$\int_z\int_\zeta \alpha(\zeta, z) = \int_\zeta\int_z \alpha(\zeta, z)$$

if α has bidegree $(2n, 2n)$.

Let $B(\eta)$ be the BM form as above and consider the mapping $\eta\colon \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ defined by $(\zeta, z) \mapsto \eta = \zeta - z$. Then $\eta^* B(\zeta, z)$

is a form that we simple write as $B(\zeta - z)$. Practically this means that each occurence of η_j in $B(\eta)$ shall be replaced by $\zeta_j - z_j$, each ocurrence of $d\eta_j$ shall be replaced by $d(\zeta_j - z_j)$ etc.

Proposition 4.1. *For any form $\psi(\zeta, z)$ of total bidegree (n, n) in $\mathbb{C}^n \times \mathbb{C}^n$, we have*

$$-\int_\zeta \int_z \bar\partial\psi(\zeta, z) \wedge B(\zeta - z) = \int_z \psi(z, z) \tag{4.2}$$

Here $\psi(z, z)$ means the pullback of ψ to the diagonal $\Delta = \{(z, z);\ z \in \mathbb{C}^n\} \subset \mathbb{C}^n \times \mathbb{C}^n$, i.e., $i^*\psi$, where $i\colon \mathbb{C}^n \to \mathbb{C}^n \times \mathbb{C}^n$, $z \mapsto (z, z)$.

Notice that $d = d_\zeta + d_z$ and $\bar\partial = \bar\partial_\zeta + \bar\partial_z$.

Proof. Since $\bar\partial$ commutes with pullbacks of holomorphic mappings, by a complex-linear change of variables on $\mathbb{C}^n \times \mathbb{C}^n$, keeping in mind that the orientation is preserved, and that Fubini's theorem holds, the integral on the left hand side of (4.2) becomes

$$-\int_z \int_\zeta \bar\partial\psi(\zeta + z, \zeta) \wedge B(\zeta) =$$
$$\int_z \Big[\int_\zeta \bar\partial_\eta\psi(\zeta + z, z) \wedge B(\zeta)\Big] + \int_\zeta \Big[\int_z \bar\partial_z\psi(\zeta + z, z)\Big] \wedge B(\zeta).$$

In the first inner integral, for degree reasons only components of ψ which have bidegree $(0, 0)$ in ζ can give a contribution, and in view of (3.1) the inner integral therefore becomes $-\psi(z, z)$. In the inner integral in the second term for degree reasons one can replace $\bar\partial_z$ by d_z, and then the integral vanishes by Stokes' theorem. $\square$

Now let $b(\zeta - z) = \eta^* b(\zeta, z) = \sum_1^n \bar\eta_j d\eta_j / 2\pi i|\eta|^2$ and let

$$s(\zeta, z) = \sum_1^n s_j(\zeta, z) d(\zeta_j - z_j)$$

be a form in $\Omega \times \Omega$ such that $\sum s_j(\zeta_j - z_j) = 1$ outside $\Delta \subset \Omega \times \Omega$, and $s(\zeta) = b(\zeta - z)$ in a neighborhood of Δ. Such a form will be called *admissible*. Precisely as before, $0 = \sum_j \eta_j \bar\partial s_j$ and therefore

$$K = s \wedge (\bar\partial s)^{n-1}$$

is $\bar\partial$-closed outside Δ. Let $K_{p,q}$ be the component of bidegree (p, q) in z, and consequently $(n - p, n - q - 1)$ in ζ.

Lemma 4.2. *If f is a smooth (p, q)-form, then*

$$\int_{\zeta \in D} K_{p,q-1}(\zeta, z) \wedge f(\zeta)$$

is a smooth $(p, q-1)$*-form in* D.

Proof. Fix a point z^0. If ω is a small enough neighborhood of z^0, then $s = b$ for all $\zeta \in \omega$ and z close to z^0. Take a cutoff function χ in ω such that $\chi = 1$ in a neighborhood of z^0, and consider the decomposition

$$u(z) = \int_D (1-\chi) K \wedge f + \int \chi K \wedge f.$$

The first term is smooth in a neighborhood of z^0 since there is no singularity in the integral. On the other hand, for z close to z^0 the second integral is just

$$\pm \int (\chi f)(\zeta) \wedge B(\zeta - z),$$

and by an argument analogous to the proof of Proposition 2.4 it follows that u is as smooth as f is. □

Theorem 4.3 (Koppelman's formula). *If* f *is a smooth* (p, q)*-form, then*

$$f(z) = \bar\partial_z \int_D K_{p,q-1} \wedge f + \int_D K_{p,q} \wedge \bar\partial f + \int_{\partial D} K_{p,q} \wedge f, \quad z \in D.$$

It is clear that if we can make the boundary integral disappear, then for each f such that $\bar\partial f = 0$, we get a solution to $\bar\partial u = f$.

Proof. Let us first assume that $f(\zeta)$ has compact support and let $\psi(z)$ be a test form of bidegree $(n-p, n-q)$. Since $\bar\partial K = 0$ and $K = B(\zeta - z)$ near Δ it is clear that (4.2) holds for K instead of $B(\zeta - z)$. Therefore,

$$\int_z \psi(z) \wedge f(z) = -\int\int \bar\partial(\psi(z) \wedge f(\zeta)) \wedge K =$$
$$-\int_z \bar\partial_z \psi(z) \wedge \int_\zeta f \wedge K - (-1)^{p+q} \int_z \psi(z) \wedge \int_\zeta \bar\partial f \wedge K.$$

In the first term we can integrate by parts in the z-integral. After moving f and $\bar\partial f$ to the right in the ζ-integrals we then get the equality

$$\int \psi(z) \wedge f(z) = \int_z \psi(z) \wedge \bar\partial_z \int_\zeta K \wedge f + \int_z \psi(z) \wedge \int_\zeta K \wedge \bar\partial f$$

which is equivalent to the theorem in case f has compact support. The general case can now be deduced, e.g., by replacing f by $\chi_k f$ where $\chi_k \nearrow \chi_D$ and take limits, see Exercise 6, or by mimicking the proof of Proposition 2.2. □

5. The local $\bar\partial$-equation

We are now in position to prove the local solvability of the $\bar\partial$-equation which is usually referred to as the Dolbeault-Grothendieck lemma. This is the counterpart of the Poincaré lemma for d.

THEOREM 5.1. *Assume that f is a smooth (p,q)-form that is $\bar\partial$-closed in a the unit ball $\mathbb{B}$. Then there is a smooth $(p,q-1)$-form u in $r\mathbb{B}$, $r<1$, such that $\bar\partial u = f$.*

We will use the notation $\bar\zeta\cdot d(\zeta-z)$ for $\sum_1^n \bar\zeta_j d(\zeta_j - z_j)$, etc.

PROOF. We may assume that f is defined and closed in the unit ball $\mathbb{B}$. Let χ be a cutoff function in $\mathbb{B}$ that is identically 1 in a neighborhood of the closure of $r\mathbb{B}$, $r<1$. Then

$$s(\zeta,z) = \chi(\zeta)b(\zeta-z) + (1-\chi(\zeta))\frac{\bar\zeta\cdot d(\zeta-z)}{(|\zeta|^2 - \bar\zeta\cdot z)2\pi i}$$

is an admissible form for z in $r\mathbb{B}$, and for ζ close to ∂B it is holomorphic in z. (One can extend it to an admissible form for $z\in\mathbb{B}$ as well by taking $\tilde\chi(z)s + (1-\tilde\chi(z))b(\zeta-z)$ where $\tilde\chi$ is identically 1 in a neighborhood of the support of χ; but this is uninteresting for us, since we just bother about z in $r\mathbb{B}$.)

If $q>0$ it follows that $K_{p,q}=0$ if $z\in r\mathbb{B}$ and ζ is close to $\partial\mathbb{B}$, since then no $d\bar z$ can occur. Therefore the boundary integral vanishes and we get

$$f(z) = \bar\partial_z \int_{\mathbb{B}} K_{p,q-1}\wedge f + \int_{\mathbb{B}} K_{p,q}\wedge \bar\partial f, \quad z\in r\mathbb{B}.$$

If in addition $\bar\partial f = 0$ in $\mathbb{B}$ we thus get a solution in $r\mathbb{B}$. □

We also have a generalization of Proposition 2.5.

THEOREM 5.2. *Suppose that f is a smooth $\bar\partial$-closed $(0,q)$-form in $\mathbb{C}^n$ with compact support. Then there is a solution to $\bar\partial u = f$ with compact support if $q<n$. If $q=n$ such a solution exists if and only if*

$$\int f(\zeta)\wedge\zeta^\alpha d\zeta_1\wedge\ldots\wedge d\zeta_n = 0 \tag{5.1}$$

for all multiindices $\alpha\geq 0$.

A similar statement holds for (p,q)-forms, since $\mathcal{E}_{p,q}(\mathbb{C}^n)\simeq \oplus'_{|I|=p}\mathcal{E}_{0,q}(\mathbb{C}^n)$.

PROOF. If $q=n$ and there is a solution u with compact support, then (5.1) follows by Stokes' theorem.

We may assume that f has its support in $r\mathbb{B}$ for some $r < 1$. Choose a cutoff function χ as in the preceding proof, and let

$$s(\zeta, z) = \chi(z)b(\zeta - z) + (1 - \chi(z))\frac{-\bar{z} \cdot d(\zeta - z)}{2\pi i(|z|^2 - \bar{z} \cdot \zeta)};$$

this is the same s as in the preceding proof but with z and ζ interchanged. Since f has compact support no boundary integral occurs and we thus have $\bar{\partial} u = f$ with

$$u(z) = \int K_{p,q-1} \wedge f.$$

Moreover, for z close to $\partial\mathbb{B}$,

$$K(\zeta, z) = \pm\frac{1}{(2\pi i)^n}\frac{\bar{z} \cdot d(\zeta - z) \wedge (d\bar{z} \cdot d(\zeta - z))^{n-1}}{(|z|^2 - \bar{z} \cdot \zeta)^n},$$

so $K_{0,q} = 0$ for $q < n - 1$ and

$$K_{0,n-1} = \pm\frac{1}{(2\pi i)^n}\frac{\bar{z} \cdot d\zeta \wedge (d\bar{z} \cdot d\zeta)^{n-1}}{(|z|^2 - \bar{z} \cdot \zeta)^n}$$

for z close to $\partial\mathbb{B}$. Since we can expand the kernel in an absolutely convergent series for $\zeta \in r\mathbb{B}$ and ζ close to $\partial\mathbb{B}$, it follows that $u(z) = 0$ close to $\partial\mathbb{B}$ if (5.1) holds, and thus u has compact support. $\square$

Remark 5.1. With the notation in Koppelman's formula one can define the kernel K as $s \wedge (ds)^{n-1}$ instead. It is then still true, see Exercise 15, that $dK = 0$ outside Δ. One can then prove the slightly more general Koppelman formula

$$f(z) = d_z \int_D K \wedge f + \int_D K \wedge df + \int_{\partial D} K \wedge f. \tag{5.2}$$

When restricting to the component K' of K that is $(n, n-1)$ in $dz, d\zeta$ we get back the previous Koppelman formula, but (5.2) also contains other relations that sometimes are useful, see Exercise 18. $\square$

6. Positive forms

Fix a point $z \in X$ and let $dV(z)$ be the volume form (at z) with respect to some holomorphic coordinate system. Recall that if w is another coordinate systen then $dV(w) = c dV(z)$ for some $c > 0$.

An (n, n)-form α is *positive*, $\alpha \geq 0$, if $\alpha = c dV(z)$ where $c \geq 0$ and strictly positive if $c > 0$. In what follows we only discuss (weak) positivity and leave it to the reader to formulate the corresponding definitions and statements for strict positivity.

A (p,p)-form ω is *positive* if for all $(1,0)$-forms α_j,

$$\omega \wedge i\alpha_{p+1} \wedge \bar{\alpha}_{p+1} \wedge \ldots \wedge i\alpha_n \wedge \bar{\alpha}_n \geq 0. \tag{6.1}$$

Clearly the set of positive (p,p)-forms is closed under positive linear combinations.

Notice that each p-dimensional complex subspace V of $T_z(X)$ is $T_z(Y)$, where Y is a p-dimensional complex submanifold. In fact, V is determined by $n-p$ linearly independent $(1,0)$-forms $\alpha_{p+1}, \ldots, \alpha_n$, and if we choose a coordinate system z such that $dz_j|_z = \alpha_j$ for $j = p+1, \ldots, n$, then $Y = \{z_{p+1} = \cdots = z_n = 0\}$ will do.

Lemma 6.1. *A (p,p)-form ω is positive at z if and only if it is positive, restricted (pulled-backed) to each complex p-dimensional submanifold through z.*

Proof. Suppose the condition in the lemma holds, and assume that $\alpha_{p+1}, \ldots, \alpha_n$ are given. If they are linearly dependent then clearly (6.1) holds. Otherwise, we can choose a coordinate system z as above. Then $z_1, \ldots, z_p$ is a coordinate system on Y and hence by assumption $\omega|_Y = cidz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge idz_p \wedge d\bar{z}_p$ with $c \geq 0$, i.e.,

$$\omega = cidz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge idz_p \wedge d\bar{z}_p + \ldots, \tag{6.2}$$

where $\ldots$ denote terms with some dz_j or $d\bar{z}_j$ for $j > p$. It follows that (6.1) holds. Thus $\omega \geq 0$.

Conversely, if $\omega \geq 0$ and Y is given, then it follows that (6.2) must hold with $c \geq 0$, and thus the pull-back of ω to Y is positive. $\square$

Lemma 6.2. *We claim that any (k,k)-form is a linear combination of forms like $a_1 \wedge \bar{a}_1 \wedge \ldots a_k \wedge \bar{a}_k$.*

This follows by repeated use of the identity

$$\begin{aligned} 4dz_j \wedge d\bar{z}_k = (dz_j + dz_k) \wedge \overline{(dz_j + dz_k)} - (dz_j - dz_k) \wedge \overline{(dz_j - dz_k)} \\ + i(dz_j + idz_k) \wedge \overline{(dz_j + idz_k)} - i(dz_j - idz_k) \wedge \overline{(dz_j - idz_k)} = \\ \sum_{\ell \in \mathcal{Z}_4} i^\ell (dz_j + i^\ell dz_k) \wedge \overline{(dz_j + i^\ell dz_k)} \end{aligned}$$

Proposition 6.3. *If $\omega \geq 0$, then ω is real, i.e., $\bar{\omega} = \omega$.*

Proof. It follows from the preceding proof that if $\alpha_{p+1}, \ldots, \alpha_n$ are arbitrary $(1,0)$-forms, then

$$\begin{aligned} &\omega \wedge i\alpha_{p+1} \wedge \bar{\alpha}_{p+1} \wedge \ldots \wedge i\alpha_n \wedge \bar{\alpha}_n \\ &\qquad = cdV = \bar{c}dV = \bar{\omega} \wedge i\alpha_{p+1} \wedge \bar{\alpha}_{p+1} \wedge \ldots \wedge i\alpha_n \wedge \bar{\alpha}_n. \end{aligned}$$

This implies that $\omega = \bar{\omega}$ in view of Lemma 6.2. $\square$

PROPOSITION 6.4. *A $(1,1)$-form $\omega = i\sum_{jk} \alpha_{jk} dz_j \wedge d\bar{z}_k$ is positive if and only if (α_{jk}) is a positively semi-definite Hermitian matrix. It is strictly positive if and only if (α_{jk}) is (strictly) positively definite.*

PROOF. If ω is positive it is real, and hence the matrix α_{jk} is Hermitian. After a linear change of coordinates, we may assume that it is diagonal, i.e.,

$$\omega = i\sum_k \alpha_k dz_k \wedge d\bar{z}_k. \tag{6.3}$$

It is now clear that ω is positive if and only if $\alpha_k \geq 0$. □

PROPOSITION 6.5. *If $\omega \geq 0$ is $(1,1)$, $\gamma \geq 0$ is (p,p), then $\alpha \wedge \omega$ is positive.*

PROOF. We may assume that ω is as in (6.3). Since it is clear that $\gamma \wedge i dz_k \wedge d\bar{z}_k$ is positive for each fixed k, the sum of them must be as well. □

It is important here that ω is $(1,1)$. The conclusion is not true in general.

LEMMA 6.6. *If a_j are $(1,0)$-forms, then*

$$ia_1 \wedge \bar{a}_1 \wedge \ldots \wedge ia_p \wedge \bar{a}_p = i^{p^2} a_1 \wedge \ldots \wedge a_p \wedge \bar{a}_1 \wedge \ldots \wedge \bar{a}_p.$$

PROOF. First notice that $a_p \wedge\ ldots \wedge a_1 = (-1)^{p(p-1)/2} a_1 \wedge \ldots \wedge a_p$ since one has to move the factor a_p $p-1$ steps to the right, then move a_{p-1} $p-2$ steps to the right etc. Therefore the right hand side is equal to

$$(-1)^{-\frac{p(p-1)}{2}} i^{p^2} a_p \wedge \ldots \wedge \alpha_1 \wedge \bar{a}_1 \wedge \ldots \wedge a_p = i^p a_p \wedge \ldots \wedge a_1 \wedge \bar{a}_1 \wedge \ldots \wedge a_p,$$

which is equal to the left hand side. □

7. Hermitian metrics and Kähler metrics

Now suppose that

$$\omega = i\sum_{jk} \alpha_{jk} dz_j \wedge d\bar{z}_k$$

is a smooth strictly positive form and fix a point p. Then ω induces a positively definite Hermitian form

$$h(\xi, \eta) = -i\omega(\xi, \bar{\eta})$$

on $T_{1,0}$ at p and on $T_{0,1}$ such that the decomposition $T^{\mathbb{C}} = T_{1,0} \oplus T_{0,1}$ is orthgogonal. Here $T^{\mathbb{C}}$ denotes the complexification of the real tangent space, i.e., $T^{\mathbb{C}} = T^{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. In practice this means that given a basis of the real tangent space $T^{\mathbb{R}}$, e.g., $\partial/\partial x_1, \partial/\partial y_1, \ldots, \partial/y_n$, then $T^{\mathbb{C}}$ is the complex space spanned by these basis elements.

We claim that h restricted to $T^{\mathbb{R}}$ defines a Riemannian metric. In fact, at a given point we choose holomorphic coordinates z such that (6.3) holds. Then it is easily checked that

$$h = \sum \alpha_j (dx_j \otimes dx_j + dy_j \otimes dy_j).$$

We say that ω is a Kähler metric if in addition $d\omega = 0$. Later on we will see that ω is Kähler if and only if at each point p we can choose holomorphic coordinates z, with $z(p) = 0$, such that $\omega = \sum_j dz_j \wedge d\bar{z}_j + \mathcal{O}(|z|^2)$.

Our Hermitian metric (induced by) ω on T^* extends to a Hermitian metric on the exterior algebra ΛT^* in the following way. Let $*$ denote the Hodge operator om the underlying real cotangent space and extend it complex-linearly to T^*. If $e_1, \ldots, e_n$ is an ON-basis for $T^*_{1,0}$, then $*e_1 =$????? etc.

$$\langle f, g \rangle \, \omega_n = f \wedge *\bar{g} \wedge \omega_{n-p}$$

Any Riemannian metric, in particular any Hermitian metric (induced by a) ω extends to a metric on all differential forms in the usual way, i.e., if e_j is an ON basis for T^*, then e_I, $|I| = k$ is ON in the space of k forms.

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Lemma 7.1. *If ω metric on X and Z is a complex submanifold, then the pullback ω_Z of ω to Z defines the induced Riemannian metric on Z.*

Proof. ???? $\square$

As a consequence we get the formula

$$area(Z) = \int_Z \omega_k \tag{7.1}$$

if Z has dimension k.

In standard metric in $\mathbb{C}^n$ β, we find that the local area of Z is the sum of the areas of the projections onto the various k-dimensional coordinate planes.

Corresponding volume form is ω_n. Moreover, if f, g are $(p,0)$ then we have the very useful formula

$$\langle f, g\rangle\, \omega_n = i^{p^2} f \wedge g \wedge \omega_p.$$

Obs on any X one can find a Hermitian metric by a partition of unity.

Aven area av ∂D etc.

8. Currents

Currents are indispensable in complex analysis. From the naive formal point of view they are just differential forms with distribution coefficients, and hence quite simple generalizations of distributions. However they provide analytic representations of geometric objects like submanifolds and subvarieties, and intrinsic generalizations of these objects.

8.1. Definition of currents. Let X be a subset of $\mathbb{C}^n$ or a complex manifold. Let $\mathcal{D}_{p,q}(X)$ be the space of (p,q)-forms with compact support, and with the topology such that $\phi_k \to 0$ if(f) all supports of ϕ_k are contained in a fixed compact set K and moreover $\phi_k \to 0$ with all its derivatives tends uniformly to 0.

Remark 8.1. If z is local coordinates, and $\phi = \sum'_{I,J} \phi_{IJ} dz_I \wedge d\bar{z}_J$, then we cam take $|\phi| = \sum'_{IJ} |\phi_{IJ}|$ as local pointwise norm of ϕ; it is easy to see that it is essentially independent of the choice of local coordinates, and so the topology of $\mathcal{D}_{p,q}$ is well-defined. However, a more elegant way is to fix a global hermitian metric ω and let $|\phi|$ be the corresponding pointwise norm. $\square$

Notice that if $f \in (L^1_{\rm loc})_{p,q}(X)$, i.e., f is a (p,q)-form with locally integrable coefficients, then it defines a continuous linear functional on $D_{n-p,n-q}(X)$,

$$\phi \mapsto \int f \wedge \phi,$$

where we use the canonical orientation of X. Notice that f is determined by this functional, i.e., $f = 0$ if and only if the functional vanishes. A *current* u of bidegree (p,q) on X, $u \in \mathcal{C}_{p,q}(X)$, is a continuous linear functional on $D_{n-p,n-q}(X)$. If u is a (p,q)-current and $\psi \in \mathcal{E}_{p',q'}(\Omega)$, that is, ψ is a smooth (p',q')-form, then $\psi \wedge u$ is a $(p+p', q+q')$-current defined by

$$\psi \wedge u.\phi = (-1)^{(p+q)(p'+q')} u.(\psi \wedge \phi), \quad \phi \in \mathcal{D}_{n-p-p',n-q-q'}(X).$$

Thus $\mathcal{C}_\bullet(X)$ is a module over the ring $\mathcal{E}_\bullet(X)$. Moreover, if $X' \subset X$ we have a restriction mapping $\mathcal{C}_\bullet(X) \subset \mathcal{C}_\bullet(X')$ in the obvious way.

In this formalism a usual distribution should be considered as a (n,n)-current since it acts on test functions.

If we fix a (say holomorphic) coordinate system z, then a (p,q)-current μ can be written

$$\mu = \sideset{}{'}\sum_{|I|=p,|J|=q} \mu_{IJ} dz_I \wedge d\bar{z}_J,$$

where μ_{IJ} are distributions in the usual sense, defined by

$$\mu_{IJ}.\phi = \pm\mu.\phi dz_{I^c} \wedge d\bar{z}_{I^c},$$

where $I^c = \{1,2,\dots,n\}\backslash I$, ordered so that I, I^c coincides with $\{1,\dots,n\}$ after an even number of permutations.

Proposition 8.1. *Let Ω_j be open subsets of X and $\Omega = \cup\Omega_j$. Then we have*

(i) If $u \in \mathcal{C}_\bullet(\Omega)$ and $u = 0$ in Ω_k for each k, then $u = 0$.

(ii) If we have $u_j \in \mathcal{C}_\bullet(\Omega_j)$ such that $u_j = u_k$ in $\Omega_j \cap \omega_k$, then there is a $u \in \mathcal{C}_\Omega$ such that $u = u_k$ in Ω_k.

Proof. The proof is obtained by a partition of unity and we only sketch it. There is a locally finite partition of unity subordinate to Ω_k, that is, a collection of cutoff functions ϕ_j, with $\operatorname{supp}\phi_j \subset \Omega_{k_j}$, such that the sum $\sum \phi_j$ is locally finite and equal to 1 in Ω. To see (i), take a test form ϕ. Then $u.\phi = \sum_j u.\chi_j\phi = 0$, since $\chi_j\phi$ has support in Ω_{k_j}. To see (ii), let

$$u.\phi = \sum_\ell u_{k_\ell}.(\chi_\ell\phi),$$

which is well-defined since the sum is finite for each fixed ϕ. For a fixed j and $\phi \in \mathcal{D}(\Omega_j)$ we have, since $u_{k_\ell} = u_j$ in $\Omega_{k_\ell} \cap \Omega_j$,

$$u.\phi = \sum_\ell u_{k_\ell}.\chi_\ell\phi = \sum_\ell u_j.\chi_\ell\phi = u_j.\phi$$

as wanted. $\square$

The proposition means that $\mathcal{C}_\bullet(X)$ is a sheaf of $\mathcal{E}$-modules, see Appendix 9.

The simplest examples of currents are $L^1_{\text{loc},\bullet}(X)$, in particular all smooth forms are currents. If Z is a smooth analytic submanifold of complex codimension p, then it defines a (p,p)-current

$$[Z].\phi = \int_Z \phi.$$

Similarly any smooth oriented submanifold V defines a current $[V]$.

A locally finite complex measure μ defines a (n,n)-current by

$$\mu.\phi = \int \phi d\mu, \quad \phi \in \mathcal{D}_{n,n}(X).$$

REMARK 8.2. Notice that the definition here is invariant and only depends on the orientation induced by the complex structure on X. □

From Proposition 8.1 follows that there is a maximal open subset of X in which the current $u \in \mathcal{D}'_\bullet$ vanishes, and the complement is called the support of u, denoted $\operatorname{supp} u$.

One important feature of currents, precisely as for usual distributions, is that differentiation is a continuous operation: Given $u \in \mathcal{C}_{p,q}(X)$ we define $\bar\partial u$ by

$$\bar\partial u.\phi = (-1)^{p+q+1} u.\bar\partial\phi, \quad \phi \in \mathcal{D}_{n-p,n-q-1}(X). \tag{8.1}$$

The choice of sign in (8.1) is chosen so that it is compatible with the usual operation in case u is (given by) a smooth form. By ???? above it follows that $\bar\partial u$ restricted to an open set ω only depends on the restriction of u to ω. In the same way ∂ and d are defined, and of course we still have that $d = \partial + \bar\partial$ and $0 = \partial^2 = \bar\partial^* = \partial\bar\partial + \bar\partial\partial$.

LEMMA 8.2. *If D is a smoothly bounded domain in X, then*

$$d[D] = -[\partial D].$$

PROOF. In fact,

$$d[D].\phi = -[D].d\phi = -\int_D d\phi = -\int_{\partial D} \phi = -[\partial D].\phi.$$

□

If Z is an complex submanifold of codimension p, thus $[Z]$ is a d-closed (p,p)-current.

We can now express (3.3) as

$$\bar\partial B(\zeta) = [0].$$

If a is a fixed point we have by translation that $\bar\partial B(\zeta - a) = [a]$. Moreover, if we consider $B(\zeta - z)$ as a current on $\mathbb{C}^n \times \mathbb{C}^n$ we have, cf., (4.2),

$$\bar\partial B(\zeta - z) = [\Delta].$$

We know that $\bar\partial K = 0$ outside Δ; since $K = B(\zeta - z)$ in a neighborhood of Δ we thus have that $\bar\partial K = [\Delta]$.

Let $f\colon \Omega \to \Omega'$ be a smooth proper (i.e., the invers image of each compact set is compact) mapping. Then the pull-back f^* maps $\mathcal{D}_\bullet(\Omega') \to \mathcal{D}_\bullet(\Omega)$ and thus we get a dual mapping

$$f_*\colon \mathcal{C}_\bullet(\Omega) \to \mathcal{C}_\bullet(\Omega'),$$

called the *push-forward*, defined by

$$f_* u.\phi = u.f^*\phi.$$

If f is holomorphic, then f_* preserves codegree; i.e., if Ω and Ω' have dimensions m and n, respectively, then

$$f_*\colon \mathcal{C}_{m-p,m-q}(\Omega) \to \mathcal{C}_{n-p,n-q}(\Omega).$$

If f is holomorphic it immediately follows that

$$f_*\bar\partial u = \bar\partial f_* u.$$

It is easily checked that if f is a diffeomorphism, then $f_* = (f^{-1})^*$.

8.2. Convolution of forms and currents. Given forms f, g in say the Schwartz class $\mathcal{S} = \mathcal{S}(\mathbb{C}^n)$ (i.e., their coefficients when expresses in the standard coordinates are in $\mathcal{S}$) we can define the convolution

$$f * g(z) = \int_\zeta f(\zeta - z) \wedge g(\zeta),$$

where as before $f(\zeta - z) = \pi^* f(\zeta, z)$ and $\pi(\zeta, z) = \zeta - z$. This definition is completely real, but again we will profit from the fact that our underlying space has *even* real dimension, and leave it to the interested reader to find out what happens in the odd-dimensional case. As in the usual case, $f * g$ will be a new form with coefficients Notice that if ψ is in $\mathcal{S}$, then

$$\int_z f * g(z) \wedge \psi(z) = \int_z \int_\zeta g(\zeta) \wedge \psi(z + \zeta) \tag{8.2}$$

and similarly as in the usual case we can take (8.2) as the definion of $f * g$ when they are currents, and one of them has compact support.

The following facts are easily verified:

$$
\begin{aligned}
(8.3)\quad & \deg f * g = \deg f + \deg g - 2n \\
& \text{if } f \in \mathcal{S}_{p,q}, g \in \mathcal{S}_{p',q'}, \text{ then } f * g \in \mathcal{S}_{p+p'-n,q+q'-n} \\
& f * g = (-1)^{\deg f \cdot \deg g} g * f \\
& (f * g) * h = f * (g * h) \\
& [0] * f = f \\
& d(f * g) = df * g + (-1)^{\deg f} f * dg \\
& \bar\partial(f * g) = \bar\partial f * g + (-1)^{\deg f} f * \bar\partial g.
\end{aligned}
$$

EXAMPLE 8.1. Let ϕ be a (n, n)-form with compact support such that $\int \phi = 1$, and let $\phi_\epsilon(z) = \phi(z/\epsilon)$. Then

$$\phi_\epsilon \to [0]$$

in the current sense. □

8.3. The local $\bar\partial$-equation for currents. nalle We can now prove the Dolbeault-Grothendieck lemma for currents.

PROPOSITION 8.3. *If f is a $\bar\partial$-closed (p, q)-current with $q \geq 1$, then locally $\bar\partial u = f$ has a current solution. If $q = 0$, then f is holomorphic.*

PROOF. Since we can multiply f with a cutoff function, we may assume that f is $\bar\partial$-closed in say the unit ball $\mathbb{B}$, and has support in $2\mathbb{B}$. If B is the BM-kernel we know that $\bar\partial B = [0]$ and hence since f has compact support, $\bar\partial B * f = f - B * \bar\partial f$ by (8.2). Moreover, since $\bar\partial f = 0$ in $\mathbb{B}$,

$$B * \bar\partial f(z) = \int_\zeta B(\zeta - z) \wedge \bar\partial f(\zeta)$$

for $z \in \mathbb{B}$ and is smooth in z there. Furthermore, it is $\bar\partial$-closed there since both the other terms are. Thus we can solve $\bar\partial v = B * \bar\partial f$ in $r\mathbb{B}$ in view of Theorem 5.1, and hence $\bar\partial(B * f + v) = f$ in $r\mathbb{B}$. □

8.4. de Rham and Dolbeault cohomology. Let X be any complex manifold, for instance an open subset of $\mathbb{C}^n$. In view of Theorem 5.1 it follows that consider the sheaf complex

$$\mathcal{E}_{0,0} \xrightarrow{\bar\partial} \mathcal{E}_{0,1} \xrightarrow{\bar\partial} \dots \xrightarrow{\bar\partial} \mathcal{E}_{0,n} \to 0 \tag{8.4}$$

is a resolution of the sheaf $\mathcal{O}$, and since in addition all the sheaves $\mathcal{E}_{0,*}$ are modules over $\mathcal{E}$, we say then that the resolution is fine, it follows from abstract sheaf theory that the sheaf cohomology groups $H^k(X, \mathcal{O})$ are represented by the cohomology of the induced complex

$$\mathcal{E}_{0,0}(X) \xrightarrow{\bar\partial} \mathcal{E}_{0,1}(X) \xrightarrow{\bar\partial} \dots \xrightarrow{\bar\partial} \mathcal{E}_{0,n}(X) \to 0. \tag{8.5}$$

of $\mathcal{O}(X)$-modules. See Appendix 9 below.

Moreover, by Proposition 8.3 we have as well the fine resolution

$$\mathcal{C}_{0,0} \xrightarrow{\bar\partial} \mathcal{C}_{0,1} \xrightarrow{\bar\partial} \dots \xrightarrow{\bar\partial} \mathcal{C}_{0,n} \to 0 \tag{8.6}$$

and thus also the cohomology of the corresponding complex of $\mathcal{O}(X)$-modules of global sections will represent the sheaf cohomology groups $H^k(X, \mathcal{O})$. Finally, since we have natural sheaf mappings $i\colon \mathcal{E}_{0,*} \to \mathcal{C}_{0,*}$ that commute with $\bar\partial$, it follows that the natural mappings

$$\frac{\operatorname{Ker}_{\bar\partial}\mathcal{E}_{0,k}(X)}{\bar\partial\mathcal{E}_{0,k-1}(X)} \to \frac{\operatorname{Ker}_{\bar\partial}\mathcal{C}_{0,k}(X)}{\bar\partial\mathcal{C}_{0,k-1}(X)}$$

are indeed ismorphisms. In particular we have: If f is a smooth form in X such that $\bar\partial u = f$ has a current solution, then there is also a smooth solution.

There are completely analogous statements for the de Rham complex by replacing $\mathcal{O}$ by the constant sheaf $\mathbb{C}$, $\bar\partial$ by d, and $\mathcal{E}_{0,k}$ and $\mathbb{C}_{o,k}$ by $\mathcal{E}_k$ and $\mathbb{C}_k$, respectively.

9. Appendix I: Some facts about sheaf theory

10. Appendix II: Some notions in algebraic (analytic) geometry

The local ring $\mathcal{O}_x$ Noetherian, blabla eventuellt primary decomposition ??? radikal ???

Recall that an ideal J is radical if $\phi^N \in J$ implies that $\phi \in J$. Thus we can represent the geometric object Z both as a certain ideal and as a current.

In this section we state some basic facts about analytic sets. Complete proofs and further results can be found in, e.g.., [**?**], and [**?**].

Weierstrass polynomial. Assume that $f(z, w)$ is holomorphic in a neighborhood of the origin in $\mathbb{C}^{n-1}_z \times \mathbb{C}_w$ and that $f(0,0) = 0$. If f does not vanish identically on the w-axis, i.e., $w \mapsto f(0, w)$ is not identically zero, then we have a unique representation

$$f = Wa$$

where a is nonvanishing,

$$W = w^r + a_{r-1}(z)w^{r-1} + \cdots + a_1(z)r + a_0(z);$$

where $a_j(0) = 0$. The function W is called a Weierstrass polynomial. Notice that after a generic linear change of variables any nontrivial f is of this form.

Let X be a complex manifold. A subset A is called an analytic set (or an analytic variety) if each point has a neighborhood $\mathcal{U}$ such that $A \cap \mathcal{U}$ is the common zero set of a (finite) collection of holomorphic functions in U.

A point $z \in A$ is regular if there is a neighborhood $\mathcal{U}$ of z such that $A \cap U$ is a submanifold of $\mathcal{U}$. The set of regular points is denoted A_{reg}. The complement $A \setminus A_{reg}$ is called the singular part of A. It is a non-trivial fact that $A \setminus A_{reg}$ is in itself an analytic set.

An analytic set A is *irreducible* in X if there are no proper analytic subsets A_1 and A_2 such that $A = A_1 \cup A_2$ and $A_j \subset_{\neq} A$.

We will also consider germs V_x of analytic varieties at x. Notice that finite unions and intersections of such germs are welldefined and are again germs at x. Such a germ V_x is irreducible if it cannot be written as the union $v_x^1 \cup V_x^2$ where $v_x^j \neq V_x$.

If x is a regular point of A, then A is irreducible at x.

If A is any analytic set, then it is irreducible in X if and only if A_{reg} is connected. More generally we have:

THEOREM 10.1. *Given any analytic set A, let $\tilde{A}_j$ be the connected components of A_{reg}, and let A_j be the closure of $\tilde{A}_j$ in A (or equivalently in X). Then each A_j is an irreducible analytic set and $A = \cup A_j$. Moreover, this union is locally finite.*

The varieties A_j are called the irreducible components of A.

EXAMPLE 10.1. Let $V = \{zw = 0\}$ in $\mathbb{C}^2$. Then $V_{reg} = V \setminus \{0, 0)\}$ which is disconnected, and we have $V = V_1 \cup V_2$, where $V_1 = \{w = 0\}$ and $V_2 = \{z = 0\}$. □

EXAMPLE 10.2. Let A be the zero set of $z^3 - w^2$ in $\mathbb{C}^2$. Notice that $A_{reg} = A \setminus \{(0, 0)\}$ by the implicit function theorem, or by direct inspection. We claim that however V is irreducible at $(0, 0)$. Of course this follows from Theorem??? since V_{reg} is connected. However it can be seen directly. Notice that V is parametrized by $\gamma(t) = (t^2, t^3)$, $t \in \mathbb{C}$.

If $V = V_1 \cup V_2$ and $V_1 \subset_{\neq} V_2$, then one could find a function ϕ_1 in a neighborhood $\mathcal{U}$ of $(0,0)$ that vanishes on V_2 but is not identically zero on V_1 in any neighborhood of $(0,0)$. In the same way one can find ϕ_2 with the same property but with V_1 and V_2 interchanged. Thus $\phi_1\phi_2$ vanishes identically on V, and hence $\gamma^*\phi_1 \cdot \gamma^*\phi_2 \equiv 0$ in $\gamma^{-1}(\mathcal{U})$. Thus one of the factors must vanish identically in a neighborhood of 0, and hence one of the ϕ_j must vanish on V in a neighborhood of $(0,0)$, which is a contradiction.

Notice that "above" any $z \in \mathbb{C}_z \setminus \{0\}$ there are presicely two values $w = \pm\sqrt{z^3}$ such that (z, w) lies on V. □

Example 10.3. Consider the curve $t \mapsto (t(t-2), t^2(t-2)) = (z, w)$ in $\mathbb{C}^2$. A simple computation reveals that it is the zero set of the cubic $w^2 - 2zw - z^3$. In fact, since $t = w/z$ we get the relation $z = (w/z)(w/z-2)$. One can show that A is globally irreducible. However, locally at the origin in $\mathbb{C}^2$ it is two curves intersecting transversally, so A is not irreducible at 0. In fact, if $\sqrt{z+1}$ denotes the branch that is close to 1 when z is close to 0, then we have

$$w^2 - 2zw - z^3 = (w - z(1 - \sqrt{1+z}))(w - z(1 + \sqrt{1+z})).$$

□

We say that an irreducible A has codimension p it the manifold A_{reg} has codimension p; since it is connected, the definition is meaningful. For a general A we say it has codimension p if p is the minimal codimension of any of its irreducible components.

If A is irreducible, and ϕ holomorphic; then either $A \subset \{\phi = 0\}$ is empty or has codimension $p + 1$.

However, in general, one needs more than p functions to define a variety A of codimension p. If p is enough we say that A is a complete intersection. In general, if A has pure dimension p one can always locally find $f_1, \ldots, f_p$, defining a complete intersection, so that A is a union of irreducible components of the intersection.

Obs if A is irreducible with codim p and f is holo, then the codim of $A \cap \{f = 0\}$ is $p+1$ or the intersection is empty. Thus, if $f_1, \ldots f_m$ is a complete intersection, then also each subset of these functions must be. (so-called regular sequence!)

For each germ of variety V_x at x we get an ideal

$$I_V \subset \mathcal{O}_x = \{\phi \in \mathcal{O}_x;\ \phi = 0 \text{ on } V_x\}.$$

Conversely, for each ideal $J \subset \mathcal{O}_x$ we get a germ of a variety

$$V_x(J) = \{z;\ \phi(z) = 0 \text{ for all } \phi \in J\}.$$

Clearly, $J \subset I_{V(J)}$ for any ideal $J \subset \mathcal{O}_x$, and Hilbert's Nullestellensatz states that

$$\sqrt{J} = I_{V(J)}.$$

Here $\sqrt{J}$ is the radical, i.e., the set of all $\phi \in \mathcal{O}_x$ such that $\phi^N \in J$ for some power N.

PROPOSITION 10.2. *Let A be the germ of an analytic variety at x. Then A_x is irreducible if and only if $I_A \subset \mathcal{O}_x$ is a prime ideal in $\mathcal{O}_x$ iff there are arbitrary small neighborhood s in which A_{reg} is connected.*

Thus we have a 1-1 correspondence between prime ideals in $\mathcal{O}_x$ and germs of irreducible varieties at x.

Recall that each $f \in \mathcal{O}_x$ has a unique factorization $f = \phi_1^{a_1} \cdots \phi_m^{a_m}$, where ϕ_j are prime elements and a_j are positive integers. Then the hypersurface (i.e., variety with codimension 1) $V(f)$ has the irreducible components $Z_j = V(\phi_j)$. We say that f has the *order* a_j on $V(\phi_j)$.

Any variety Z of pure codimension p is a union of irreducible components of a complete intersection.

EXAMPLE 10.4. The curve $t \mapsto (t^3, t^4, t^5) = (z, w, u)$ is defined by the three binomials $z^4 - w^3, z^5 - u^3, w^5 - u^4$ but one can prove that not just two of them are enough. In fact, the set cannot, even locally at 0, be defined by less than three holomorphic functions, i.e., the local prime ideal at least three generators.

However, V is an irreducible component of the complete intersection $z^4 - w^3, z^5 - u^3$. Besides V it also contains for instance the curve $t \mapsto (t^3, e^{i2\pi/3}t^4, e^{i2\pi/3}t^5)$. $\square$

Suppose that V is the germ of a irreducible variety at $0 \in \mathbb{C}^n$ oc codimension p. After a generic linear change of coordinates, we may assume that $\mathbb{C}^n = \mathbb{C}^{n-p}_z \times \mathbb{C}^p_w$, there is a polydisk $\Delta = \Delta' \times \Delta''$, a positive nubmer r and a variety $Y \subset \Delta'$ such that V is a branched covering of Δ':

$$\pi\colon V \cap \Delta \to \Delta'$$

such that for each $z \in \Delta' \setminus Y$ the fiber $\pi^{-1}(z)$ has precisely r points, for any z $\pi^{-1}(z)$ has at most r points, outside Y, the projection is locally a biholomorphism.

Moreover, $V \cap (\Delta \setminus \pi^{-1}(Y))$ is connected, and contained in V_{reg}.

Analytic space

Structure sheaf ablbalbla

Blowing up along a subvariety, normalization, Hironaka resolution in various formulations,

and examples .....

11. Exercises

EXERCISE 1. Let V be a real $2n$-dimensional vector space (think of V as the cotangent space of $\mathbb{C}^n$ at some fixed point p) and consider the complex $2n$-dimensional vector space $V^{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$, the so-called complexification of V. The notion of wedge product etc extends by (complex-)linearity from V to $V^{\mathbb{C}}$. Let $v_1, \ldots, v_k$ be elements in $V^{\mathbb{C}}$. Show that they are lineraly independent over $\mathbb{C}$ if and only if $v_1 \wedge \ldots \wedge v_k \neq 0$.

EXERCISE 2. Let $n = 1$ and assume that $f = f_1 d\bar{z}$ has compact support. Show that there is a solution u to $\bar{\partial} u = f$ with compact support if and only if

$$\int f \wedge z^k dz = 0$$

for all integers $k \geq 0$.

EXERCISE 3. Show that Hartogs' phenomenon immediately implies that $\bar{\partial} u = f$ has a solution with compact support if f is a $\bar{\partial}$-closed $(0,1)$-form.

EXERCISE 4. Let $\Psi(z)$ be a reasonable function in the unit ball in $\mathbb{C}^n$ (or rather defined in a neighborhood of the closure) and let $\tilde{\psi}(z,w)$ be the function in the ball in $\mathbb{C}^{n+1}$ defined by $\tilde{\psi}(z,w) = \phi(z)$. Show that

$$\int_{|z|^2+|w|^2<1} (1-|z|^2-|w|^2)^{\alpha} \tilde{\psi}(z,w) dV(z,w) = \frac{\pi}{\alpha+1} \int_{|z|<1} (1-|z|^2)^{\alpha+1} \psi(z) dV(z). \tag{11.1}$$

EXERCISE 5. Prove, by (11.1), that the volume of the ball in $\mathbb{C}^n$ is $\pi^n/n!$.

EXERCISE 6. Let h be smooth a $(n, n-1)$-form and let χ_k be a sequence of cutoff functions in D that tends to the characteristic function χ_D. Show that

$$\bar{\partial}\chi_k \wedge h \to -\int_{\partial D} h.$$

EXERCISE 7. Assume that we know (3.4) for all f with compact support in D. Let χ_k be a sequence of cutoff functions in D that tends to the characteristic function χ_D. If f if arbitrary, apply (3.4) to $\chi_k f$ and deduce (3.4) for f.

EXERCISE 8. Show that the linear term in the Taylor expansion of a function $\rho(\zeta)$ at a is $2\mathrm{Re}\,\langle\partial\rho(\zeta), \zeta - a\rangle$. Prove (3.5),

EXERCISE 9. Assume that f is locally integrable in Ω and that

$$\int f \wedge \bar{\partial}\psi = 0$$

for all smooth $(n, n-1)$-forms with compact support. Show that in fact f is holomorphic; to be precise, there is a holomorphic function F in Ω that is equal to f almost everewhere.

EXERCISE 10. Assume that f is an entire function, i.e., $f \in \mathcal{O}(\mathbb{C}^n)$ and such that $|f(z)| = \mathcal{O}(|z|^\gamma)$. Show that f is a polynomial of degree at most γ.

EXERCISE 11. Prove: *Assume that $f \in \mathcal{O}(\Omega)$ and Ω is connected. Then for each interior point a, $|f(a)| < \sup_\Omega |f|$ unless f is constant.*

EXERCISE 12. Let $\{\rho < 0\}$ be a domain in $\mathbb{R}^N$. Show that a $(N-1)$-form α defined in a neighborhood of $\partial D = \{\rho = 0\}$ represents the surface measure on ∂D (with the orientation such that ???) if and only if

$$d\rho \wedge \alpha/|d\rho| = dV.$$

EXERCISE 13. Sometimes it is instructive to compare to the real case, by letting "holomorphic" correspond to "constant" (i.e., homogeneous solution to the Poincare' operator d). Prove analogues of Propositions 2.5 and 2.6 in $\mathbb{R}^n$ for $n > 1$ and compare to the case $n = 1$.

EXERCISE 14. Antag $n = 2$. Suppose that σ and s are $(1,0)$-forms outside 0 such that $s \cdot \zeta = \sigma \cdot \zeta = 1$. Show that

$$\bar{\partial}\sigma \wedge s = \sigma \wedge \bar{\partial}\sigma - s \wedge \bar{\partial}s.$$

Give a new proof of the CFL formula in this case, assuming that it is proved for $s = b$. Try to generalize to higher dimensions.

EXERCISE 15. Let $s = \sum_1^n s_j d\eta_j$ be a form in $\mathbb{C}^{2n}$ such that $\sum_j s_j\eta_j = 1$. Show that
$$\omega(s,\eta) = s \wedge (ds)^{n-1}$$
is d-closed where it is defined. Prove formula (5.2) in Remark 5.1 above. (First show that $ds_1, \dots, ds_n, d\eta_1, \dots, d\eta_n$ must be linearly dependent!)

EXERCISE 16. Let u and v be (p,p)-forms. We say that $u \le v$ if(f) $v - u \ge 0$. Let ω be a positive $(1,1)$-form and let u be any smooth $(1,1)$-form. Show that $u \le C\omega$ for some number C.

EXERCISE 17. Assume that $w(z)$ is a holomorphic coordinate system in $\mathbb{B} \setminus \overline{(1/2)\mathbb{B}}$. Prove that it has an extension to a coordinate system in $\mathbb{B}$.

EXERCISE 18. Use (5.2) to prove that ?????????????????

EXERCISE 19. Assume that $f\colon \Omega \to \Omega'$ is holomorphic and proper and that $f(0) = 0$. Show that $f_*[0] = [0]$.

Let ω be the Cauchy kernel in $\mathbb{C}$ with pole at the origin, and let $K_\sigma = f_*^\sigma \omega$, where $f(\tau) = \tau\sigma$ and $\sigma \in \mathbb{C}^n$, $\sigma \ne 0$. Show that
$$\bar\partial K_\sigma = [0].$$

Define
$$K = \int_{|\sigma|=1} K_\sigma \, d\sigma,$$
where $d\sigma$ is normalized surface nmeasure, and prove that $\bar\partial K = [0]$.

Show that in fact K is the BM kernel.

12. Notes

CHAPTER 2

Integral representation with weights

In this chapter we shall now consider various modifications of the simple integral representations we encountered in the previous chapter. They will admit balbalblaa. It turns out to be very useful to have a functional calculus for forms of even degree.

1. Functional calculus for forms of even degreee

Let E be an m-dimensional vector space and recall that $\Lambda^k E$ consists of all alternating multilinear forms on the dual space E^*. If $v \in E^*$ we define contraction (or interior multiplication) with v, $\delta_v \colon \Lambda^{k+1} E \to \Lambda^k E$, by

$$(\delta_v \omega)(u_1, \dots, u_k) = \omega(v, u_1, \dots, u_k).$$

It is readily checked that this is an alternating form and therefore an element in $\Lambda^k E$. Clearly δ_v is complex-linear in v.

To get a more hands-on idea how δ_v acts, let us choose a basis e_j for E, with dual basis e_j^*, such that $v = e_1^*$. Then $\delta_v(e_1 \wedge e_J) = e_J$ if $1 \notin J$. Thus

$$\delta_v(\alpha \wedge \beta) = \delta_v \alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge \delta_v \beta, \tag{1.1}$$

if $\alpha = e_J$ and $\beta = e_K$. By linearity, then (1.1) holds for arbitrary forms.

Now let $\omega_1, \dots, \omega_m$ be even forms, i.e., in $\oplus_\ell \Lambda^{2\ell} E$, and let $\omega_j = \omega_j' + \omega_j''$ be the decomposition in components of degree zero and positive degree, respectively. Notice that $\wedge$ is commutative for even forms. Thus if $p(z) = \sum_\alpha c_\alpha z^\alpha = \sum_\alpha c_\alpha z_1^{\alpha_1} \cdots z_m^{\alpha_m}$ is a polynomial, then we have a natural definition of $p(\omega)$ as the form $\sum_\alpha c_\alpha \omega_1^{\alpha_1} \wedge \ldots \wedge \omega_m^{\alpha_m}$. However, it is often convenient to use more general holomorphic functions.

Now $\omega' = (\omega_1', \dots, \omega_m')$ is a point in $\mathbb{C}^m$ and for f holomorphic in some neighborhood of ω' we define

$$f(\omega) = \sum_\alpha f^{(\alpha)}(\omega')(\omega'')_\alpha, \tag{1.2}$$

where we use the convention that

$$w_\alpha = \frac{w_1^{\alpha_1} \wedge \dots w_m^{\alpha_m}}{\alpha_1! \cdots \wedge \alpha_m!}.$$

Thus $f(\omega) = f(\omega' + \omega'')$ is defined as the formal power series expansion at the point ω'. Since the sum is finite, $f(\omega)$ is a well-defined form, and if ω depends continuously (smoothly, holomorphically) on some parameter(s), $f(\omega)$ will do as well.

If $f(z) - g(z) = \mathcal{O}((z-\omega')^M)$ for a large enoough M, then $f(\omega) = g(\omega)$.

LEMMA 1.1. *Suppose that $f_k \to f$ in a neighborhood of $\omega' \in \mathbb{C}^m$ and that $\omega_k \to \omega$. Then $f_k(\omega_k) \to f(\omega)$.*

PROOF. In fact, by the Cauchy estimate, Ch 1, Example 3.3, $f_k^{(\alpha)} \to f^{(\alpha)}$ uniformly for each α in a slightly smaller neighborhood . Therefore, α, $f_k^{\alpha)}(\omega_k') \to f^{(\alpha)}(\omega_k')$ for each α. It follows that

$$f_k(\omega_k) - f(\omega) = f_k(\omega_k) - f(\omega_k) + f(\omega_k) - f(\omega) \to 0$$

since only a finite number of derivatives come into play. $\square$

Clearly

$$(af + bg)(\omega) = af(\omega) + bg(\omega), \quad a, b \in \mathbb{C},$$

and moreover we have

PROPOSITION 1.2. *If p is a polynomial, then the definition above of $p(\omega)$ coincides with the natural one. If f, g are holomorphic in a neighborhood of ω', then*

$$(fg)(\omega) = f(\omega) \wedge g(\omega). \tag{1.3}$$

If f is holomorphic in a neighborhood of ω' (possibly $\mathbb{C}^r$-valued) and h is holomorpic in a neighborhood of $f(\omega')$, then

$$(h \circ f)(\omega) = h(f(\omega)). \tag{1.4}$$

If v is in E^*, then

$$\delta_v f(\omega) = \sum_1^m \frac{\partial f}{\partial z_j}(\omega) \wedge \delta_v \omega_j, \tag{1.5}$$

and if ω depends on a parameter, then

$$df(\omega) = \sum_1^m \frac{\partial f}{\partial z_j}(\omega) \wedge d\omega_j. \tag{1.6}$$

PROOF. For the first statement, with no loss of generality, we may assume that $\omega' = 0$, and $p(z) = z^\beta$. Then $p^{(\alpha)}(0)(\omega'')^\alpha$ vanishes for $\alpha \neq \beta$ and equals $(\omega'')^\beta$ for $\alpha = \beta$. By linearity the first statement follows.

Now (1.3) clearly holds for polynomials, and since we can approximate f, g with polynomials f_k, g_k in $\mathcal{O}(\{0\})$, the general case follows from Lemma 1.1. One can obtain (1.4) in a similar way, noting that if $\tau_k = f_k(\omega)$ and $h_k \to h$ in a neighborhood of $f(\omega')$, then $h_k(\tau_k) \to h(\tau) = h(f(\omega))$, and $h_k(\tau_k) = (h_k \circ f_k)(\omega) \to (h \circ f)(\omega)$.

The remaining statements also clearly hold for polynomials and hence in general. $\square$

EXAMPLE 1.1. Since

$$\frac{1}{1-z} = 1 + z + z^2 +$$

in a the unit disk, if ω is an even form and $|\omega'| < 1$, we have

$$\frac{1}{1-\omega} = 1 + \omega + \omega^2 + \omega^3 + \cdots ,$$

and $(1-\omega)[1/(1-\omega)] = 1$. If in addition $\omega' = 0$ we have

$$\frac{1}{1-\omega} = 1 + \omega + \omega^2 + \omega^3 + \cdots \omega^n.$$

If ω_1 and ω_2 are even forms, then

$$e^{\omega_1+\omega_2} = e^{\omega_1} \wedge e^{\omega_2}.$$

In fact, if $f(z_1, z_2) = z_1 + z_2$, $\pi_j(z_1, z_2) = z_j$, then $(\exp \circ f) = (\exp \pi_1)(\exp \pi_2)$, and hence by (1.3) and (1.4),

$$LHS = \exp(f(\omega_1, \omega_2)) = (\exp \circ f)(\omega_1, \omega_2) = \pi_1(\omega_1, \omega_2) \wedge \pi_2(\omega_1, \omega_2) = RHS,$$

since (clearly) $\pi_j(\omega_1, \omega_2) = \omega_j$.

Of course, in both cases one can easily check the statement directly as well. $\square$

2. A general Cauchy-Fantappiè-Leray formula

We have already seen that the CFL formula is sort of a substitute for the Cauchy formula in several variables. We shall now see that the proper generalization of the Cauchy kernel is a certain cohomology class.

If U is an open subset of $\mathbb{C}^n$, or any complex manifold, then

$$H^{p,q}_{\bar\partial}(U) = \frac{\{f \in \mathcal{E}_{p,q}(U);\ \bar\partial f = 0\}}{\{\bar\partial h;\ h \in \mathcal{E}_{p,q-1}(U)\}}$$

is the Dolbeault cohomology group (or rather space) of bidegree (p, q). In case $q = 0$ the denominator is interpreted as 0, so $H^{p,0}(U)$ is just the space of holomorphic $(p, 0)$-forms in U. The interest of these spaces of course depends on the fact that they are non-trivial in general, i.e., there are $\bar\partial$-equations that are not solvable.

We need some more notation. Let $\mathcal{E}_{p,q}(U)$ denote the space of smooth (p, q) forms in the open set $U \subset \mathbb{C}^n$ and, for any integer m, let $\mathcal{L}^m(U) = \oplus_{k=0}^n \mathcal{E}_{k,k+m}(U)$. For instance, $u \in \mathcal{L}^{-1}(U)$ can be written $u = u_{1,0} + \ldots + u_{n,n-1}$, where the index denotes bidegree in dz.

Fix a point $a \in \mathbb{C}^n$, let $\delta_{z-a} \colon \mathcal{E}_{p,q}(U) \to \mathcal{E}_{p-1,q}(U)$ be contraction with the vector field

$$2\pi i \sum_1^n (z_k - a_k)\frac{\partial}{\partial z_k}.$$

We claim that

$$\delta_{z-a}\bar\partial f = -\bar\partial\delta_{z-a} f. \tag{2.1}$$

In fact, by linearity it is enough to check for f of the form $f = \phi\gamma$, where ϕ is a function and $\gamma = dz_I \wedge d\bar z_J$. We have $\bar\partial\delta f = \bar\partial(\phi\delta\gamma) = \bar\partial\phi \wedge \delta\gamma$ since $\delta\gamma$ is $\bar\partial$-closed, and $\delta\bar\partial f = \delta(\bar\partial\phi \wedge \gamma) = -\bar\partial\phi \wedge \delta\gamma$, since $\bar\partial\phi$ is a $(0, 1)$-form.

We now let

$$\nabla = \nabla_{z-a} = \delta_{z-a} - \bar\partial.$$

Then

$$\nabla_{z-a} \colon \mathcal{L}^m(U) \to \mathcal{L}^{m+1}(U).$$

The usual wedge product extends to a mapping $\mathcal{L}^m(U) \times \mathcal{L}^{m'}(U) \to \mathcal{L}^{m+m'}(U)$, such that $g \wedge f = (-1)^{mm'} f \wedge g$, and ∇_{z-a} satisfies the same formal rules as the usual exterior differentiation, i.e., ∇_{z-a} is a anti-derivation,

$$\nabla_{z-a}(f \wedge g) = \nabla_{z-a} f \wedge g + (-1)^m f \wedge \nabla_{z-a} g, \quad f \in \mathcal{L}^m(U). \tag{2.2}$$

Moreover, in view of (2.1) we have

$$\nabla_{z-a}^2 = \nabla_{z-a} \circ \nabla_{z-a} = 0.$$

Notice that the Cauchy kernel $u(z) = dz/2\pi i(z - a)$ satisfies

$$2\pi i(z - a)u(z) = dz \quad \text{and} \quad \bar\partial u = [a], \tag{2.3}$$

where $[a]$ denotes the current integration (evaluation) at a. In order to generalize Cauchy's formula to higher dimensions it is natural to look for forms u that satisfy the second equation in (2.3), since each such solution gives rise to a representation formula by Stokes' theorem, cf., Chapter 1. However, to find such solutions in several variables it turns out to be appropriate to look for multivariable analogues to (2.3) outside the point a to begin with.

Notice that (2.3) can be written

$$\nabla_{z-a}u(z) = 1 - [a]. \tag{2.4}$$

If $n > 1$, (3.17) means that

$$\delta_{z-a}u_{1,0} = 1, \quad \delta_{z-a}u_{k+1,k} - \bar\partial u_{k,k-1} = 0, \quad \bar\partial u_{n,n-1} = [a]. \tag{2.5}$$

EXAMPLE 2.1. Let $s = \sum_1^n s_j dz_j$ be a $(1,0)$-form in U such that $\delta_{z-a}s = 2\pi i \sum_j s_j(z_j - a_j) = 1$. Then clearly we must have $a \notin U$. Since the component of zero degree of the form $\nabla_{z-a}u$ is nonvanishing, we can define the form

$$u = \frac{s}{\nabla s}$$

and we claim that

$$\nabla_{z-a}u = 1 \tag{2.6}$$

in U. In fact, by the functional calculus for forms we have

$$\nabla_{z-a}u = \frac{\nabla_{z-a}s}{\nabla_{z-a}s} + \frac{s}{(\nabla_{z-a}s)^2}\nabla^2_{z-a}s = 1$$

since $\nabla^2_{z-a} = 0$.

More explicitly

$$u = \frac{s}{\nabla_{z-a}s} = \frac{s}{1-\bar\partial s} = s \wedge [1 + \bar\partial s + (\bar\partial s)^2 + \cdots + (\bar\partial s)^{n-1}] =$$
$$s + s\wedge\bar\partial s + s\wedge(\bar\partial s)^2 + \cdots + s\wedge(\bar\partial s)^{n-1}.$$

The sceptical reader of course can confirm (2.6) by a direct computation. Notice that highest order term is precisely the CFL form in Corollary 3.3 in Ch. 1. □

PROPOSITION 2.1. *If*

$$b(z) = \frac{1}{2\pi i}\frac{\partial|z|^2}{|z|^2},$$

then the form

$$u_{BM} = \frac{b}{\nabla_z b} = \frac{b}{1-\bar\partial b} = b \wedge \sum_1^n (\bar\partial b)^{k-1},$$

is locally integrable in $\mathbb{C}^n \setminus \{0\}$ *and satisfies* (3.17) *(with* $a = 0$*).*

We will refer to u_{BM} as the (full) Bochner-Martinelli form.

PROOF. A simple computation yields

$$u_{k,k-1} = b \wedge (\bar\partial b)^{k-1} = \frac{\partial |z|^2 \wedge (\bar\partial \partial |z|^2)^{k-1}}{(2\pi i)^k |z|^{2k}},$$

so $u_k = \mathcal{O}(|z|^{-(2k-1)})$ and hence locally integrable. We already know that $\bar\partial u_{n,n} = [0]$; this is precisely Lemma 3.1 in Ch. 1, so what remains is to verify that

$$-\int \bar\partial\phi \wedge u_{k,k-1} = \int \phi \wedge \delta_z u_{k+1,k}, \quad \phi \in \mathcal{D}_{n-k,n-k}(\mathbb{C}^n). \tag{2.7}$$

However,

$$-\int_{|z|>\epsilon} \bar\partial\phi \wedge u_{k,k-1} = \int_{|z|=\epsilon} \phi \wedge u_{k,k-1} + \int_{|z|>\epsilon} \phi \wedge \delta_z u_{k+1,k}.$$

Moreover, since $k < n$, $u_{k,k-1} = \mathcal{O}(|z|^{-(2n-3)})$, and hence the boundary integral tends to zero when $\epsilon \to 0$. Thus (2.7) follows. □

LEMMA 2.2. *Suppose that $a \notin U$. If f is any form in U such that $\nabla_{z-a} f = 0$, then there is a form w such that $\nabla_{z-a} w = f$.*

PROOF. In fact, $u(z) = u_{BM}(z-a)$ is smooth in U and $\nabla_{z-a} u = 1$. Thus $\nabla_{z-a}(u \wedge f) = f$. □

We are now ready to prove the main result of this section, stating that the proper generalization of the Cauchy form from one variable is a certain cohomology class ω_{z-a}.

PROPOSITION 2.3. *Suppose that $a \in D$ and $a \notin U \supset \partial D$. If $u \in \mathcal{L}^{-1}(U)$ and $\nabla_{z-a} u = 1$, then $\bar\partial u_{n,n-1} = 0$. All such forms $u_{n,n-1}$ define the same Dolbeault cohomology class ω_{z-a} in U and any representative for ω_{z-a} occurs in this way. For any representative we have that*

$$\phi(a) = \int_{\partial D} \phi(z) u_{n,n-1}, \quad \phi \in \mathcal{O}(\overline{D}). \tag{2.8}$$

PROOFS. If $\nabla_{z-a} u = 1$ then $\bar\partial u_n = 0$. If u' is another solution then $\nabla_{z-a}(u - u') = 0$ and since $a \notin U$ there is a solution to $\nabla_{z-a} w = u - u'$, and hence $\bar\partial w_{n,n-2} = u'_{n,n-1} - u_{n,n-1}$. If u is a fixed solution and ψ is a $(n, n-2)$ form, then $u' = u - \nabla_{z-a}\psi$ is another solution, and $u'_{n,n-1} = u_{n,n-1} + \bar\partial\psi$.

If $u'_{n,n-1} - u_{n,n-1} = \bar\partial w_{n,n-2}$ in U and ϕ is holomorphic, then

$$d(\phi w_{n,n-2}) = \phi u'_{n,n-1} - \phi u_{n,n-1}.$$

Therefore, Stokes' theorem, applied to the compact manifold ∂D, implies that (2.8) is unchanged if u_n is replaced by u'_n. If $u(z) = u_{BM}(z-$

a) is the Bochner-Martinelli form (translated by a), then we know that (2.8) holds, and hence it holds in general.

Since the Bochner-Martinelli form u is a Cauchy form, (2.8) and (2.9) follow immediately. Since any other u_n is cohomologous in U, (2.8) and (2.9) follow in general. $\square$

Notice that ∇_{z-a} is defined on currents as well. We have an analogous result for nonsmooth forms, and leave the proof as an exercise, see Exercise 23

PROPOSITION 2.4. *With the same notation as in the previous proposition, let u be a current solution to $\nabla_{z-a}u = 1$ in U. If χ is a cutoff function that is 1 in a neighborhood of a and such that the support of $\bar\partial\chi$ is contained in U, then*

$$\phi(a) = -\int \bar\partial\chi \wedge \phi u_{n,n-1}, \tag{2.9}$$

for ϕ that are holomorphic in some neighborhood of the support of χ.

With the choice of u from Example 2.1, (2.8) is just the CFL formula. However, (2.8) admits other realizations.

EXAMPLE 2.2. If we have several $(1,0)$-forms $s^1, \ldots, s^n$ such that $\delta_{z-a}s^j = 1$ we can get a solution u to $\nabla_{z-a}u = 1$ by letting $u_1 = s^1$, $u_{k+1} = s^{k+1} \wedge \bar\partial u_k$. Then $s^n \wedge \bar\partial s^{n-1} \wedge \ldots \wedge \bar\partial s^1$ is a representative for ω_{z-a}; the corresponding representation formula appeared in [**?**]. $\square$

EXAMPLE 2.3. Let us define the current $u = u_{1,0} + \cdots + u_{n,n-1}$ in $\mathbb{C}^n$ by

$$u_1 = \frac{1}{2\pi i}\frac{dz_1}{z_1 - a_1}, \quad u_k = \left(\frac{1}{2\pi i}\right)^k \frac{dz_k}{z_k - a_k} \wedge \bar\partial u_{k-1}.$$

The products are well-defined since they are just tensor products of distributions. From Proposition 2.4 we get the representation formula

$$f(a) = -(2\pi i)^{-1} \int_{z_n} \bar\partial_{z_n}\chi(\ldots, a_{n-1}, z_n) \wedge f(\ldots, a_{n-1}, z_n)\frac{dz_n}{z_n - a_n}.$$

Of course, this formula follows immediately from the one-variable Cauchy formula. $\square$

If we have a solution to $\nabla_{z-a}u = 1$ in $\Omega \setminus \{a\}$, that has a current extension across a it is natural to ask whether (3.17) holds.

PROPOSITION 2.5. *Suppose that $u \in \mathcal{L}^{-1}(\Omega \setminus \{a\})$ solves $\nabla_{z-a}u = 1$ in $\Omega \setminus \{a\}$ and that $|u_k| \lesssim |z-a|^{-(2k-1)}$. Then u is locally integrable and (3.17) holds.*

PROOF. If u^1 and u^2 both satisfy the growth condition, then $u^1 \wedge u^2 = \mathcal{O}(|z-a|^{-(2n-2)})$ and $\nabla_{z-a}(u^1 \wedge u^2) = u^2 - u^1$ pointwise outside a. Hence it holds in the current sense. If (3.17) holds for one of them it thus holds for both; taking one of them as the Bochner-Martinelli form, the proposition follows. $\square$

EXAMPLE 2.4. Let $s(z)$ be a smooth $(1,0)$-form in Ω such that $|s(z)| \leq C|z-a|$ and $|\delta_{z-a}s(z)| \geq C|z-a|^2$. Then $u = s/\nabla_{z-a}s$ satisfies the hypothesis and therefore (3.17) holds. $\square$

EXAMPLE 2.5. Any (locally integrable) form u in a neighborhood of a such that (3.17) holds, can be extended to a global such form in Ω. In fact, if v is any solution to $\nabla_{z-a}v = 1$ outside a, then $\nabla_{z-a}(u-v) = 0$ where it is defined, and hence we have w such that $\nabla_{z-a}w = u - v$ there. If χ is an appropriate cutoff function, then $u' = \chi u + (1-\chi)v + \bar\partial\chi \wedge w$ is a global form that coincides with u close to a. $\square$

3. Weighted representation formulas

We now introduce weighted formulas. Let z be a fixed point in $\Omega \subset \mathbb{C}^n$. A smooth form $g \in \mathcal{L}^0(\Omega)$ such that $\nabla_{\zeta-z}g = 0$ and $g_{0,0}(z) = 1$ will be referred to as a *weight* (with respect to the point z).

THEOREM 3.1. *If g is a weight in Ω, $z \in D \subset\subset \Omega$, and $\nabla_{\zeta-z}u = 1$ in a neighborhood U of ∂D, then*

$$\phi(z) = \int_{\partial D} (g \wedge u)_{n,n-1}\phi + \int_D g_{n,n}\phi, \qquad \phi \in \mathcal{O}(\overline{D}). \tag{3.1}$$

Notice that

$$(u \wedge g)_{n,n-1} = \sum_{k=1}^{n} u_{k,k-1} \wedge g_{n-k,n-k}.$$

One can replace $u \wedge g$ in the theorem by any solution v to $\nabla_{\zeta-z}v = g$ in U, see Exercise **??**.

COROLLARY 3.2. *If g is a weight (with respect to z) with compact support in Ω, then*

$$\phi(z) = \int \phi g_{n,n}, \qquad \phi \in \mathcal{O}(\Omega). \tag{3.2}$$

PROOF. First suppose that u is chosen so that (3.17) holds (with $a = z$). Then

$$\nabla_{\zeta-z}(u \wedge g) = (1 - [z]) \wedge g = g - [z]$$

since $[z] \wedge g = [z]g_{0,0} = [z]$ and $g_{0,0}(z) = 1$. Thus $d(u \wedge g)_{n,n-1} = \bar\partial(u \wedge g)_{n,n-1} = g_n - [z]$ in the current sense and hence

$$\int_{\partial D} (u \wedge g)_n \phi = \int_D d[(u \wedge g)_n \phi] = \int_D ([z] - g_n))\phi = \phi(z) - \int_D g\phi,$$

and so (3.1) holds for this particular choice of u. If now u' is an arbitrary solution to $\nabla_{\zeta - z} u' = 1$ in U, then

$$\nabla_{\zeta-z}(u \wedge u' \wedge g) = u' \wedge g - u \wedge g,$$

and hence

$$d(u \wedge u' \wedge g)_{n,n-2} = (u \wedge g)_{n,n-1} - (u' \wedge g)_{n,n-1}.$$

Thus the general formula follows from Stokes' theorem. $\square$

The collection of weights clearly is closed under convex combinations and (wedge) products. More generally, if $g^1, \dots, g^m$ are weights, and $G(\lambda_1, \dots, \lambda_m)$ is holomorphic on the image of $z \mapsto (g^1_0, \dots, g^m_0)$, and $G(1, \dots, 1) = 1$, then $g = G(g^1, \dots, g^m)$ is another weight. If q is any smooth form in $\mathcal{L}^{-1}(\Omega)$ then $g = 1 + \nabla_{z-a} q$ clearly is a weight. In fact, any weight is of this form, see Exercise **??**.

Example 3.1. Let q be a $(1,0)$-form in D, and assume that $G(\lambda)$ is holomorphic on the image of $\zeta \mapsto \delta_{\zeta-z} q(\zeta)$ and $G(0) = 1$. Then

$$g = G(\nabla_{\zeta-z} q) = G((\delta_{\zeta-z} - \bar\partial)q) = \sum_{k=0}^{n} G^{(k)}(\delta_{\zeta-z} q)(-\bar\partial q)^k / k! \tag{3.3}$$

is a weight.

In case g is a weight as in Example 3.1 and u is a form as in Proposition 2.5, then $g_{n,n}$ and $(u \wedge g)_{n,n-1}$ are precisely the kernels P and K from [**?**]. $\square$

Example 3.2. Let χ be a cutoff function in Ω that is 1 in a neighborhood of z and let u be a (smooth outside z) Cauchy form. Then $h = \chi - \bar\partial\chi \wedge u$ is a smooth weight that has compact support in Ω. If u depends holomorphically of z (close to some fixed point say) then h will do as well. Notice that (3.1) with this choice of weight gives us back (2.9). For further reference also notice that $w = (1 - \chi)u$ is smooth in Ω and that $\nabla_{\zeta-z} w = 1 - h$. $\square$

Example 3.3. Let $\mathbb{B}$ be the ball and let χ be a cutoff function in $\mathbb{B}$ that is identically 1 in a neighborhood of the closure of $r\mathbb{B}$ and take

$$s = \frac{\partial |\zeta|^2}{2\pi i (|\zeta|^2 - \bar\zeta \cdot z)}.$$

Then

$$g = \chi - \bar\partial\chi \wedge \frac{s}{\nabla_{\zeta - z} s} = \chi - \bar\partial\chi \wedge [s + s \wedge \bar\partial s + \cdots + s \wedge (\bar\partial)^{n-1}]$$

is a weight with compact support that depends holomorphically on $z \in r\mathbb{B}$. $\square$

EXAMPLE 3.4. Let g be a current in Ω with compact support such that $\nabla_{\zeta - z} g = 0$. Moreover, assume that g is smooth in a neighborhood of z and that $g_0(z) = 1$. Then (3.2) still holds; this follows by exactly the same proof. If $z \in D$, then a possible choice of weight (with respect to z) is

$$g_D = \chi_D - \bar\partial\chi_D \wedge u,$$

where u is a Cauchy form in a neighborhood of ∂D. Then (3.2) is just formula (2.8).

If g is any smooth weight, then $g_D \wedge g$ is a new weight, and in this case (3.2) becomes (3.1). $\square$

As we have seen, the CFL formula represents a holomorphic function in a domain D in terms of its values on ∂D. However, this requires that the function has some reasonable boundary values. To admit a representation for a larger class of functions one can use a weighted formula. Here we will exemplify with the ball; in a subsequent section we will consider more general domains.

EXAMPLE 3.5 (Weighted Bergman projections in the ball). Notice that

$$1 + \nabla_{\zeta - z} \frac{\partial |\zeta|^2}{2\pi i (1 - |\zeta|^2)} = \frac{1 - \bar\zeta \cdot z}{1 - |\zeta|^2} - \frac{1}{2\pi i} \bar\partial \frac{\partial |\zeta|^2}{1 - |\zeta|^2}.$$

Therefore, as long as $z, \zeta \in \mathbb{B}$, for any complex α,

$$g = \Big(\frac{1 - \bar\zeta \cdot z}{1 - |\zeta|^2} - \frac{1}{2\pi i} \bar\partial \frac{\partial |\zeta|^2}{1 - |\zeta|^2} \Big)^{-\alpha},$$

is welldefined, and in fact a weight. If $\operatorname{Re}\alpha$ is large (in fact > 1 is enough), then g vanishes on $\partial\mathbb{B}$ (for fixed z of course) and is then a weight with compact support. More specifically, if

$$\omega = \frac{i}{2} \partial \frac{\bar\partial |\zeta|^2}{1 - |\zeta|^2}, \quad \omega_k = \omega^k / k!,$$

then

$$g_{n,n} = c_\alpha \Big(\frac{1 - |\zeta|^2}{1 - \bar\zeta \cdot z} \Big)^{\alpha + n} \omega_n,$$

where

$$c_\alpha = (-1)^n n! \frac{1}{\pi^n} \frac{\Gamma(-\alpha + 1)}{\Gamma(n+1)\Gamma(-\alpha - n + 1)}.$$

Using that $\Gamma(n+1)=n!$ and $\Gamma(\tau+1)=\tau\Gamma(\tau)$ one gets

$$c_\alpha = \frac{1}{\pi^n}\frac{\Gamma(n+\alpha)}{\Gamma(\alpha)}.$$

Moreover,

$$\omega_n = \big(\frac{i}{2}\big)^n\Big(\frac{\partial\bar\partial|\zeta|^2}{1-|\zeta|^2}+\frac{\partial|\zeta|^2\wedge\bar\partial|\zeta|^2}{(1-|\zeta|^2)^2}\Big)_n =$$
$$\big(\frac{i}{2}\big)^n\frac{(1-|\zeta|^2)(\partial\bar\partial|\zeta|^2)_n+\partial|\zeta|^2\wedge\bar\partial|\zeta|^2\wedge(\partial\bar\partial|\zeta|^2)_{n-1}}{(1-|\zeta|^2)^{n+1}} =$$
$$\frac{dV(\zeta)}{(1-|\zeta|^2)^{n+1}},$$

where we have used (???) and the fact that

$$\partial|\zeta|^2\wedge\bar\partial|\zeta|^2\wedge(\partial\bar\partial|\zeta|^2)_{n-1} = |\zeta|^2(\partial\bar\partial|\zeta|^2)_n. \tag{3.4}$$

To see this, notice that

$$(\partial\bar\partial|\zeta|^2)_{n-1} = \sum_{j=1}^n\bigwedge_{k\neq j} d\zeta_k\wedge d\bar\zeta_k$$

and

$$\partial|\zeta|^2\wedge\bar\partial|\zeta|^2 = \sum_\ell \bar\zeta_\ell d\zeta_\ell\wedge\sum_\nu \zeta_\nu d\bar\zeta_\nu.$$

It is now easy to see that (3.4) holds.

It is clear that all terms in g will vanish on the boundary if $\operatorname{Re}\alpha$ is large. Summing up we get, for $\operatorname{Re}\alpha$ large the representation

$$\phi(z) = \frac{\Gamma(n+\alpha)}{\pi^n\Gamma(\alpha)}\int_{|\zeta|<1}\frac{\phi(\zeta)(1-|\zeta|^2)^{\alpha-1}dV(\zeta)}{(1-\bar\zeta\cdot z)^{n+\alpha}}, \quad \phi\in\mathcal{O}(\overline{\mathbb{B}}). \tag{3.5}$$

However, $(1-|\zeta|^2)=(1+|\zeta|)(1-|\zeta|)\sim 2(1-|\zeta|)$, i.e., roughly speaking the distance from ζ to the boundary, and hence the right hand side has meaning as a convergent integral for all α with $\operatorname{Re}\alpha>0$. Moreover, it depends holomorphically on α, and by the uniqueness theorem the equality must hold for all α with $\operatorname{Re}\alpha>0$.

We will see later that it holds for all ϕ that belongs to $L^2((1-|\zeta|^2)^{\alpha-1}dV)$. □

We conclude this section with another type of application that will be elaborated further on, which is called a ?? division formula. Roughly speaking, given holomorphic functions $f_1,\dots,f_m$ and ϕ one looks for holomorphic functions $u_1, \dots, u_m$ such that

$$\phi = \sum_1^m f_j u_j,$$

that is, an explicit realization of ϕ as a member of the ideal $(f) = (f_1, \dots, f_m)$ generated by f_j. Of course this is not always possible, but in the first place we will assume that the functions f_j have no common zeros. balbal

LEMMA 3.3. *Suppose that Ω is a convex domain in $\mathbb{C}^n$ and f is holomorphic. Then there are holomorphic functions $h^j(\zeta, z)$ in $\Omega \times \Omega$, $j = 1, \dots, n$, such that*

$$\sum_1^n (\zeta_j - z_j) h^j(\zeta, z) = f(\zeta) - f(z).$$

If f is a polynomial we can choose h^j as polynomials.

Notice that if we define the $(1,0)$-form $h = \sum_1^n h^j(\zeta, z) d\zeta_j / 2\pi i$, then

$$\delta_{\zeta - z} h = f(\zeta) - f(z).$$

PROOF. We have

$$f(\zeta) - f(z) = \int_0^1 \frac{d}{dt} f(z + t(\zeta - z)) dt = \int_0^1 \sum_1^n \frac{\partial f}{\partial w_j}(z + t(\zeta - z))(\zeta_j - z_j) dt,$$

so we can take

$$h^j(\zeta, z) = \int_0^1 \frac{\partial f}{\partial w_j}(z + t(\zeta - z)) dt.$$

It is clear from this formula that h^j is a polynomial if f is. $\square$

EXAMPLE 3.6 (Berndtsson's division formula). Suppose we have given holomorphic $f_1, \dots, f_m$ in a convex domain Ω, and assume for simplicity that they have no common zeros, i.e.,

$$|f|^2 = \sum_1^n |f_j|^2 > 0.$$

Let h_j be Hefer forms, so that

$$\delta_{\zeta - z} h_j = f_j(\zeta) - f_j(z).$$

Moreover, let

$$\sigma_j = \frac{\bar{f}_j}{|f|^2}.$$

If $f = (f_1, \dots, f_m)$ etc then

$$f \cdot \sigma = 1. \tag{3.6}$$

Now

$$\nabla_{\zeta - z}(f(z)\cdot\sigma - h\cdot\bar\partial\sigma) = -f(z)\cdot\bar\partial\sigma - (f - f(z))\cdot\bar\partial\sigma = -f\cdot\bar\partial\sigma = -\bar\partial(f\cdot\sigma) = 0,$$

in view of (3.6). Moreover, the value of the scalar term is 1 at $\zeta = z$. Therefore, for each natural number μ,

$$g = (f(z)\cdot\sigma - h\cdot\bar\partial\sigma)^\mu$$

is a weight. We now claim that

$$(h\cdot\bar\partial\sigma)^\nu = 0$$

if $\nu \geq \min(m, n+1)$. In fact, if $\nu \geq n+1$, then it vanishes since we have too many differentials of $d\bar\zeta$. If $\nu \geq m$ it vanishes since $\sum f_j\bar\partial\sigma_j = 0$ so $\bar\partial\sigma_1, \dots, \bar\partial\sigma_m$ are linearly dependent, cf., ???? in Ch. 1. If $\mu \geq \min(m, n+1)$ we thus have

$$g = f(z)\cdot\sigma(-h\cdot\bar\partial\sigma)^{m-1} + \cdots = f(z)\cdot A(\zeta, z).$$

If g' is any weight with compact support, cf., Example 3.10, we then get the representation

$$\phi(z) = f(z)\cdot\int A(\zeta, z)\wedge g', \quad \phi\in\mathcal{O}(\Omega).$$

If we are able to choose g' so that it depends holomorphically on z, we thus get a holomorphic realization of ϕ as an element in the ideal (f) generated by f. □

As an application of the preceding example we prove the following well-known balbla

A compact subset of $\mathbb{C}^n$ is called polynomially convex if for each $z \notin K$ there is a polynomial p such that $|p(z)| > \sup_K |p|$.

We leave it as an exercise to check that a compact subset of the plane $\mathbb{C}$ is polynomially convex if and only if $\mathbb{C}\setminus K$ is connected; i.e., K has no "holes". In several variables, however, polynomial convexity is much more involved. Given any compact subset K there is a smallest polynomially convex set $\hat K$ that contains K; $\hat K$ is called the polynomially convex hull of K. There are deep open questions about there are interesting open questions ???? balbalblabla

THEOREM 3.4. *Assume that $K \subset \mathbb{C}^n$ is polynomially convex. Then any $\phi \in \mathcal{O}(K)$ can be approximated uniformly by polynomials on K.*

PROOF. Take $\phi \in \mathcal{O}(K)$ and let U be an open neighborhood of K such that $\phi \in \mathcal{O}(U)$. Moreover, let D be a domain with smooth boundary such that $K \subset D \subset\subset U$. Alternatively one chooses a cutoff function χ in $\mathcal{U}$ that is 1 in a neighborhood of K.

For any point z on ∂D (or on $\operatorname{supp} \chi\bar{\partial}\chi$) one can find a polynomial f_z such that $|f_z(z)| > 1$ and $|f_z| < 1$ on K. By continuity $|f_z| > 1$ in a neighborhood ω_z of z. By compactness we can select a finite number of polynomials $f_1, \ldots, f_m$ such that $|f_j| > 1$ in ω_j, $|f_j| < 1$ on K and $\cup\omega_j \supset \partial D$ (or $\supset \operatorname{supp} \chi\bar{\partial}\chi$). Replacing each f_j with a large power (here we use the fact that the polynomials constitute an algebra) we may assume that $|f| > 1$ on ∂D and $|f| < 1/2$ on K. Moreover, possibly after adding an additional (e.g., constant) function we may assume that $|f| > 0$.

Since f_j are polynomials we can choose Hefer forms h_j that are polynomials in z; in fact the construction described above provides such h_j. Letting u^z be the BM form with pole at z, we get the representation

$$\phi(z) = \int_D (f(z)\sigma - h \cdot \bar{\partial}\sigma)^\mu \phi + \int_{\partial D} (f(z)\sigma - h \cdot \bar{\partial}\sigma)^\mu \wedge u\phi.$$

Thus the first integral is a polynimal in z, so we have to show that the boundary integral can be made arbitrarily small.

Since there is a positive distance between K and ∂D, we have that $|u| \leq C$ for $z \in K$ and $\zeta \in \partial D$. Moreover, $\sigma(\zeta)| \leq 1/|f(\zeta)|$ and hence

$$|(f(z)\sigma - h \cdot \bar{\partial}\sigma)^\mu| \leq C \sum_{k=0}^{n} |f(z) \cdot \sigma|^{\mu-k} |h \cdot \bar{\partial}\sigma|^k \frac{\mu!}{(\mu-k)!k!} \leq C\frac{\mu^n}{2^\mu}$$

for $z \in K$ and $\zeta \in \partial D$, where C does not depend on μ. Since we can choose μ as large as we want, the theorem follows. $\square$

4. Weighted Koppelman formulas

Let X be a domain in $\mathbb{C}^n$ and let

$$\eta = \zeta - z = (\zeta_1 - z, \ldots, \zeta_z - z_n)$$

in $X_\zeta \times X_z$. Let E be the subbundle of $T^*(X \times X)$ spanned by $T^*_{0,1}(X \times X)$ and the differentials $d\eta_1, \ldots, d\eta_n$. In this section all forms will take values in ΛE. We let δ_η denote formal interior multiplication with

$$2\pi i \sum_1^n \eta_j \frac{\partial}{\partial \eta_j},$$

on this subbundle, i.e., such that $(\partial/\partial\eta_j) d\eta_k = \delta_{jk}$. Moreover, we let

$$\nabla_\eta = \delta_\eta - \bar{\partial}.$$

Let

$$b = \frac{\eta \cdot d\eta}{2\pi i |\zeta|^2} = \frac{\sum_j (\bar{\zeta}_j - \bar{z}_j) d\eta_j}{2\pi i |\zeta - z|^2} = \frac{\partial |\zeta|^2}{2\pi i |\zeta|^2}.$$

and consider the Bochner-Martinelli form

$$u = \frac{b}{\nabla_\eta b} = b + b \wedge (\bar{\partial} b) + \cdots + b \wedge (\bar{\partial} b)^{n-1}.$$

Notice that

$$u_{k,k-1} = b \wedge (\bar{\partial} b)^{k-1} = \frac{1}{(2\pi i)^k} \frac{\partial |\zeta|^2 \wedge (\bar{\partial}\partial |\zeta|^2)^{k-1}}{|\eta|^{2k}}$$

so that

$$u_{k,k-1} = \mathcal{O}(1/|\zeta|^{2k-1}). \tag{4.1}$$

PROPOSITION 4.1. *The form $u = b/\nabla_\eta b$ is locally integrable in $\mathbb{C}^n \times \mathbb{C}^n$ and it solves*

$$\nabla_\eta u = 1 - [\Delta] \tag{4.2}$$

in the current sense.

We already know that $\bar{\partial} u_{n,n} = [\Delta]$, so only has to check bab;a;ab as in balbabla.

PROPOSITION 4.2. *If u is any smooth form in $X \times X$ such that $\nabla_\eta u = 1$ and such that* (3.1) *holds locally at the diagonal. Then* (3.2) *holds in the current sense.*

Is proved precisely as ??? in ?????.

EXAMPLE 4.1. Assume that $s(\zeta, z)$ is a smooth form in $X \times X$ such that

$$|s| \leq C|\zeta|, \qquad |\langle s, \eta \rangle| \geq C|\eta|^2 \tag{4.3}$$

uniformly locally at the diagonal. Then

$$u = \frac{s}{\nabla_\eta s} = \frac{s}{2\pi i \langle s, \eta \rangle} + \cdots + \frac{s \wedge (\bar{\partial} s)^{n-1}}{(2\pi i)^n \langle s, \eta \rangle^n}$$

fufills the hypotheses in Proposition 3.2. $\square$

4.1. Henkin-Ramirez formulas. Recall that we solved $\bar\partial$ in smaller ball $r\mathbb{B}$ by patching together BM with an form that was holomorphic in z for ζ close to the boundary. The same can be done in a strictly psc domain by means of the form $H \cdot d\eta$. However, to get a solution in the whole domain the patching must be done infinitesmally close to the boundary.

Now let D be strictly psc and let $H(\zeta, z)$ be the vector-valued ablbaa introduced in ?????, smooth in a nbh of $\overline{D \times D}$. Recall that

$$2\text{Re } \langle H(\zeta, z), \eta\rangle \geq \rho(z) - \rho(\zeta) + \delta|\eta|^2. \tag{4.4}$$

We now choose

$$s(\zeta, z) = \overline{\langle H(\zeta, z), \eta\rangle} H(\zeta, z) \cdot d\eta - \rho(\zeta)\bar\eta \cdot d\eta$$

and claim that (3.3) holds in $D \times D$. In fact, clearly $|s| \leq C|\eta|$. Moreover,

$$\langle s, \eta\rangle = |\,\langle H(\zeta, z), \eta\rangle\,|^2 - \rho(\zeta)|\eta|^2.$$

If $\zeta \in D$, then $-\rho(\zeta) > 0$ so (3.3) holds.

Also notice that even if $\zeta \in \partial D$ we have that $\langle s, \eta\rangle = 0$ if and only if $z = \zeta$, since then $2\text{Re } \langle H, \eta\rangle \geq -\rho(z) + \delta|\eta|^2$.

If we now form the CFL kernel

$$K = \frac{1}{(2\pi i)^n} \frac{s \wedge (\bar\partial s)^{n-1}}{\langle s, \eta\rangle^n}$$

then $\bar\partial K = [\Delta]$ and we thus have the Koppelman formula ?????. as in Theorem 4.3 in Ch. 1. However, notice that when $\zeta \in D$, s is parallell to $H \cdot d\eta$, so holomorphic in z and so $K_{p,q} = 0$ on ∂D for $q > 0$. We therefore have

$$f(z) = \bar\partial_z \int_D K_{p,q-1} \wedge f + \int_D K_{p,q} \wedge f, \quad z \in D,\ q \geq 1,$$

whereas for a function v we have

$$v(z) = \int_D K_{p,0} \wedge \bar\partial v + \int_{\partial D} K_{p,0} v.$$

Notice that the last term is precisely a CFL integral with a kernel that is holomorphic in z so it is a projection $\mathcal{E}_{p,0}(\overline{D}) \to \mathcal{O}(D)$. We can write as

$$f = \bar\partial K f + K(\bar\partial f)$$

and

$$v = K(\bar\partial v) + Pv.$$

If f is a $\bar\partial$-closed form, smoth on $\overline{D}$, then Kf is a solution in D to $\bar\partial v = f$.

Henkin-Ramirez formulas.
ablbalblalbaba

Example in the ball. Then

$$s(\zeta, z) = (|\zeta|^2 - \zeta \cdot \bar{z})\bar{\zeta} \cdot d\eta + (1 - |\zeta|^2)\bar{\eta} \cdot d\eta,$$

and

$$\langle s, \eta \rangle = |1 - \bar{\zeta} \cdot z|^2 - (1 - |\zeta|^2)(1 - |z|^2).$$

4.2. Weighted Koppelman formulas. Now let g be a form with values in E, cf., the previous section, such that $\nabla_\eta g = 0$ and $g_0 = 1$ on the diagonal $\Delta \subset X \times X$. If u is a locally integrable form that satisfies (3.2) holds, then

$$\nabla_\eta (g \wedge u) = g - [\Delta].$$

If we let $K = (g \wedge u)_n$ and $P = g_n$ we thus have

$$\bar{\partial} K = [\delta] - P$$

which as in Ch 1 Section ???, leads to the Koppelman formula
(4.5)

$$f(z) = \bar{\partial} \int_D K_{p,q-1} \wedge f + \int_D K_{p,q} \wedge \bar{\partial} f + \int_{\partial D} K_{p,q} \wedge f - \int_D P_{p,q} \wedge f, \quad f \in \mathcal{E}_{p,q}(\overline{D}).$$

In order to obtain a solution formula for $\bar{\partial}$ we must get rid of the last two terms.

5. Further examples of integral repesentation

5.1. Integral representation in $\mathbb{C}^n$.

Example 5.1 (The Fock space). Let $\mathcal{F}$ be the Fock space, i.e., the space of locally square integrable functions in $\mathbb{C}^n$ such that

$$\|f\|^2_{\mathcal{F}} = \int_{\mathbb{C}^n} |f|^2 e^{-|\zeta|^2} \frac{dV(\zeta)}{\pi^n}$$

is finite.

Consider the weight

$$g = e^{-\nabla_{\zeta - z} \frac{1}{2\pi i} \partial |\zeta|^2} = e^{\bar{\cdot} z - |\zeta|^2 + \frac{1}{\pi} \frac{i}{2} \partial\bar{\partial} |\zeta|^2}$$

so that

$$g_n = e^{\bar{\cdot} z - |\zeta|^2} \frac{dV(\zeta)}{\pi^n}.$$

Since for fixed z, $\zeta \mapsto e^{\zeta\cdot\bar{z}}$ is in $\mathcal{F}$, in fact ???, the integral

$$Pf(z) = \int_{\mathbb{C}^n} e^{\bar{\cdot}z - |\zeta|^2} f(\zeta) \frac{dV(\zeta)}{\pi^n}$$

is defined for each $f \in \mathcal{F}$ by Cauchy-Schwarz inequality. In fact we have

PROPOSITION 5.1. *Let $A_{\mathcal{F}} = \mathcal{F} \cap \mathcal{O}(\mathbb{C}^n)$. Then $A_{\mathcal{F}}$ is a closed subspace of $\mathcal{F}$ and $P\colon \mathcal{F} \to A_{\mathcal{F}}$ is the orthogonal projection.*

PROOF. Clearly $A_{\mathcal{F}}$ must be a closed subspace. We first prove that $Pf(z) = f(z)$ if $f \in A_{\mathcal{F}}$. Let u be the BM-form with pole at z. Moreover let $\chi(t)$ be a function that is identically 1 for $t < 1 + \epsilon$ and 0 for $t > 2 - \epsilon$. Then by ????

$$f(z) = \int \chi(|\zeta|/R) g_n f + \int \bar{\partial}\chi(|\zeta|/R) \wedge (u \wedge g)_n f,$$

if $R > |z|$. Clearly the first term tends to $Pf(z)$ when $R \to \infty$. Now $u_k = \mathcal{O}(|\zeta|^{-2k+1})$ and $\bar{\partial}\chi(|\zeta|/R) = \mathcal{O}(1/R)$, so that the second term is bounded by a constant times

$$\int_{R<|\zeta|<2R} \frac{1}{R} |f| e^{-|\zeta|^2} e^{\mathrm{Re}\,\zeta\cdot z}.$$

blablba Cauchy-Schwarz blablabla.

If now f is in the orthogonal complement of $A_{\mathcal{F}}$, then in particular f is orthogonal to each function $\zeta \mapsto \exp(\zeta \cdot \bar{z})$ and hence $Pf = 0$. Thus the proposition is proved. $\square$

$\square$

EXAMPLE 5.2. Now let ϕ be any convex function in $\mathbb{C}^n$, and recall that then

$$2|re\,\langle \partial\phi(\zeta), \zeta - z\rangle \geq \phi(\zeta) - \phi(z). \tag{5.1}$$

In analogy withthe previous example we take

$$g = e^{-\nabla_{\zeta - z} \frac{1}{2\pi i}\partial\phi} = e^{-\langle \partial(\zeta), \zeta - z\rangle + \frac{1}{\pi}\frac{i}{2}\partial\bar{\partial}\phi}.$$

Then

$$g_n = e^{-\langle \partial(\zeta), \zeta - z\rangle} dV_\phi,$$

where

$$dV_\phi = \frac{1}{\pi^n}\big(\frac{i}{2}\partial\bar{\partial}\phi\big)_n,$$

which is non-negative form by ?????.

We now claim that as in the previous example, if f has a growth at most like $e^{\phi/2}$ then

$$Pf(z) = \int_{\mathbb{C}^n} e^{-\langle\partial(\zeta),\zeta - z\rangle} f(\zeta) f V_\phi(\zeta)$$

has as well. More precisely we have that for each z,

$$e^{\phi(z)/2}|Pf(z)| \leq \int_{\mathbb{C}^n} e^{-\phi(\zeta/2)}|f(\zeta)|dV_\phi(\zeta).$$

This follows immediately from (5.1). $\square$

We conclude with representation of and projections onto polynomials.

Example 5.3. Let m be any natural number and let

$$g = \Big(1 - \nabla_{\zeta - z}\frac{\partial|\zeta|^2}{2\pi i(1+|\zeta|^2)}\Big)^{m+n} = \Big(\frac{1+\bar{\zeta}\cdot z}{1+|\zeta|^2} + \frac{1}{\pi}\omega\Big)^{m+n},$$

where

$$\omega = \frac{i}{2}\partial\bar{\partial}\log(1+|\zeta|^2)$$

so that

$$\omega_n = \frac{dV(\zeta)}{(1+|\zeta|^2)^{n+1}}.$$

Then

$$g_n = \frac{(1+\bar{\zeta}\cdot z)^m}{(1+|\zeta|^2)^{m+n+1}}\frac{1}{\pi^n}dV(\zeta).$$

Again we let

$$P_m f(z) = \int_{\mathbb{C}^n} g_n f$$

provided that the integral exists. As in Example ??? one now can check that

Proposition 5.2. *If* $L^2_m = L^2((1+|\zeta|^2)^{-m}))$ *and* $\mathcal{P}_m = L^2_m \cap \mathcal{O}(\mathbb{C}^n)$, *then* $\mathcal{P}_m$ *is a closed subspace and* $P_m \colon L^2_m \to \mathcal{P}_m$ *is the orthogonal projection. Moreover,* $\mathcal{P}_m$ *precisely consists of the holomorphic polynomials of degree ar most* m.

$\square$

5.2. Further examples of representation formulas. Integral formulas in strictly pseudoconvex domains with weights with polynomial decay at the boundary were first used in [**?**] and [**?**] (decay of order one) and in [**?**].

Example 5.4. Sometimes it is of interest to compare two different weights. Since $(1+\lambda)^{r+1}-(1+\lambda)^r=\lambda(1+\lambda)^r$ we have that (using the notation from the preceding example)

$$g^{r+1}-g^r=\nabla_{z-a}(h/\rho)\wedge(1+\nabla_{z-a}(h/\rho))^{-r},$$

so it follows that $g_n^r-g_n^{r+1}=\bar\partial R^r$, where

$$R^r=\begin{pmatrix}-r\\ n-1\end{pmatrix}\frac{(-\rho)^{-1+r}}{v^{n-1+r}}h\wedge(-\bar\partial h)^{n-1}.$$

Since the kernels vanish on the boundary we get the formula

$$\int_D g^{r+1}(z,a)f(z)-\int_D g^r(z,a)f(z)=\int_D R^r(z,a)\wedge\bar\partial f,\quad f\in\mathcal{E}(\overline{D}),$$

which expresses the difference of two holomorphic projections with different weights as an integral of $\bar\partial f$. In the ball case this formula was obtained in [**?**] for integer values of r. □

Example 5.5. Suppose that $a\in D$ and that $\{U_j\}$ is an open cover of a neighborhood U of ∂D, $a\notin U$, and that we have local solutions $u_j\in\mathcal{L}^{-1}(U_j)$ to $\nabla_{z-a}u_j=1$. Let $\{\phi_j\}$ be a partition of unity subordinate to this cover. Then $\Phi u=\sum_j\phi_j u_j$ and $\bar\partial\Phi\wedge u=\sum_j\bar\partial\phi_j\wedge u_j$ are global forms. Since $\nabla_{z-a}(\Phi u)=1-\bar\partial\Phi\wedge u$ therefore

$$v=\Phi u/\nabla_{z-a}(\Phi u)=\Phi u\wedge\sum(\bar\partial\Phi\wedge u)^k \tag{5.2}$$

is a solution to $\nabla_{z-a}v=1$ in U. If s_j are $(1,0)$-forms in U_j such that $\delta_{z-a}s_j\neq 0$, then we can take $u_j=s_j/\nabla_{z-a}s_j$. The corresponding representation formula was obtained in [**?**]. This is *not* the same as just normalizing s_j so that $\delta_{z-a}s_j=1$, then piece together to a global $(1,0)$-form $\Phi s=\sum\phi_j s_j$ and take the global (Cauchy-Fantappie-Leray) solution $v=\Phi s/\nabla_{z-a}(\Phi s)$ to $\nabla_{z-a}v=1$ in U. In the former formula each $\bar\partial s_j$ occurs together with s_j which implies that the the kernel is unaffected if any s_j is multiplied by a scalar function. This homogeneity property is however lost in the second case. For a cohomological interpretation of (5.2), see Section **??**. □

Example 5.6. Let E be a lineally convex compact set in $\mathbb{C}^n$. This means that through each point outside E there is a complex hyperplane that does not intersect E. Let us furthermore assume that $0\in E$. We want to obtain a representation for $f\in\mathcal{O}(E)$ as a superposition of functions like $\Pi_1^n(1-\alpha_j\cdot z)^{-1}$, where α_j belong to the dual complement; this means that the hyperplane $1-\alpha_j\cdot z=0$ does not intersect E.

Locally in $\mathbb{C}^n\setminus E$ we can find $(1,0)$-forms $s_j(z)$ such that $\delta_{z-a}s_j(z)\neq 0$ for all $a\in E$. If we let $u_j=s_j(z)/\nabla_{z-a}s_j(z)$ and piece together to a solution v to $\nabla_{z-a}v=1$ as in Example 5.5 we get a representation formula

$$f(a)=\int_{\partial D}K(z,a)f(z),\quad a\in E, \tag{5.3}$$

if $D\supset E$ and $f\in\mathcal{O}(\overline{D})$, where $K=v_n$. Moreover, $K(z,a)$ is a locally finite sum of terms like $\gamma(z)\Pi_{j=1}^n(1-\alpha_j(z)\cdot a)^{-1}$ so (5.3) indeed provides the desired decomposition of f.

The decomposition formula is particularily simple if we choose a fine enough open cover so that each $s_j(z)$ can be chosen to be holomorphic (e.g., constant!), because then $\nabla_{z-a}s(z)=\delta_{z-a}s(z)$. □

Example 5.7 (Weighted Cauchy-Weil formulas). Let D be an analytic polyhedron, i.e., assume that we have functions $\psi_1,\dots,\psi_m$, that are holomorphic in a neighborhood Ω of $\overline{D}$ such that $D=\{z\in\Omega;\ |\psi_j(z)|<1,\ j=1,\dots,m\}$. By Hefer's theorem we can find $(1,0)$-forms h_j, holomorphic in z as well as a, such that $\delta_{z-a}h_j=\psi_j(z)-\psi_j(a)$ for z and a in a neighborhood of $\overline{\Omega}$. Taking $q_j(z)=\overline{\psi_j(z)}h_j(z,a)/(1-|\psi_j(z)|^2)$ then $g=\Pi_1^m(1+\nabla_{z-a}q_j)^{-r_j}$ will vanish on ∂D, and we get the weighted Cauchy-Weil formula

$$f(a)=\int_D P^r(z,a)f(z),\quad a\in D,\quad f\in\mathcal{O}(\overline{D}) \tag{5.4}$$

where P^r is the kernel

$$c\sum_{|I|=n}{}^{'}\prod_{j\notin I}\left(\frac{1-|\psi_j(z)|^2}{1-\overline{\psi_j(z)}\psi_j(a)}\right)^{r_j}\bigwedge_{\ell\in I}r_\ell\frac{(1-|\psi_\ell(z)|^2)^{r_j-1}}{(1-\overline{\psi_\ell(z)}\psi_\ell(a))^{r_j+1}}\wedge\bar\partial\psi_\ell\wedge h_\ell.$$

By analytic continuation in r_j, (5.4) must hold for $r_j>0$. If $\partial\psi_I\wedge\ldots\wedge\partial\psi_{I_n}\neq 0$ on $D_I=\{z\in\overline{D};\ |\psi_{I_1}(z)|=\cdots=|\psi_{I_n}(z)|=1\}$ (with an appropriate orientation) for each multiindex I of length n, then we can let $r_j\searrow 0$ and recover the classical Cauchy-Weil formula

$$f(a)=c\sum_{|I|=n}{}^{'}\int_{D_I}f(z)\bigwedge_{\ell\in I}\frac{h_\ell}{\psi_\ell(z)-\psi_\ell(a)}.$$

□

6. Fourier transforms of forms and currents

Our main tool is the Fourier transformation of vectorvalued currents. Roughly speaking the Fourier transform of a (p,q)-form (or current) $f = f_{IJ}(z)dz^J \wedge d\bar{z}^J$ will be $\pm \hat{f}_{IJ}(\zeta)d\zeta^{J'} \wedge d\bar{\zeta}^{I'}$, where $\hat{f}_{IJ}$ is the usual Fourier transform of the coefficient f_{IJ} and I' and J' denote complementary indices. The idea with such a Fourier transformation is quite natural and appeared already in [**?**], and occurs in [**?**]; this definition is quite different from ours below but equivalent. Another definition, but again equivalent, is introduced and used in [**?**]. Our definition makes it possible to give simple arguments for the basic results that we need. Let

$$\omega = \omega(z,\zeta) = 2\pi i \mathrm{Re}\, z\cdot\bar{\zeta} + \mathrm{Re}\,(dz \wedge d\bar{\zeta}) = \\ = \pi i(z\cdot\bar{\zeta} + \bar{z}\cdot\zeta) + (dz\wedge d\bar{\zeta} + d\bar{z}\wedge d\zeta)/2,$$

where $dz \wedge d\bar{\zeta} = \sum dz_j \wedge d\bar{\zeta}_j$ etc. Since ω has even degree, $\exp(-\omega)$ is welldefined, and for a form $f(z)$ with coefficients in $\mathcal{S}(\mathbb{C}^n)$ we let

$$\mathcal{F}f(\zeta) = \int_z e^{-\omega(z,\zeta)} \wedge f(z). \tag{6.1}$$

Since we have an even real dimension it is immaterial whether we put all differentials of $d\zeta, d\bar{\zeta}$ to the right or to the left before performing the integration, and thus $\mathcal{F}f(\zeta)$ is a welldefined form with coefficients in $\mathcal{S}$. To reveal a more explicit form of the condensed definition (6.1) let us assume that $f \in \mathcal{S}_{p,q}$. Then

$$\mathcal{F}f(\zeta) = \int_z e^{-2\pi i \mathrm{Re}\, z\cdot\bar{\zeta}} \wedge \sum_{k=0}^{\infty}(-\mathrm{Re}\,(dz\wedge d\bar{\zeta}))^k/k! \wedge f(z) = \\ \int_z e^{-2\pi i \mathrm{Re}\, z\cdot\bar{\zeta}} \wedge (-\mathrm{Re}\,(dz\wedge d\bar{\zeta}))^{2n-p-q}/(2n-p-q)! \wedge f(z)$$

for degree reasons, and hence $\mathcal{F}f$ is an $(n-q, n-p)$-form. In what follows we let $\hat{f}$ mean the same as $\mathcal{F}f$.

Proposition 6.1. *We have the inversion formula*

$$f(z) = (-1)^n \int_\zeta e^{\omega(z,\zeta)} \wedge \hat{f}(\zeta). \tag{6.2}$$

Of course it can be deduced from the inversion formula for the usual Fourier transform, but we prefer to repeat one of the wellknown argument in the form formalism.

PROOF. Take $\psi \in \mathcal{S}_{0,0}$ such that $\psi(0) = 1$. Then

$$\int_\zeta \hat{f}(\zeta) \wedge e^{\omega(w,\zeta)} = \lim_{\epsilon \to 0} \int_\zeta \psi(\epsilon\zeta)\hat{f}(z) \wedge e^{\omega(w,\zeta)} =$$

$$= \lim_{\epsilon \to 0} \int_\zeta \int_z \psi(\epsilon\zeta)\hat{f}(z) \wedge e^{\omega(w-z,\zeta)}.$$

Making the change of variables $\zeta \mapsto \zeta/\epsilon$, $z \mapsto z + w$, the right hand double integrals becomes (the mapping is orientation preserving, so no minus sign appears)

$$\int_\zeta \int_z \psi(\zeta)\hat{f}(z+w) \wedge e^{-\omega(z,\zeta/\epsilon)}$$

and since $\omega(z, \zeta/\epsilon) = \omega(z/\epsilon, \zeta)$ another change of variables $\zeta \mapsto \epsilon\zeta$ gives

$$\int_\zeta \int_z \psi(\zeta)\hat{f}(w+\epsilon z) \wedge e^{-\omega(z,\zeta)}$$

which tends to $c_n f(w)$, where

$$c_n = \int_z \int_\zeta \psi(\zeta)e^{-\omega(z,\zeta)}.$$

Taking for instance $\psi(\zeta) = \exp(-|\zeta|^2)$, a simple computation reveals that $c_n = (-1)^n$. □

PROPOSITION 6.2. *In analogy with the usual case we have the formula*

$$\widehat{f * g} = \hat{f} \wedge \hat{g}. \tag{6.3}$$

PROOF. In fact, by definition,

$$\widehat{f * g}(w) = \int_z e^{-\omega(z,w)} \wedge f * g(z) = \int_z \int_\zeta e^{-\omega(z,w)} \wedge f(\zeta - z) \wedge g(\zeta).$$

If we now make the change of variables, $t = \zeta - z$, $\zeta = \zeta$ and use that ω is bi-linear, we get

$$\int_t \int_\zeta e^{-\omega(\zeta,w)} \wedge e^{-\omega(t,w)} \wedge f(t) \wedge g(\zeta) = \hat{f}(w) \wedge \hat{g}(w).$$

□

Let δ_{z-a} denote contraction with the vector field $2\pi i(z-a) \cdot \frac{\partial}{\partial z}$ for $a \in \mathbb{C}^n$ and let $\nabla_{z-a} = \delta_{z-a} - \bar{\partial}_z$. Moreover, let $\hat{\nabla}_\zeta = \delta_\zeta - \bar{\partial}_\zeta$. Then $(\nabla_z - \nabla_\zeta)^2 = 0$, and since

$$\omega = \frac{1}{2}(\nabla_z - \hat{\nabla}_\zeta)(\bar{\zeta} \cdot dz - \bar{z} \cdot d\zeta)$$

it follows that

$$(\nabla_{z-a} - \hat{\nabla}_\zeta)\omega(z,\zeta) = -\pi i a \cdot d\bar{\zeta}. \tag{6.4}$$

PROPOSITION 6.3. *If $a \in \mathbb{C}^n$, then*

$$\mathcal{F}(\nabla_{z-a} f(z)) = -(\hat{\nabla}_\zeta + A)\mathcal{F}f, \tag{6.5}$$

where

$$A\phi = \pi i a \cdot d\bar{\zeta} \wedge \phi$$

for forms $\phi(\zeta)$.

Identifying bidegrees we also get that

$$\mathcal{F}(\delta_{z-a} f) = -(\bar{\partial}_\zeta + A)\mathcal{F}f \quad \text{and} \quad \mathcal{F}(\bar{\partial} f) = \delta_\zeta \mathcal{F}f.$$

PROOF. By (6.4) we have that

$$\begin{aligned}(\nabla_{z-a} - \hat{\nabla}_\zeta)(e^{-\omega} \wedge f(z)) = A(e^{-\omega} \wedge f) + e^{-\omega} \wedge (\nabla_{z-a} - \hat{\nabla}_\zeta)f = \\ = A(e^{-\omega} \wedge f) + e^{-\omega} \wedge \nabla_{z-a} f.\end{aligned}$$

Integrating with respect to z we get (6.5), since $\int \nabla_{z-a} g = 0$ for forms g in $\mathcal{S}$. □

It is readily verified that Propositions 6.1 and 6.3 hold for X-valued forms (and currents, see below) and commuting n-tuples of operators a. This is checked by applying functionals on both sides of each equality.

We now want to extend the Fourier transform to currents in $\mathcal{S}'$, and to this end we first notice that

$$(-1)^n \int_z u(z) \wedge f(z) = \int_\zeta \hat{f}(-\zeta) \wedge \hat{u}(\zeta), \tag{6.6}$$

for $u, f \in \mathcal{S}$. To see this, just notice that both sides are equal to

$$\int_z \int_\zeta u(z) \wedge \hat{f}(\zeta) \wedge e^{\omega(z,\zeta)} = \int_z \int_\zeta u(z) \wedge \hat{f}(-\zeta) \wedge e^{-\omega(z,\zeta)}.$$

Moreover, one easily checks that if $\check{f}(z) = f(-z)$, then $\mathcal{F}\check{f}(\zeta) = \mathcal{F}f(-\zeta)$. Any $u \in \mathcal{S}$ defines an element in $\mathcal{S}'$ by

$$u.f = \int_z u(z) \wedge f(z), \quad f \in \mathcal{S}.$$

For a general $u \in \mathcal{S}'$ it is therefore natural to define $\hat{u}$ by the formula

$$\hat{u}.\hat{f} = (-1)^n u.\check{f}, \quad f \in \mathcal{S}.$$

It is routine to extend δ_{z-a}, $\bar{\partial}$ etc to $\mathcal{S}'$, and verify that Proposition 6.3 still holds for currents $u \in \mathcal{S}'$.

REMARK 6.1. One can check that

$$\int \overline{\hat{u}(\zeta)} \wedge \phi(\zeta) = \int \overline{u(z)} \wedge \hat{\phi}(-z) \tag{6.7}$$

since the conjugates of both sides are equal to

$$\int_\zeta \int_z e^{-\omega(z,\zeta)} \wedge u(z) \wedge \bar{\phi}(\zeta),$$

in view of the equality $\omega(z,\zeta) = \overline{\omega(\zeta,-z)}$. One can then define the Fourier transform of currents by means of formula (6.7) instead. □

LEMMA 6.4. *If* $[0]$ *denotes the current integration at the point* 0, *then*

$$\mathcal{F}[0](\zeta) = 1 \quad \textit{and} \quad \mathcal{F}1(\zeta) = (-1)^n[0](\zeta). \tag{6.8}$$

PROOF. In fact, for $f \in \mathcal{S}_{0,0}$ we have

$$\widehat{[0]}.\hat{f} = (-1)^n \int_z [0](z) \wedge \check{f}(z) = (-1)^n f(0) = \int_\zeta \hat{f}(\zeta),$$

where the last equality follows from the inversion formula (6.2), holding in mind that $\hat{f}$ is a (n,n)-form. In a similar way we have

$$\hat{1}.\hat{f} = (-1)^n \int_z \check{f}(z) = (-1)^n \int_z f(z) = (-1)^n \hat{f}(0),$$

since in this case f is a (n,n)-form. □

We say that $u \in \mathcal{L}^{-1}(\mathcal{S}', \mathbb{C}^n)$ is a Cauchy current if

$$\nabla_z u = 1 - [0]. \tag{6.9}$$

From Lemma 6.4 and Proposition 6.3 it follows that u is a Cauchy current if and only if

$$\hat{\nabla}_\zeta \hat{u}(\zeta) = 1 - (-1)^n[0].$$

For instance, if $b(z) = \partial|z|^2/2i$, then

$$B(z) = \frac{b(z)}{\nabla_z b(z)} = \sum_{\ell=1}^{n} \frac{b(z) \wedge (\bar{\partial} b(z))^{\ell-1}}{(\delta_z b(z))^\ell} \tag{6.10}$$

is a Cauchy current. In fact, since $\delta_z b(z) \neq 0$ outside 0 it follows that $\nabla_z B(z) = 1$ there, and the behaviour at 0 is easily checked. We will refer to $B(z)$ as the Bochner-Martinelli form. It follows that $\hat{\nabla}_\zeta \hat{B}(\zeta) = 1 - (-1)^n[0]$, and more precisely we have that

PROPOSITION 6.5. *If $B(z)$ is the Bochner-Martinelli form* (6.10), *then*

$$\hat{B}(\zeta) = \frac{b(\zeta)}{\hat{\nabla}_\zeta b(\zeta)} = \sum_{\ell=1}^{n} (-1)^{\ell-1} b(\zeta) \wedge (\bar{\partial} b(\zeta))^{\ell-1}.$$

In fact, one can verify that $B(z)$ is the only Cauchy current that is rotation invariant and 0-homogeneous. Since these properties are preserved by $\mathcal{F}$, the proposition follows. Alternatively, one can use the wellknown formulas for Fourier transforms of homogeneous functions in $\mathbb{R}^{2n}$. However, we prefer to give a direct argument which reflects the handiness of our formalism. We begin with a lemma of independent interest.

LEMMA 6.6. *If $\beta(z) = (i/2)\partial\bar{\partial}|z|^2$, then*

$$\mathcal{F}(e^{-\pi|z|^2+\beta(z)}) = e^{-\pi|\zeta|^2-\beta(\zeta)}. \tag{6.11}$$

PROOF. It is convenient to use real coordinates, so let $z = x + iy$ and $\zeta = \xi + i\eta$. Then $\beta(z) = dx \wedge dy$ and $\omega(z,\zeta) = 2\pi i(x \cdot \xi + y \cdot \eta) + dx \wedge d\xi + dy \wedge d\eta$. Now,

$$1 = \int_{x,y} e^{-\pi(x^2+y^2)+dx\wedge dy} = e^{\pi(\xi^2+\eta^2)} \int_{x,y} e^{-\pi(x^2+y^2)-2\pi i(x\cdot\xi+y\cdot\eta)+dx\wedge dy}$$

by an application of Cauchy's theorem. By the translation invariance of the Lebesgue integral we can make the change of variables $x \mapsto x+\eta$, $y \mapsto y - \xi$, in the last integral which yields

$$e^{\pi(\xi^2+\eta^2)-d\xi\wedge d\eta} \int_{x,y} e^{-\pi(x^2+y^2)+dx\wedge dy-\omega}$$

and so the lemma follows. □

If $\beta_k = \beta^k/k!$, then the lemma thus means that

$$\mathcal{F}\big[e^{-\pi|z|^2}\beta_k(z)\big](\zeta) = e^{-\pi|\zeta|^2}(-1)^{n-k}\beta_{n-k}(\zeta),$$

for each k.

PROOF OF PROPOSITION 6.5. From (the remark after) Proposition 6.3 (and taking conjugates) we get that

$$\mathcal{F}(e^{-\pi|z|^2+\beta(z)} \wedge \pi\bar{z} \cdot dz) = -e^{-\pi|\zeta|^2-\beta(\zeta)} \wedge \pi\bar{\zeta} \cdot d\zeta.$$

Noting that $-\pi|z|^2 + \beta(z) = -\nabla_z b(z)$ and $-\pi|\zeta|^2 - \beta(\zeta) = -\hat{\nabla} b(\zeta)$, and using the homogeneity property of the Fourier transformation, we get that

$$F(e^{-t\nabla_z b(z)} \wedge b(z)) = -e^{-(1/t)\hat{\nabla}_\zeta b(\zeta)} \wedge b(\zeta)/t^2,$$

and integrating in t over the positive real axis we get Proposition 6.5. □

7. The $\bar\partial$-equation in $\mathbb{C}^n$

We will now consider the

Consider

We will discuss the $\bar\partial$-equation $\bar\partial u = f$ in $\mathbb{C}^n$ when the right-hand side has polynomial growth and polyonomial decay. In both cases, f is in $\mathcal{S}'$ so via the Fourier transformation the equation is transformed to $\delta_\zeta \hat{u} = \hat{f}$.

Our principal results are for simplicity we only formulate for L^∞ estimates, similar holds for L^1, left as an exercise; L^2 more special ????

Theorem 7.1. *If f is a*

that $\hat{f}$ has a meaning and hence the equation

8. Exercises

Exercise 20. Let ω be a form of even degree and suppose that f is holomorphic in a neighborhood of $\overline{D}$ and $\omega' \in D$. Show that

$$f(\omega) = \frac{1}{2\pi i}\int_D \frac{f(\zeta)d\zeta}{\zeta - \omega}.$$

Are there some multivariable analogue?

Exercise 21. Show that

$$\xrightarrow{\nabla_{z-a}} \mathcal{L}^{m-1}(U) \xrightarrow{\nabla_{z-a}} \mathcal{L}^m(U) \to \dots \tag{8.1}$$

is a complex in U that is exact if and only $a \notin U$.

Exercise 22. Show the following variant of Theorem 3.17:

If $\nabla_{\zeta - z} v = g - [z]$ in Ω, then

$$\phi(z) = \int_{\partial D} \phi v_n + \int_D \phi g_n - \int_D \bar\partial\phi \wedge v_n, \qquad \phi \in \mathcal{E}(\overline{D}). \tag{8.2}$$

Exercise 23. Prove Proposition 2.4.

Exercise 24. Let u be the current from Example 2.3. Let $a = 0$ and notice that u and u_{BM} are cohomologous in $\mathbb{C}^n \setminus \{0\}$. Use this to compute the volume of the unit ball in $\mathbb{C}^n$.

Exercise 25. Show that $u \wedge g$ in Theorem **??** can be replace by any v in U such that $\nabla_{\zeta - z} v = g$ in U.

Exercise 26. Show that any weight g (with respect to z) can be written $g = 1 + \nabla_{\zeta - z} q$, for some q in L^{-1}. Hint: It is easy to find a current solution q but more involved to find a smooth solution.

EXERCISE 27. Suppose that $g = G(\nabla_{z-a}q)$ and $h = H(\nabla_{z-a}q)$ are two weights as in Example ?? above. If $\lambda A(\lambda) = H(\lambda) - G(\lambda)$ then

$$h - g = \nabla_{z-a}q \wedge A(\nabla_{z-a}q) = \nabla_{z-a}(q \wedge A(\nabla_{z-a}q));$$

thus we get an explicit solution to $\nabla_{z-a}w = h - g$.

EXERCISE 28. Show that there is a smallest polynomially convex set $\hat{K}$ that contains a given compact set K.

EXERCISE 29. Show that $\hat{K}$ is contained in the convex hull of K, and that strict inclusion can occur.

EXERCISE 30. Antag att $D = \rho < 0\}$ with ρ smooth and $d\rho \neq 0$ on ∂D. Moreover, assume that

$$H\rho_z(v, v) > 0$$

for all real $v \in T_z(\partial D)$, i.e., all real v such that $d\rho.v = 0$. Show that D is strictly convex.

Norguet formler ??????

9. Comments on Chapter ??

Integral formulas in strictly pseudoconvex domains with weights with polynomial decay at the boundary were first used in [?] and [?] (decay of order one) and in [?].

CHAPTER 3

Integral representation in strictly pseudoconvex domains

Strictly pseudoconvex domains in $\mathbb{C}^n$ have several good properties which make it accessible to obtain various generalizations of function theory as in one variable for finitly connected domains, such as the disk, the annulus, etc.

For instance, a spsc domain D can be exhausted by compactly included spsc, it can be approximated from outside by larger spsc domains, and there are "good" integral representations. This implies that there is a good theory for boundary values, function spaces, estimates for the $\bar\partial$-equation. ETCETC approximate holo ϕ in say $L^2(D)$ or with bondary values in $L^2(\partial D)$ by functions holo in a neighborhood of $\overline{D}$; things that are often quite simple in one variable.

The model case of a spsc domain is the unit ball $\mathbb{B}$.

(Det som gar i boll gar i str psc, aven svagt konvext andlig typ Fornaess, statt i utveckling sa inskranker till str psc har)

1. Weighted Bergman spaces in the ball

For $\alpha > 0$, let

$$dV_\alpha = c_\alpha(1-|\zeta|^2)^{\alpha-1}dV(\zeta)$$

where c_α is chosen so that

$$\int_{|\zeta<1} dV_\alpha = 1,$$

cf., ???? in Section ???, and let $L^2_\alpha = L^2(dV_\alpha)$. We define the weighted Bergman space

$$B_\alpha = L^2_\alpha \cap \mathcal{O}(\mathbb{B}).$$

Moreover, for $\phi \in C\overline{B})$ we define

$$P_\alpha(z) = \int_{|\zeta|<1} \frac{\phi(\zeta)dV_\alpha}{(1-\bar\zeta\cdot z)^{n+\alpha}}.$$

Clearly $P_\alpha\phi$ is holomorphic, and from ??? we recall that $P_\alpha\phi = \phi$ if $\phi \in \mathcal{O}(\overline{\mathbb{B}})$.

THEOREM 1.1. *(i) The Bergman space B_α is a closed subspace of L^2_α.*
(ii) $\mathcal{O}(\overline{\mathbb{B}})$ is dense in B_α.
(iii) The operator P_α has a continuous extension $P_\alpha\colon L^2_\alpha \to B_\alpha$ and is in fact the orthogonal projection.

For the proof we will use the following lemma, whose proof will be discussed in a somwhat more general context in the next section.

LEMMA 1.2. *If $r > 0$ and $\gamma > 0$, then*

$$\int_{|\zeta|<1} \frac{(1-|\zeta|^2)^{\gamma-1}}{|1-\bar\zeta\cdot z|^{n+\gamma+\epsilon}} \le C_{\gamma,\epsilon}\frac{1}{(1-|z|^2)^{\epsilon}}.$$

PROOF OF THEOREM 1.1. If $f_k \to f$ in L^2_α, then $f_k \to f$ in $L^2_{\rm loc}(\mathbb{B})$ so f_j is a Cauchy sequence in $L^2_{\rm loc}$ and hence in $\mathcal{E}(\mathbb{B})$ as a consequence of the Cauchy estimates, cf., ???. Therefore, $f_k \to f$ in $\mathcal{E}(\mathbb{B})$ and since f_j are holomorphic, therefore f is holomorphic.

To see (ii), let $f_r(\zeta) = f(r\zeta)$ for $r < 1$. Then $f_r \in \mathcal{O}(\overline{\mathbb{B}})$ and we claim that $f_r \to f$ in L^2_α. For simplicity we assume that $\alpha = 1$ and leave the general case as an exercise. Clearly $f_r(\zeta) \to f(\zeta)$ pointwise in $\mathbb{B}$ when $r \to 1$. Moreover, notice that

$$\int_{|\zeta|<1} |f_r(\zeta)|^2 = r^{-2n}\int_{|\zeta|<r} |f(\zeta)|^2 \to \int_{|\zeta|<1} |f(\zeta)|^2$$

by the monotone convergence theorem. An application of Fatous's theorem to the positive sequence

$$2|f|^2 + 2|f_r|^2 - |f - f_r|^2$$

now yields

$$\int_{|\zeta|<1} |f - f_r|^2 \to 0.$$

For (ii) we also assume $\alpha = 1$. By Cauchy-Schwarz' inequality and Lemma 1.2 we have

$$\Big|\int \frac{\phi(\zeta)}{(1-\bar\zeta\cdot z)^{n+1}}\Big|^2 \le \int \frac{(1-|\zeta|^2)^{\epsilon}|\phi(\zeta)|^2}{|1-\bar\zeta\cdot z|^{n+1}} \int \frac{(1-|\zeta|^2)^{-\epsilon}}{|1-\bar\zeta\cdot z|^{n+1}} \le$$
$$C(1-|z|^2)^{-\epsilon}\int \frac{(1-|\zeta|^2)^{\epsilon}|\phi(\zeta)|^2}{|1-\bar\zeta\cdot z|^{n+1}}.$$

Another application of Lemma 1.2 now gives

$$\int |P\phi(z)|^2 \le C\int |\phi|^2$$

and thus the first statement in (iii) is proved. Since $P_\alpha\phi = \phi$ for $\phi \in \mathcal{O}(\overline{\mathbb{B}})$ it no follows from (ii) that P_α maps $B_\alpha \to B_\alpha$ and hence is a projection $L^2_\alpha \to B_\alpha$. It remains to see that it is orthogonal.

For $\phi, \psi \in \mathcal{L}^2$,

$$\Big(\int_z \int \zeta \frac{|\phi(\zeta)||\psi(z)|}{|1-\bar\zeta\cdot z|^{n+1}}\Big)^2 \le$$

$$\int_z |\psi(z)|^2 \int_z \Big(\int_\zeta \frac{|\phi(\zeta)|}{|1-\bar\zeta\cdot z|^{n+1}}\Big)^2 \le C\int|\psi|^2\int|\phi|^2.$$

Since the kernel $p(\zeta, z) = 1/(1-\bar\zeta\cdot z)$ is Hermitian, i.e., $p(z,\zeta) = \overline{p(\zeta, z)}$, by Fubini's theorem we get $(\phi, P\psi) = (P\phi, \psi)$. Thus P is self-adjoint and hence indeed the orthogonal projection. $\square$

1.1. Convexity. Recall that a C^2 function $h(t)$ of one variable is convex if $h'' \ge 0$ and strictly convex if $h'' > 0$. It is easily seen that if h is convex in a neighborhood of the interval $[a, b]$, then

$$h(ta + (1-t)b) \ge th(a) + (1-t)h(b), \quad t \in (0,1). \tag{1.1}$$

We say that a function $f(x)$ in $C^2(\Omega)$, $\Omega \subset \mathbb{R}^N$ is (strictly) convex if for each $x \in \Omega$ and $\alpha \in \mathbb{R}^N$, $h(t) = f(x + t\alpha)$ is (strictly) convex where it is defined. Since

$$h''(0) = \sum_{j\ell} \frac{\partial^2 f}{\partial x_j \partial x_\ell}(x)\alpha_j\alpha_\ell = Hf_x(\alpha,\alpha),$$

f is thus convex if and only if the qudratic form (Hessian) $Hf_x(\alpha,\alpha)$ is positively semidefinit at each point x. Moreover, since the sphere $\{\alpha;\ |\alpha| = 1\}$ is compact, we find that f is strictly convex if and only if

$$Hf_x(\alpha,\alpha) \ge c(x)|\alpha|^2,$$

where $c(x) > 0$ in Ω.

Now assume that f is strictly convex in $\mathbb{R}^N$. Then $\{f < c\}$ is a convex set for each $c \in \mathbb{R}$ (in view of (1.1)), and if we assume that $\{f < 0\}$ is bounded, then $\{f < 0\}$ is compactly included in $\Omega = \{f < \epsilon\}$ for some $\epsilon > 0$, see Exercise ??. By Taylor's formula,

$$f(y) - f(x) = (y-x)\cdot\nabla f(x) + Hf_{x+\theta(y-x)}(y-x, y-x), \quad y, x \in \Omega$$

and since f is strictly convex we thus have

$$\delta|y-x|^2 \le f(y) - f(x) - (y-x)\cdot\nabla f(x) \le C|y-x|^2, \quad x, y \in \Omega.$$

We have already seen, ???, that one has a simple CFL representation formula for holomorphic functions in a convex domain in $\mathbb{C}^n$.

We are now going to find weighted such formulas. To this end we assume that $\rho(\zeta)$ is a smooth convex function in $\mathbb{C}^n$ and $D = \{\rho < 0\}$ is bounded, and $d\rho \neq 0$ on ∂D so that ∂D is smooth by the implicit function theorem. (It is enough to assume ρ is C^3 or even C^2 but to avoid technicalities we assume C^∞.) It is easy to see that any convex domain D admits such a convex defining function.

balblablablaba argument !!

1.2. Plurisubharmonic functions. Let u be a real smooth function in $\mathbb{C}^n$. Already in Example ??in Ch. 1 we noticed that the linear form in the Taylor expansion at ζ in complex notation is

$$2\mathrm{Re}\,\langle \partial u(\zeta), z-\zeta\rangle\,,$$

which also is equal to $-2\mathrm{Re}\,\delta_{\zeta-z}\partial u(\zeta)$, and the Hessian is in complex notation

$$Hu_z(h,h) = \mathrm{Re}\sum_{j\ell} u_{\ell j}(\zeta)h_j\bar h_\ell + \sum_{j\ell} u_{j\bar\ell}(\zeta)h_j\bar h_\ell, \quad h\in\mathbb{C}^n,$$

where

$$u_{j\ell} = \frac{\partial^2 u}{\partial z_j z_\ell}, \quad u_{j\bar\ell} = \frac{\partial^2 u}{\partial z_j \bar z_\ell}.$$

Now assume that u is strictly convex at z, i.e., $Hu_z(h,h)\geq c|h|^2$. Replacing h by $e^{i\theta}h$ this implies that

$$\sum_{j\ell} u_{j\bar\ell}(\zeta)h_j\bar h_\ell - \Big|\sum_{j\ell} u_{\ell j}(\zeta)h_j\bar h_\ell + \sum_{j\ell} u_{j\bar\ell}(\zeta)h_j\bar h_\ell\Big| \geq c|h|^2, \quad h\in\mathbb{C}^n,$$

and in particular

$$Lu_z(h,h) = \sum_{j\ell} u_{j\bar\ell}(z)h_j\bar h_\ell \geq c|h|^2, \quad h\in\mathbb{C}^n. \tag{1.2}$$

The Hermitian form $Hu_z(h,h)$ is called the Levi form of u at z. A real function $u\in C^2(\Omega)$ such that (1.2) holds for all $z\in\Omega$, i.e., Lu_z is positive definite for all z, is called strictly plurisubharmonic, spsh. If it is just semi-definite u is called plurisubharmonic, psh.

Convexity of a function is preserved under real linear mappings but not under more general mappings. However, (strict) plurisubharmonicity is a complex invariant notion. In fact, if h is identified with the $(1,0)$ vector $\sum_1 h_j(\partial/\partial z_j)$ at z, then

$$Lu_z(h,h) = \partial\bar\partial u(h,\bar h), \tag{1.3}$$

so u is strictly psh if and only if $\partial\bar\partial u(h,\bar h) > 0$ for all nonzero $(1,0)$ vectors, and this is an invariant condition.

It is easily checked that u is (s)psh if and only if

$$v(\tau) = u(z + \tau h)$$

is (strictly) subharmonic in the plane (where it is defined), for all $z \in \Omega$ and $h \in \mathbb{C}^n$. Recall that v is (strictly) subharmonic if

$$\Delta v(z) = 4\frac{\partial^2}{\partial\tau\partial\bar{\tau}}(z)$$

is (strictly) positive.

Now assume that ϕ is C^2 on the image of u. Then by the chain rule,

$$\phi(u)_j = \phi'(u)u_j, \quad \phi(u)_{j\bar{k}} = \phi''(u)u_j u_{\bar{k}} + \phi'(u)u_{j\bar{k}},$$

and hence

$$\sum_{jk} \phi(u)_{j\bar{k}} h_j \bar{h}_k = \sum_{jk} \phi''(u)u_j u_{\bar{k}} h_j \bar{h}_k + \sum_{jk} \phi'(u)u_{j\bar{k}} h_j \bar{h}_k,$$

so that

$$\sum_{jk} \phi(u)_{j\bar{k}} h_j \bar{h}_k = \phi''(u)\Big|\sum_j u_j h_j\Big|^2 + \phi'(u)\sum_{jk} u_{j\bar{k}} h_j \bar{h}_k. \tag{1.4}$$

If now u is psh and ϕ is increasing and convex, then $\phi \circ u$ is psh; moreover, if in addition u is spsh and $\phi' > 0$, then $\phi \circ u$ is spsh.

We say that an (n, n)-form ω is positive (at a given point) if $\omega = cdV(z)$ with $c \geq 0$.

Proposition 1.3. *If u is psh then $(i\partial\bar{\partial}u)^n$ is positive.*

Proof. Let $z(w)$ be a holomorphic change of coordinates. From (1.3) (or by a direct computation) follows that

$$\sum_{jk} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} h_j \bar{h}_k = \sum_{i\ell} \frac{\partial^2 u}{\partial w_i \partial \bar{w}_\ell} \eta_i \bar{\eta}_\ell,$$

where $\eta = (\partial w/\partial z)h$. By a complex-linear change of coordinates we can therefore assume that, at a given point z, the Levi form is diagonal, i.e.,

$$i\partial\bar{\partial}u = i\sum_j \alpha_j dz_j \wedge d\bar{z}_j.$$

It follows that

$$(i\partial\bar{\partial}u)_n = \alpha_1 \cdots \alpha_n dV(z).$$

$\square$

EXAMPLE 1.1. If u is spsh and negative, then

$$\log(1/-u)$$

is strictly psh. In fact, for $t<0$, the function $t\mapsto\log(1/(-t)$ is convex and strictly increasing. $\square$

1.3. Weighted formulas in (strictly) convex domains. Here we will use the notation $\langle\bar\zeta,z\rangle=\bar\zeta\cdot z$ Let ρ be a real function in $\mathbb{C}^n$. Already in Example ??in Ch. 1 we noticed that the linear form in the Taylor expansion at ζ in complex notation is $2\mathrm{Re}\,\langle\partial\rho(\zeta),z-\zeta\rangle$ which also is equal to $-2\mathrm{Re}\,\delta_{\zeta-z}\partial\rho(\zeta)$. In that example we get a representation formula boundary integral, and as we have seen it is independent of the choice of defining function, see Exercise ?. We shall now construct weighted formulas in convex domains.

Let ρ be smooth convex function in $\mathbb{C}^n$, $D=\{\rho<0\}$ and $d\rho\neq 0$ on ∂D. From the preceding subsection we know that

$$\delta\le|z-\zeta|^2\le\rho(z)-\rho(\zeta)-2\mathrm{Re}\,\langle z-\zeta,\partial\rho(\zeta)\rangle\le C|z-\zeta|^2, \tag{1.5}$$

for ζ,z in say $\{\rho\le 1\}$ if ρ is strictly convex and at least

$$0\le\rho(z)-\rho(\zeta)-2\mathrm{Re}\,\langle z-\zeta,\partial\rho(\zeta)\rangle$$

if ρ is merely convex.

In analogy with the ball case we now introduce the weight

$$g=\Big(1+\nabla_{\zeta-z}\frac{\partial\rho}{2\pi i(-\rho)}\Big)^{-\alpha}=\Big(\frac{-\rho+\langle\partial\rho,\zeta-z\rangle}{-\rho}-\frac{1}{\pi}\omega\Big)^{-\alpha}.$$

where

$$\omega=\frac{i}{2}\partial\bar\partial\log(1/(-\rho)).$$

It follows from Example **??** that ω_n is positive. As in the ball case we then get

$$g_n=\frac{1}{v(\zeta,z)^{n+\alpha}}dV_\alpha(\zeta),$$

where

$$v(\zeta,z)=\langle\rho(\zeta),\zeta-z\rangle-\rho(\zeta)$$

and

$$dV_\alpha(\zeta)=c_\alpha(-\rho(\zeta))^{\alpha-1}(i\partial\bar\partial\rho)_n+i\partial\rho\wedge\bar\partial\rho\wedge(i\partial\bar\partial\rho)_{n-1}$$

In view of ????, it is defined for $\zeta,z\in D$, and thus we have, cf ball case,

$$\phi(z)=\int_D\frac{\phi(\zeta)dV_\alpha(\zeta)}{v(\zeta,z)^{n+\alpha}},\quad\phi\in\mathcal{O}(\overline{D}).$$

It follows from Example **??** that ?? is strictly positive. ??????

Notice that

$$(i\partial\bar\partial\rho)_n + i\partial\rho\wedge\bar\partial\rho\wedge(i\partial\bar\partial\rho)_{n-1}$$

is strictly positive on $\overline{D}$. balbalal se som i beviset av ????.

PROPOSITION 1.4. *Suppose that D is convex, $\phi\in\mathcal{O}(D)$, and $\int_D|\phi|dV_\alpha<\infty$. Then*

$$\phi(z)=\int_D\frac{\phi(\zeta)dV_\alpha(\zeta)}{v(\zeta,z)^{n+\alpha}}.$$

SKETCH OF PROOF. If $\phi\in\mathcal{O}(\overline{D})$, then it follows from blalabla, for large α and hence for any $\alpha>0$ by analytic continuation. Now we can apply this result, for a fixed z, to the slightly smaller convex domains $D_\epsilon=\{\rho+\epsilon<0\}$. Thus

$$\phi(z)=\int_{\rho+\epsilon<0}\frac{\phi(-\rho-\epsilon)^{\alpha-1}dV_{\rho+\epsilon}}{(\langle\partial\rho,\zeta-z\rangle-\rho-\epsilon)^{n+\alpha}}$$

Then let $\epsilon\to 0$. □

General convex domains estimates etc etc balbla not fully understood so far, lots of research currently. Therefore we will in the sequel restrict to strictly convex domains, i.e., domains defined by a strictly convex function ρ.

PROPOSITION 1.5. *Suppose that D is strictly convex and ρ is a smooth strictly convex defining function. Then,*
(i) $v(\zeta,z)$ is smooth in a neighborhood of $\overline{D\times D}$ and $z\mapsto v(\zeta,z)$ is holomorphic.

$$-\rho(\zeta)-\rho(z)+\delta|\zeta-z|^2\le 2\mathrm{Re}\,v(\zeta,z)\le-\rho(\zeta)-\rho(z)+C|\zeta-z|^2. \tag{1.6}$$

and

$$d_\zeta\mathrm{Im}\,v(\zeta,z)|_{\zeta=z}=-d^c\rho(z)=d_\zeta\mathrm{Im}\,v(z,\zeta)|_{\zeta=z} \tag{1.7}$$

Here

$$d^c=i(\bar\partial\rho-\partial),$$

Notice that d^cu is a real form if u is real. Moreover, notice that

$$dd^cu=??\partial\bar\partial u$$

and that

$$d^cu\wedge du=??\partial u\wedge\bar\partial u.$$

If u is real and $\partial u\neq 0$ therefore $d^cu\wedge du\neq 0$. In fact, at a given point we may assume that $\partial u=dz_1$ after a linear change of coordinates.

Notice that (1.6) in particular implies that if $\zeta, z \in \overline{D}$, then $v(\zeta, z) = 0$ if and only if $\zeta = z$ and $z \in \partial D$.

PROOF. To see (1.7).

$$d_\zeta(\langle \partial\rho(\zeta), \zeta - z\rangle - \rho(\zeta))|_{\zeta=z} = (\partial\rho - d\rho)|_z = -\bar\partial\rho(z),$$

and hence

$$d_\zeta 2\mathrm{Re}\, v = d_\zeta \frac{1}{i}(v - \bar v) = -\frac{1}{i}(\bar\partial\rho - \partial\rho) = -d^c\rho.$$

The second equality in the same way □

1.4. Strictly pseudoconvex domains. Suppose that $D = \{\rho < 0\}$. We say that a tangent vector v at $z \in \partial D$ is complex tangential to ∂D if $d\rho.v = d^c\rho.v = 0$. If v is $(1,0)$, then it is complex tangential as soon as it is tangential, i.e., $d\rho.v = 0$ (since then automatically also $d^c v = 0$). $v \in T^{\mathbb{C}}_{1,0}(\partial D)$

Expressed in complex coordinates, the $(1,0)$-vector $v = \sum_1^n v_j \frac{\partial}{\partial z_j}$ is complex-tangential if and only if $\sum_1^n \rho_j v_j = 0$.

We say that D is strictly psc for each $z \in \partial D$,

$$\partial\bar\partial\rho(h, h) \geq c|h|^2, \quad h \in T^{\mathbb{C}}_{1,0}(\partial D), \tag{1.8}$$

i.e.,

$$\sum_{jk} \rho_{j\bar k} h_j \bar h_k \geq c|h|^2, \quad \sum_j h_j\rho_j = 0. \tag{1.9}$$

If D is defined by aa spsh function then clearly D is spsh, but the condition (1.8) is easily seen to be independent of the choince of defining function, and hence an intrinsic property of ∂D. In fact, if ρ' is another defining function then $\rho' = \alpha\rho$, where $\alpha > 0$ on ∂D, and for $z \in \partial D$,

$$\rho'_{j\bar k} = \alpha\rho_{j\bar k} + \alpha_{\bar k}\rho_j + \alpha_j\rho_{\bar k}.$$

It is now clear that (1.9) holds for ρ' if(f) it holds for ρ.

PROPOSITION 1.6. *If D is strictly psc, then there is a defining function ρ that is strictly psh in a neighborhood of ∂D.*

In fact, one can easily extend it to a s psh function in a neighborhood of $\overline{D}$, see ????.

PROOF. Let ρ be any smooth defining function and consider $\phi \circ \rho$. In view of (1.4), if ϕ is convex enough, $\phi \circ \rho$ will be s psh. □

COROLLARY 1.7. *If D is strictly psc, then it can be approximated from inside and outside by str psc domains.*

In particular a strictly convex domain is str psc; and there is a local converse.

PROPOSITION 1.8. *If D is str psc, then locally at $z \in \partial D$ one can choose holomorphic coordinates so that ∂D is str convex.*

It is not true that a weakly psc domain is locally biholo with a convex domain. !!!

PROOF. Suppose ρ is a s psh defining function. In view of ??? enough to find local change of coordinates at $z = 0$, $z(\zeta)$, such that

$$\frac{\partial^2 \rho}{\partial \zeta_j \partial \zeta_k}(0) = 0,$$

since then balbala.

After a linear change of coordinates $z' = Az$ we may assume that

$$\sum_{jk} \rho_{z_j \bar{z}_k}(0) h_j \bar{h}_k \geq |h|^2,$$

and that $\rho_{z_1}(0) = 1$, $\rho_{z_j}(0) = 0$, $j > 1$.

We now make the change of coordinates

$$z_1(\zeta) = \zeta_1 - \sum_{jk} \rho_{z_j z_k}(0) \zeta_j \zeta_k / 2, \quad z_\ell = \zeta_\ell, \ \ell \geq 2.$$

Then

$$\rho_{\zeta_j} = \sum_k \rho_{z_k} \frac{\partial z_k}{\partial \zeta_j}$$

and so

$$\rho_{\zeta_j \zeta_\ell} = \sum_m \sum_k \rho_{z_k z_m} \frac{\partial z_k}{\partial \zeta_j} \frac{\partial z_m}{\partial \zeta_\ell}$$

and evaluated at 0, it becomes

$$\rho_{\zeta_j \zeta_\ell}(0) = \rho_{\zeta_j \zeta_\ell}(0) - \rho_{\zeta_j \zeta_\ell}(0) = 0,$$

since

$$\frac{\partial z_k}{\partial \zeta_j}(0) = \delta_{jk}.$$

□

THEOREM 1.9 (Fornaess etc). *Let $D = \{\rho < 0\}$ be strictly psc, and suppose that ρ is spsh. There is a neighborhood $\mathcal{U}$ of $\overline{D \times D}$ and a (vector-valued) smooth function $H(\zeta, z)$ in U, holomorphic in z, such that $H(\zeta, z) = \partial \rho(\zeta) + \mathcal{O}(|\zeta - z|)$ and*

$$2\text{Re}\, \langle H(\zeta, z), \zeta - z \rangle \geq \rho(\zeta) - \rho(z) + \delta |\zeta - z|^2.$$

Thus $H(\zeta, z)$ is the subsitute for $H(\zeta, z)$ plays the role of $\partial\rho(\zeta)$,

In view of Proposition?? it is easy to find such an H locally; however, to piece together to a global function is non-trivial. As in the str convex case we now define

$$v(\zeta, z) = \langle H(\zeta, z), \zeta - z\rangle - \rho(\zeta),$$

and in precisely the same way Proposition **??** will hold.

We get analogous representation formulas like ?????

We will use the notation $A \sim B$ for: There are constants $C, c > 0$ such that $cA \leq B \leq CB$.

LEMMA 1.10. *If D is strictly psc, then for fixed $\gamma > 0$, $\epsilon > 0$, we have that*

$$\int_D \frac{-\rho(\zeta))^{\gamma-1}}{|v(\zeta, z)|^{n+\gamma+\epsilon}} \sim \frac{1}{(-\rho(z))^\epsilon},$$

and for $\epsilon > 0$,

$$\int_{\partial D} \frac{dS(\zeta)}{|v(\zeta, z)|^{n+\epsilon}} \sim \frac{1}{(-\rho(z))^\epsilon}.$$

If $\epsilon = 0$ then the integrals are like

$$\log(1/-\rho(z)).$$

Moreover, the same statement hold with z and ζ interchanged.

Let

$$P_\alpha\phi(z) = \int_D \frac{\phi dV_\alpha}{v^{n+\alpha}}.$$

Let $B_\alpha = L^2_\alpha \cap \mathcal{O}(D)$. Taking smaller domains $D_\epsilon = \{\rho + \epsilon < 0\}$ and taking limits we see that ??? holds for $\phi \in B_\alpha$.

Precisely as in Section ??? in the ball case can verify that P_α is a projection $P_\alpha\colon L^2_\alpha \to B_\alpha$; however in general it is not the orthogonal projection. However, we will see later on that a small modification of this projection is approximately orthogonal, in the sense that $P_\alpha - P^*_\alpha$ is compact.

So far we have just met stritcly psc domains that are conex and hence topologically trivial, i.e., contractible, i.e., hopotopy eqquivalent to a point. Here is a non-trivial example.

EXAMPLE 1.2. Notice that

$$\rho(z, w) = 4|1 - zw|^2 + |z|^2 + |w|^2 - 3$$

is strictly psh in $\mathbb{C}^2$; this is checked by a simple computation. It also follows because $|1 - zw|^2$ is psh since $1 - zw$ is holomorphic, cf. ???, and $|z|^2 + |w|^2$ is str psh. Thus

$$D = \{(z, w);\ \rho(z, w) < 0\}$$

is strictly psc since one easily checks that $d\rho \neq 0$ on ∂D. One can also invoke Sard's theorem that ensures that at least a small perturbation, i.e., $3 \pm \epsilon$ for some small ϵ will do.

Notice that the cycle

$$\gamma\colon \quad \theta \mapsto (e^{i\theta}, e^{-i\theta})$$

is contained in D. In particular, the complex line $z = 0$ lies in the complement of D. Therefore, dz/z is a smooth form in D

$$\int_\gamma \frac{dz}{z} = 2\pi i \neq 0$$

so that $H^1(D, \mathbb{C})$ is non-trivial. In particular, D is not contractible. Thus D is topologically non-trivial. In fact, D is homotopy equivalent to the cycle γ. $\square$

1.5. The Koranyi balls. We shall now describe the local geometry at the boundary of a s psc domain, and provide a proof of Lemma 1.10.

Let us first fix a point $z \in \partial D$. By Proposition **??**, $d\rho(\zeta) \wedge d_\zeta \operatorname{Im} v$ is non-zero at $\zeta = z$ and hence we can choose local real coordinates $(y, x_2, x_3, \ldots, x_{2n})$ in a neighborhood of z such that $y = -\rho(\zeta)$, $x_2 = \operatorname{Im} v(\zeta, z)$, and $x_j(z) = 0$.

This means that we have prescribed the two co-directions $d\rho|_z$ and $d^c\rho|_z$ at z, whereas the other $2n-2$ ones are chosen freely. Notice that in this way we have prescribed a $2n-2$-dimensional real subspace of the real tangent space at z. It follows that there is a constant $c > 0$ such that for all $t > 0$,

$$\{\zeta \in \overline{D};\ |v(\zeta, z)| \leq\ ct\} \subset \{(y, x);\ y + |x_2| + \sum_j x_j^2 < t\} \subset \\ \{\zeta \in \overline{D};\ |v(\zeta, z)| \leq\ t/c\}.$$

Notice that these "balls" have extension $\sim \sqrt{t}$ in the $2n-2$ directions determined by $d\rho|_z$ and $d^c\rho|_z$, the "long" directions, and extension $\sim t$ in the remaining two directions. In particular, the volume is $\sim t^{n+1}$.

Remark 1.1. For $\zeta, z \in \partial D$, let

$$d(\zeta, z) = |v(\zeta, z)| + |v(z, \zeta)|.$$

Then clearly $d(\zeta, z) = d(z, \zeta)$. It is also easy to show that there is a constant C such that

$$d(\zeta, z) \leq C(d(\zeta, z') + d(z', z)).$$

This means that d is a pseudo-metric on ∂D called the Koranyi metric. One can easily show that up to obvious equivalence, this metric is determined by the covectors $d\rho$ and $d^c\rho$.

If we choose a smooth projection $z \mapsto z' \in \partial D$ for z close to the boundary, then for any z (close to ∂D)

$$\{\zeta \in D;\ |v(\zeta, z)| \leq t\} \sim \{\zeta \in D;\ d(\zeta', z') - \rho(z) - \rho(\zeta) < t\}.$$

The Koranyi balls $B(z,t) = \{\zeta \in \partial D;\ d(\zeta, z) < t\}$, satisfy the following properties:

(i) There is a $C > 0$ such that if $B(z,t) \cap B(\zeta, s) \neq \emptyset$, then either $B(z, Ct) \supset B(\zeta, z)$ or $B(\zeta, Cs) \supset B(z,t)$.

(ii) There is a $C > 0$ such that for all $z \in \partial D$ and $t > 0$, $|B(z, 2t)| \leq C|B(z,t)|$.

These two properties make ∂D into a space of homogeneuos type, and implies that a large amount of standard harmonic analysis, including maximal function inequalities, Carleson measures, covering lemmas, etc, carry over from the Euclidean case. $\square$

Remark 1.2. Notice that in the definition of the Koranyi balls, it is important to decide from the beginning the directions that are to be "long". Regardless of the choice of "short directions, one gets an equivalent system of balls. However, if one tries to start with the short directions, different choices of long directions will give non-equivalent systems. This is easily seen already in $\mathbb{R}^2$. $\square$

Proof of Lemma 1.10. Fix a point $z_0 \in \partial D$. By uniformity of the inverse function theorem, for each $z \in D$ in a neighborhood ω_{z_0} of z_0 one can choose a local coordinate system (x, y) such that

$$y = -\rho(\zeta), \quad x_2 = \operatorname{Im}(\zeta, z), \quad x_j(z) = 0.$$

Moreover, we may assume that all functional determinants are bounded from below and from above by some uniform constants for all these z. Since ??? it is enough to verify the estimate as long as the integration in balba is performed over a fixed small neighborhood U of z_0, which we may assume is contained in $\{y < 1,\ |x_j| < 1\}$ for each $z \in \omega_{z_0}$. By compactness of ∂D, then the lemma will follow.

Therefore we have to estimate

$$\int_{y=0}^{1} \int_{|} x_j| < 1 \frac{y^{\gamma-1} dy dx_2 \ldots dx_{2n}}{(y - \rho(z) + x_2 + \sum x_j^2)^{n+\gamma+\epsilon}}.$$

Notice that for $a > 0$ and $\alpha > 0$,

$$\int_0^1 \frac{dt}{a+t^2} = \Big(\int_0^{\sqrt{a}} + \int_{\sqrt{a}}^1\Big) \frac{dt}{a+t^2} \sim \int_0^{\sqrt{a}} \frac{dt}{a^\alpha} + \int_{\sqrt{a}}^1 \frac{dt}{t^{2\alpha}} \sim \frac{1}{a^{\alpha-1/2}}.$$

In the same way,

$$\int_0^1 \frac{dt}{a+t} \sim a^{\alpha-1}.$$

Therefore the integral ??? is comparable to

$$\int_0^1 \frac{y^{\gamma-1}dy}{(y-\rho(z))^{\gamma+\epsilon}}.$$

Again decomposing the interval in $\{y < -\rho(z)\}$ and $\{y > -\rho(z)\}$ we get the desired estimate $(1/-\rho(z))^{\epsilon}$. □

2. The $\bar\partial$-equation in a strictly psc domain

In convex domains one can obtain a solution formula for $\bar\partial$ from Koppelman's formula. In $\sim$ 1970 Henkin and Ramirez independently found that each str psc domain D admits a holomorphic support function, i.e., essentially the $H(\zeta, z)$ in Section ???. In this way one gets a representation formula of CFL type in D with holomorphic kernel which makes possible function theory balbla. Henkin also observed that one obtains a solution formula for $\bar\partial$. It turns out that this formula is good for $L^p(\partial D)$ estiamates, $p > 1$.

However, a long standing question was to prove that each divisor in $D = \{\rho < 0\}$ that satisfies the so-called Blaschke condition is defined by a holomorphic function in the Nevanlinna class $N(D)$. The converse follows immediately by Jensen's formula precisely as in one variable. See Section ?????.

It was well known that one could reduce the question to the following a priori estimate:

For each smooth $\bar\partial$-closed $(0,1)$-forms on D there is solution u to $\bar\partial u = f$ in D such that

$$\int_{\partial D} |u| dS \le C \int_D |f| + \frac{1}{\sqrt{-\rho}} |\bar\partial\rho \wedge f|. \tag{2.1}$$

It turns out that if f smooth up to the boundary, there is a solution that is smooth up to the boundary as well, so statement has a meaning.

Let us point out, in the case of one variable, why a weighted formula is needed. Let f be a $(0,1)$-form in the disk D such that

$$\int_D |f| < \infty,$$

and consider the solution

$$u(z) = \frac{1}{2\pi i}\int_D \frac{d\zeta \wedge f}{\zeta - z}.$$

When $|z| = 1$ we have

$$u(z) = -\frac{1}{2\pi i}\int_D \frac{\bar{z} d\zeta \wedge f}{1 - \zeta\bar{z}}.$$

Using the second estimate in Lemma ???? we get,

$$\int_{|z|=1} |u(z)| \leq C \int_D |f| \log(1/1 - |\zeta|),$$

which is not what we wanted. However,

$$u(z) = \frac{1}{2\pi i}\int_D \frac{1 - |\zeta|^2}{1 - \bar{\zeta} z}\frac{d\zeta \wedge f}{\zeta - z}.$$

is also a solution in D to $\bar{\partial} u = f$, since the new factor is holomorphic in z and identically 1 when $\zeta = z$, see Exercise???. When $z = 1$ we thus have

$$u(z) =$$

and if we now use Lemma ??? we get the desired estimate

$$\int_{|z|=1} |u(z)| \leq C \int_D |f| \log(1/1 - |\zeta|).$$

Let us now denote

$$Kf(z) = \frac{1}{2\pi i}\int_{|\zeta|<1} \frac{d\zeta \wedge f}{\zeta - z}.$$

Show that

$$K(\bar{\partial} v)(z) = v(z) - Sv(z),$$

where

$$Sv(z) = \frac{1}{2\pi i}\int_{|\zeta|=1} \frac{v(\zeta) ds(\zeta)}{1 - \bar{\zeta} z}.$$

It is well-known that the Cauchy transform $v \mapsto Sv$, maps $L^p(T)$, $1 < p < \infty$, onto H^p, where H^p is the space of holomorphic functions in D that have boundary values in $L^p(T)$. Therefore, if there is a solution v to $\bar{\partial} v = f$ with boudary values in $L^p(T)$, then Kf will also be such a solution.

3. Solution formulas in strictly pseudoconvex domains

Let X be a domain in $\mathbb{C}^n$ and let

$$\eta = \zeta - z = (\zeta_1 - z, \dots, \zeta_n - z_n)$$

in $X_\zeta \times X_z$. Let E be the subbundle of $T^*(X \times X)$ spanned by $T^*_{0,1}(X \times X)$ and the differentials $d\eta_1, \dots, d\eta_n$. In this section all forms will take values in ΛE. We let δ_η denote formal interior multiplication with

$$2\pi i \sum_1^n \eta_j \frac{\partial}{\partial \eta_j},$$

on this subbundle, i.e., such that $(\partial/\partial\eta_j)d\eta_k = \delta_{jk}$. Moreover, we let

$$\nabla_\eta = \delta_\eta - \bar\partial.$$

Let

$$b = \frac{\eta \cdot d\eta}{2\pi i|\zeta|^2} = \frac{\sum_j(\bar\zeta_j - \bar z_j)d\eta_j}{2\pi i|\zeta - z|^2} = \frac{\partial|\zeta|^2}{2\pi i|\zeta|^2}.$$

and consider the Bochner-Martinelli form

$$u = \frac{b}{\nabla_\eta b} = b + b \wedge (\bar\partial b) + \cdots + b \wedge (\bar\partial b)^{n-1}.$$

Notice that

$$u_{k,k-1} = b \wedge (\bar\partial b)^{k-1} = \frac{1}{(2\pi i)^k} \frac{\partial|\zeta|^2 \wedge (\bar\partial\partial|\zeta|^2)^{k-1}}{|\eta|^{2k}}$$

so that

$$u_{k,k-1} = \mathcal{O}(1/|\zeta|^{2k-1}). \tag{3.1}$$

PROPOSITION 3.1. *The form $u = b/\nabla_\eta b$ is locally integrable in $\mathbb{C}^n \times \mathbb{C}^n$ and it solves*

$$\nabla_\eta u = 1 - [\Delta] \tag{3.2}$$

in the current sense.

We already know that $\bar\partial u_{n,n} = [\Delta]$, so only has to check bab;a;ab as in balbabla.

PROPOSITION 3.2. *If u is any smooth form in $X \times X$ such that $\nabla_\eta u = 1$ and such that* (3.1) *holds locally at the diagonal. Then* (3.2) *holds in the current sense.*

Is proved precisely as ??? in ?????.

EXAMPLE 3.1. Assume that $s(\zeta, z)$ is a smooth form in $X \times X$ such that

$$|s| \leq C|\zeta|, \qquad |\langle s, \eta\rangle| \geq C|\eta|^2 \tag{3.3}$$

uniformly locally at the diagonal. Then

$$u = \frac{s}{\nabla_\eta s} = \frac{s}{2\pi i\,\langle s, \eta\rangle} + \cdots + \frac{s \wedge (\bar\partial s)^{n-1}}{(2\pi i)^n \langle s, \eta\rangle^n}$$

fufills the hypotheses in Proposition 3.2. $\square$

3.1. Henkin-Ramirez formulas. Recall that we solved $\bar\partial$ in smaller ball $r\mathbb{B}$ by patching together BM with an form that was holomorphic in z for ζ close to the boundary. The same can be done in a strictly psc domain by means of the form $H \cdot d\eta$. However, to get a solution in the whole domain the patching must be done infinitesmally close to the boundary.

Now let D be strictly psc and let $H(\zeta, z)$ be the vector-valued ablbaa introduced in ?????, smooth in a nbh of $\overline{D \times D}$. Recall that

$$2\mathrm{Re}\,\langle H(\zeta, z), \eta\rangle \geq \rho(z) - \rho(\zeta) + \delta|\eta|^2. \tag{3.4}$$

We now choose

$$s(\zeta, z) = \overline{\langle H(\zeta, z), \eta\rangle} H(\zeta, z) \cdot d\eta - \rho(\zeta)\bar\eta \cdot d\eta$$

and claim that (3.3) holds in $D \times D$. In fact, clearly $|s| \leq C|\eta|$. Moreover,

$$\langle s, \eta\rangle = |\langle H(\zeta, z), \eta\rangle|^2 - \rho(\zeta)|\eta|^2.$$

If $\zeta \in D$, then $-\rho(\zeta) > 0$ so (3.3) holds.

Also notice that even if $\zeta \in \partial D$ we have that $\langle s, \eta\rangle = 0$ if and only if $z = \zeta$, since then $2\mathrm{Re}\,\langle H, \eta\rangle \geq -\rho(z) + \delta|\eta|^2$.

If we now form the CFL kernel

$$K = \frac{1}{(2\pi i)^n} \frac{s \wedge (\bar\partial s)^{n-1}}{\langle s, \eta\rangle^n}$$

then $\bar\partial K = [\Delta]$ and we thus have the Koppelman formula ?????. as in Theorem 4.3 in Ch. 1. However, notice that when $\zeta \in D$, s is parallell to $H \cdot d\eta$, so holomorphic in z and so $K_{p,q} = 0$ on ∂D for $q > 0$. We therefore have

$$f(z) = \bar\partial_z \int_D K_{p,q-1} \wedge f + \int_D K_{p,q} \wedge f, \quad z \in D,\ q \geq 1,$$

whereas for a function v we have

$$v(z) = \int_D K_{p,0} \wedge \bar\partial v + \int_{\partial D} K_{p,0} v.$$

Notice that the last term is precisely a CFL integral with a kernel that is holomorphic in z so it is a projection $\mathcal{E}_{p,0}(\overline{D}) \to \mathcal{O}(D)$. We can write as

$$f = \bar\partial K f + K(\bar\partial f)$$

and

$$v = K(\bar\partial v) + Pv.$$

If f is a $\bar\partial$-closed form, smoth on $\overline{D}$, then Kf is a solution in D to $\bar\partial v = f$.

Henkin-Ramirez formulas.
ablbalblalbaba

Example in the ball. Then

$$s(\zeta, z) = (|\zeta|^2 - \zeta\cdot\bar z)\bar\zeta\cdot d\eta + (1-|\zeta|^2)\bar\eta\cdot d\eta,$$

and

$$\langle s, \eta\rangle = |1-\bar\zeta\cdot z|^2 - (1-|\zeta|^2)(1-|z|^2).$$

4. Weighted Koppelman formulas

Now let g be a form with values in E, cf., the previous section, such that $\nabla_\eta g = 0$ and $g_0 = 1$ on the diagonal $\Delta \subset X\times X$. If u is a locally integrable form that satisfies (3.2) holds, then

$$\nabla_\eta(g\wedge u) = g - [\Delta].$$

If we let $K = (g\wedge u)_n$ and $P = g_n$ we thus have

$$\bar\partial K = [\delta] - P$$

which as in Ch 1 Section ???, leads to the Koppelman formula
(4.1)

$$f(z) = \bar\partial\int_D K_{p,q-1}\wedge f + \int_D K_{p,q}\wedge\bar\partial f + \int_{\partial D} K_{p,q}\wedge f - \int_D P_{p,q}\wedge f, \quad f\in\mathcal{E}_{p,q}(\overline{D}).$$

In order to obtain a solution formula for $\bar\partial$ we must get rid of the last two terms.

4.1. Weighted formulas in str psc domains. Let $D = \{\rho < 0\}$ be strictly psc and let H be a holomorphic support function as before. We can then define the weight

$$g = \Big(1 + \nabla_\eta \frac{H \cdot d\eta}{2\pi i(-\rho)}\Big)^{-\alpha} = \Big(\frac{-\rho + \langle H, \eta\rangle}{-\rho} - \bar\partial \frac{H \cdot d\eta}{2\pi i(-\rho)}\Big)^{-\alpha}.$$

Thus

$$g = \sum_{k=0}^{n} c_{k,\alpha} \frac{(-\rho)^{\alpha+k}}{(\langle H, \eta\rangle - \rho)^{\alpha+k}} \Big(\bar\partial \frac{H \cdot d\eta}{-\rho}\Big)^{k},$$

so if α is large enough, then $g = 0$ when $\zeta \in \partial D$, and so the boundary integral vanishes. Thus we get a solution formula regardless of the choice of form u. However, for good estimimates on the boundary we will be somewhat careful.

Let $s(\zeta, z)d\eta$ be the form used in the previous section, and let

$$\hat s(\zeta, z)\cdot = -s(z, \zeta)d\eta, \quad \hat H(\zeta, z) = H(\zeta, z).$$

We will use

$$u = \frac{\hat s}{\nabla_\eta \hat s}.$$

we already know that this choice form satisfies (3.2). If f is $\bar\partial$-closed in D, then

$$v(z) = \int_D K \wedge f$$

is a solution to $\bar\partial v = f$, with

$$K = (u \wedge g)_n.$$

We will now prove that this solution formula admits a proof of Theorem **??**. There are similar statements for higher degree forms, but here we restrict, for simplicity, to the case f is a $\bar\partial$-closed $(0,1)$-form.

Sketch of proof of Theorem **??**. Thus assume that f is a $(0,1)$-form, and assume that $z \in \partial D$.

For $z \in \partial D$, at least formally, we skip the necessary strict argument, σ is parallell to

$$\frac{\hat H \cdot d\eta}{2\pi i \left\langle \hat H, \eta \right\rangle}$$

which is holomorphic in ζ. Therefore,

$$K_{0,0} \wedge f = c \frac{\hat H \cdot d\eta}{\left\langle \hat H, \eta \right\rangle} \wedge \frac{(-\rho)^{\alpha+n-1}}{(\langle H, \eta\rangle - \rho)^{\alpha+n-1}} \Big(\bar\partial \frac{H \cdot d\eta}{-\rho}\Big)^{n-1}.$$

LEMMA 4.1. *For $z \in \partial D$ we have*

$$|K_{0,0} \wedge f| \lesssim \frac{(-\rho)^{\alpha}|f|}{|v(\zeta,z)|^{\alpha+n}} + \frac{(-\rho)^{\alpha-1}|\bar\partial\rho \wedge f|}{|v(\zeta,z)|^{\alpha+n-1/2}}.$$

From this lemma, and Lemma 1.10 in Ch.2 we get the theorem. □

Thus it remains to prove Lemma4.1.

PROOF. First notice that

$$\Big(\bar\partial\frac{H\cdot d\eta}{-\rho}\Big)^{n-1} = \frac{(\bar\partial H\cdot d\eta)^{n-1}}{(-\rho)^{n-1}} + \frac{(\bar\partial H\cdot d\eta)^{n-2}\wedge\bar\partial\rho\wedge H\cdot d\eta}{(-\rho)^{n}}$$

We recall that

$$\langle H,\eta\rangle - \rho = v(\zeta,z)$$

and hence

$$\Big\langle \hat H,\eta\Big\rangle = -v(z,\zeta)$$

since $\rho(z)=0$. From the proof of Lemma 1.10 it follows that

$$|v(z,\zeta)| \sim |v(\zeta,z)|.$$

Morever, since $H(\zeta,\zeta) = (\rho_1(\zeta),\dots,\rho_n(\zeta))$ it follows that

$$\hat H\cdot d\eta\wedge H\cdot d\eta = \mathcal{O}(|\eta|).$$

From ??? we also have that

$$|\zeta| \lesssim \sqrt{|v|}.$$

A straight forward estimate of $K_{0,0}\wedge f$ noe gives the lemma. □

Close to the boundary we can write

$$f = a + b\wedge\bar\partial\rho,$$

and the condition on f then means that

$$\int_D |b| + \frac{1}{\sqrt{-\rho}}|a| < \infty.$$

One cam measure f in the metric induced by the positive definite form

$$\Omega = -\rho(i\partial\bar\partial\log(1/-\rho)).$$

Then

$$\Omega_n \sim \frac{dV}{-\rho}$$

and

$$(|f| + \frac{1}{\sqrt{-\rho}}|\bar\partial\rho\wedge f|)dV =$$

which means that balbalblaba.

5. The Henkin-Skoda theorem

The Nevanlinna class N is the set of holomorphic functions in $D = \{\rho < 0\}$ such that

$$\sup_{\epsilon>0} \int_{\rho+\epsilon=0} \log^+ |f| dS < \infty.$$

THEOREM 5.1. *Let D be a str psc domain and Z a divisor in D. Then Z is defined by a function f in the Nevanlinna class N if and only if the Blaschke condition*

$$\int_D -\rho(\zeta)[Z](\zeta) < \infty$$

holds.

6. Exercises

1. Let $\psi(\zeta, z)$ be a C^1 function in $\Omega \times \Omega$ that is holomorphic in z and $\psi(z, z) = 1$ and f is a $(0, 1)$-form of class C^1. Show that

$$u(z) = \frac{1}{2\pi i} \int_\Omega \psi(\zeta, z) \frac{d\zeta \wedge f}{\zeta - z} \tag{6.1}$$

is a solution to $\bar{\partial} u = f$ in Ω if

$$\int_\Omega |\psi(\zeta, z)||f\zeta)| \leq C_K, \quad z \in K,$$

for each compact $K \subset \Omega$.

Conversely, show that if $\psi(\zeta, z)$ is C^1 and (6.1) is a solution for each f with compact support, then ψ is holo in z and $\psi(z, z) = 1$.

CHAPTER 4

Basic residue theory

1. Introduction

Let us first give a glimse of the content of this chapter, without giving any precise definitions and formulations. Let X be a complex manifold and let Z be an analytic subvariety, i.e., locally $Z = \{f_1 = \ldots = f_N = 0\}$ for some holomorphic functions, and let Z_{reg} be the subset of Z where it is smooth, and hence locally a complex submanifold. The set $Z_{sing} = Z \setminus Z_{reg}$ is closed in X. It was proved by Lelong 19?? that the current

$$\xi \mapsto \int_{Z_{reg}} \xi, \qquad \xi \in \mathcal{D}_{n-p,n-p}(X \setminus Z_{sing})$$

in $X \setminus Z_{sing}$ has a natural current extension $[Z]$ to X. This current, that is called the Lelong current associated with Z, turns out to be positive and closed, see Section ??. It provides an analytic representation of the geometric object Z, and for this reason one can consider closed positive currents as intrinsic generalizations of analytic subvarieties. It is easy to see that the annihilator ideal $ann\,[Z]$, i.e., the set of holomorphic functions ϕ such that $\phi[Z] = 0$, is precisely the radical ideal

$$I_Z = \{\phi;\ \phi = 0 \text{ on } Z\}.$$

Assume now that J is a more general ideal in $\mathcal{O}(X)$, say of locally pure dimension. If we want to find a similar analytic object that represents J we are led to consider so-called Coleff-Herrera currents. Let us describe the case with a principal ideal $J = (f)$ generated by the single function f. It was proved by Schwartz in -50's that there exists a distribution U such that $fU = 1$. Then $T = \bar\partial U$ is a $(0,1)$-current with the property that a holomorphic function ϕ annihilates T if and only if ϕ belongs to the ideal (f) generated by f. In fact, if ϕ is holomorphic and $\phi T = 0$, then $0 = \bar\partial(\phi U)$ so that $\phi U = \psi$ is holomorphic, and hence $\phi = f\psi$. Conversely, if $\phi = f\psi$, then it follows that $\phi T = \psi\bar\partial(fU) = \psi\bar\partial 1 = 0$.

Another question is to find an explicit such U. It turns out, but is a highly non-trivial fact, that one can define the *principal value current* $[1/f]$ (Mazzilli??)

$$\Big[\frac{1}{f}\Big].\psi = \lim_{\epsilon\to 0}\int_{|f|>\epsilon}\frac{\psi}{f}, \quad \psi\in\mathcal{D}_{1,1}. \tag{1.1}$$

The current $\bar\partial[1/f]$ the has support on Z and by Stokes' formula we have that

$$\bar\partial\Big[\frac{1}{f}\Big].\psi = \lim_{\epsilon}\int_{|f|=\epsilon}\frac{\psi}{f}, \quad \psi\in\mathcal{D}_{1,0}.$$

Often one is only interested in the action of $\bar\partial[1/f]$ on test forms that are holomorphic at 0 and then it is not necessary to take limits:

$$\bar\partial\Big[\frac{1}{f}\Big].\psi = \int_{|f|=\epsilon}\frac{\psi}{f},$$

if just ϵ is small enough.

We will consider generalizations to more general ideals. To get an idea, let

$$Q=\sum_{|\alpha|<m} a_\alpha(z)\partial_z^\alpha$$

be a holomorphic differential operator, and defined the current μ by

$$\mu.\xi=\int_{Z_{reg}} Q\xi.$$

Here we let Q act as Lie derivatives ablabla. Then this current has support on Z but it is no longer of order zero. It is a so-called Coleff-Herrera current. We will show that roughly speaking all ideals can be expressed locally as the intersections of the annihilators of a finite number of such currents. However the full proof in next chapter. We will also discuss Noetherian differential operators. We will show that for each Coleff-Herrera current one find a finite number of diff op Q_ν such that roughly speaking $\phi\in \mathrm{a}nn\,\mu$ iff $Q_\nu\phi=0$ on X for all ν. Section ????

It is also classical (Poincare-Lelong's formula) that

$$dd^c\log|f| = \bar\partial\Big[\frac{1}{f}\Big]\wedge\frac{df}{2\pi i} = [Z],$$

if $Z=\{f=0\}$ where the various irreducible components are counted with multiplicities. We will consider various generalizations of this formula.

Remark 1.1. Let a and b be say smooth, forms defined outside some exceptional set V in a manifold X, and let P be a differential operator in X such that $Pa = b$ in $X \setminus V$. Moreover, assume that a and b have a reasonable current extension A and B across V; for instance they are locally integrable, or the some sort of principle values exist. In our case V will always be an analytic subvariety and P will be d or $\bar\partial$. Then clearly $PA = B$ outside V since differential operators are local; however sometimes something extra occurs at V, a residue: That is, $PA = B + R$, where R is a current that has support on V. Called a residue current.

The simplest example is the Cauchy kernel $\omega = d\zeta/\zeta$. It is closed outside the origin, but $d\omega = [0]$ so $[0]$ is the residue current here. $\square$

Let us first consider the case of one complex variable. Then the zeros of f is a dicrete set (unless $f \equiv 0$) and so the definition of $[1/f]$ and $\bar\partial[1/f]$ is local, so we may assume $z = 0$ is an isolated zero. As long as th test form ξ is holomorphic

$$\bar\partial\big[\frac{1}{f}\big].\xi dz =$$

Sometimes one just restrict oneself to consider the action on $\psi = 1dz$, and we then get just one number, which is the classical notion of residue

$$\bar\partial\big[\frac{1}{f}\big].1dz = \mathrm{R}es_{z=0}(1/f),$$

from the one-variable theory.

Here we have a preferred coordinate z. If we want an invariant definition we have to define the residue of meromorphic $(1,0)$-forms rather than of functions.

To see that the general limit ???? exists little harder.

Since the zero set of f is discrete, the existence of (1.1) is a local problem so we may assume 0 is an isolated zero of f. Then $f(z) = z^{-m}g(z)$ g is nonvanishing, so locally $f(z) = (z\phi(z))^m$, and we can take $w = z\phi(z)$ as a new holomorphic coordinate. Thus it is enough to consider $f(z) = z^m$.

Proposition 1.1. *For each integer m and each test function $\phi \in \mathcal{D}(\mathbb{C})$ the limit*

$$\big[\frac{1}{z^m}\big].\phi dz \wedge d\bar z = \lim_{\epsilon\to 0}\int_{|z|>\epsilon}\frac{\phi d\zeta \wedge d\bar\zeta}{\zeta^m} \tag{1.2}$$

exists, and defines a current. We have the following properties:

$$z\big[\frac{1}{z^{m+1}}\big] = \big[\frac{1}{z^m}\big], \tag{1.3}$$

$$\frac{\partial}{\partial z}\big[\frac{1}{z^m}\big] = -m\big[\frac{1}{z^{m+1}}\big], \quad m > 0, \tag{1.4}$$

$$\bar\partial\big[\frac{1}{z^m}\big].\phi dz = \lim_{\epsilon\to 0}\int_{|\zeta|=\epsilon}\frac{\phi d\zeta}{\zeta^m} = \frac{2\pi i}{(m-1)!}\frac{\partial^{m-1}}{\partial z^{m-1}}\phi(0). \tag{1.5}$$

$$\bar z\bar\partial\big[\frac{1}{z^m}\big] = 0, \quad m > 0. \tag{1.6}$$

PROOF. By Taylor's formula,

$$\phi(z) = \sum_{\ell+k<m}\frac{\partial^{\ell+k}\phi}{\partial z^\ell\partial\bar z^k}(0)\frac{z^\ell\bar z^k}{\ell!k!} + \mathcal{O}(|z|^m) = p_m(z) + \mathcal{O}(|z|^m).$$

Notice that

$$I = \int_{\epsilon<|\zeta|<R}\frac{\zeta^\ell\bar\zeta^k d\zeta\wedge d\bar\zeta}{\zeta^m} = 0$$

if $\ell + k < m$. In fact, making the change of variable $\zeta \mapsto \lambda\zeta$ with $|\lambda| = 1$, we have that

$$I = \lambda^{\ell-k-m}I,$$

and since $\ell < m \le m + k$ this implies that $I = 0$. If R is large enough we therefore have

$$\int_{\epsilon<|\zeta|}\frac{\phi d\zeta\wedge d\bar\zeta}{\zeta^m} = \int_{\epsilon<|\zeta|<R}\frac{\phi d\zeta\wedge d\bar\zeta}{\zeta^m} = \int_{\epsilon<|\zeta|<R}\frac{\mathcal{O}(|\zeta|^m)}{\zeta^m},$$

which clearly has a limit when $\epsilon \to 0$. Thus (1.2) exists. The equality (1.3) is now immediate. For similar symmetry reasons as above we have that

$$\int_{|\zeta|=\epsilon}\frac{\phi d\bar\zeta}{\zeta^m} = \int_{|\zeta|=\epsilon}\mathcal{O}(1)d\bar\zeta \tag{1.7}$$

and

$$\int_{|\zeta|=\epsilon}\frac{\phi d\zeta}{\zeta^m} = \int_{|\zeta|=\epsilon}\frac{(\partial^{m-1}\phi/\partial\zeta^{m-1})(\zeta)d\zeta}{(m-1)!\zeta} + \int_{|\zeta|=\epsilon}\mathcal{O}(1)d\bar\zeta. \tag{1.8}$$

. Now (1.4) follows from an integration by parts,

$$\frac{\partial}{\partial z}\big[\frac{1}{z^m}\big]\phi dz\wedge d\bar z = -\lim_\epsilon\int_{\epsilon<|\zeta|}\frac{1}{\zeta^m}\frac{\partial\phi}{\partial\zeta}d\zeta\wedge d\bar\zeta = -\lim_\epsilon\int_{\epsilon<|\zeta|}\frac{m}{\zeta^{m+1}}\phi d\zeta\wedge d\bar\zeta,$$

since the boundary integral vanishes in view of (1.7).

The first equality in (1.5) follows by Stokes' formula, (notice the orientation!), whereas the second one follows from (**??**). $\square$

Corollary 1.2. *For a function ϕ that is holomorphic in a neighborhood of* 0 *the following are equivalent:*

(i) $\phi \in (z^m)$

(ii)

$$\phi\bar\partial\Big[\frac{1}{z^m}\Big] = 0$$

(iii)

$$\frac{\partial^\ell \phi}{\partial z^\ell}(0) = 0,\ \ \ell = 0, \dots, m-1.$$

Thus we can represent the ideal either by a generator z^m, as the annihilator of a residue current, or by so-called Noetherian differential operators. Later on we will discuss multivariable analogues to these various representations.

Proof. If (i) holds, then $\phi = \psi z^m$ where ψ is holomorphic, and so

$$\phi\bar\partial\Big[\frac{1}{z^m}\Big] = \psi\bar\partial z^m\Big[\frac{1}{z^m}\Big] = \psi\bar\partial 1 = 0,$$

according to (1.3), and thus (ii) holds. If (ii) holds, then $\bar\partial[1/z^m].\phi\xi dz = 0$ for all test forms ξdz, which by (**??**) means that (iii) must hold. Finally (iii) implies (i) by Taylor's formula. □

From (1.3) and (1.5) we also have

Corollary 1.3.

$$\bar\partial\big[\frac{1}{z^m}\big] \wedge \frac{dz^m}{2\pi i} = m[0].$$

In the several variable case we will mainly rely on another way to define the currents $[1/z^m]$ and $\bar\partial[1/z^m]$: The functions $\lambda \mapsto |z|^{2\lambda}/z^m$ and $\bar\partial|z|^{2\lambda}/z^m$, a priori just defined for $\mathrm{Re}\,\lambda >>$ large, have current-valued analytic continuations to $\mathrm{Re}\,\lambda > -1/2$; and the values at $\lambda = 0$ are precisely the principal value current $[1/z^m]$ and the residue current $\bar\partial[1/z^m]$, respectively. For technical reasons we need the following slightly more elaborated version of this statement.

Lemma 1.4. *Let v be a strictly positive smooth function in $\mathbb{C}$, ψ a test function in $\mathbb{C}$, and m a positive integer. Then*

$$\lambda \mapsto \int v^\lambda |z|^{2\lambda}\psi(z)\frac{ds \wedge d\bar z}{z^m}$$

and

$$\lambda \mapsto \int \bar\partial(v^\lambda |z|^{2\lambda}) \wedge \psi(z)\frac{dz}{z^m}$$

both have analytic continuations to $\mathrm{Re}\,\lambda > -1$, *and the values at* $\lambda = 0$ *are* $[1/z^m].\psi ds \wedge d\bar s$ *and* $\bar\partial[1/z^m].\psi ds$, *respectively. In particular, the* *second one vanishes if* $(d\bar z/\bar z)\wedge\psi = 0$*, i.e. if* $\psi(z) = \bar z\phi(s)$ *or* $\psi = d\bar z\wedge\phi$*.*

Proof. With the same notation as in the previous proof we have for a large enough R and $\operatorname{Re}\lambda$ large and fixed $\mu\in\mathbb{C}$,

$$\int v^\mu|\zeta|^{2\lambda}\frac{\phi d\zeta\wedge d\bar\zeta}{\zeta^m} = \int_{|\zeta|<R}\frac{|\zeta|^{2\lambda}(v^\mu\phi)d\zeta\wedge d\bar\zeta}{\zeta^m} = \int_{|\zeta|<R}\frac{|\zeta|^{2\lambda}\mathcal{O}(|\zeta|^m)d\zeta\wedge d\bar\zeta}{\zeta^m}$$

since

$$\int_{|\zeta|<R}\frac{|\zeta|^{2\lambda}p(\zeta)d\zeta\wedge d\bar\zeta}{\zeta^m}$$

for symmetry reasons as before. Hence the proposed analytic continuation to $\operatorname{Re}\lambda > -1/2$ exists. It is easy to see that $p(\zeta)$ and hence $\mathcal{O}(|\zeta|^m)$ are holomorphic in μ, and therefore the left hand side is holomorphic in (λ,μ) for $\operatorname{Re}\lambda > -1$, $\mu\in\mathbb{C}$, and when $\lambda = 0$ it is equal to $[1/z^m].v^\mu\phi d\zeta\wedge d\bar z$, in view of the previous proof. Taking $\mu = 0$, the statement follows. The second integral is $\bar\partial(v^\mu|z|^{2\lambda})/z^m.\psi dz$ for large $\operatorname{Re}\lambda$, and hence even at $\lambda = 0$. $\square$

2. Positive currents and Lelong currents

From now on, even if μ is a current we will frequently use the suggestive notation

$$\int\mu\wedge\psi$$

for the action of μ on the test form ψ, rather than $\mu.\psi$.

We say that an (n,n)-current μ is *positive* if

$$\int\psi\mu\geq 0$$

for all test forms ψ such that $\psi\geq 0$. Since μ is just a distribution it follows as usual that then μ has order zero, i.e., it is just a locally finite positive measure.

We say that a (p,p)-current μ is *positive* if

$$\mu\wedge ia_{p+1}\wedge\bar a_{p+1}\wedge\ldots\wedge ia_n\wedge\bar a_n\geq 0$$

for all smooth $(1,0)$-forms a_j.

Lemma 2.1. *If* μ *is a positive* (p,p)*-current, then it is real and has order zero.*

Moreover, if for fixed coordinates

$$\mu = \sum_{|I|=p,|J|=p}{}' \mu_{IJ}i^{p^2}dz_I\wedge d\bar z_J,$$

then μ_{II} are positive for each I and there is a number C only depending on n, p such that

$$|\mu_{IJ}| \leq C \sum_{|I|=p}' \mu_{II}.$$

If we fix coordinates z,

$$\mu = \sum_{|I|=p,|J|=p}' \mu_{IJ} i^{p^2} dz_I \wedge d\bar{z}_J,$$

so we have to prove that μ_{IJ} are complex measures.

Proof. Fix IJ and let L, K be complementary multiindices. Recall from Lemma 6.2 in Ch.1 that ????? so that

$$i^{p^2} dz_L \wedge d\bar{z}_K =$$

$$\bigwedge\nolimits_{s=1}^{p} i dz_{L_s} \wedge d\bar{z}_{K_s} = \bigwedge\nolimits_{s=1}^{p} \frac{i}{4} \sum_{\ell \in \mathcal{Z}_4} i^\ell (dz_{L_s} + i^\ell dz_{K_s}) \wedge \overline{(dz_{L_s} + i^\ell dz_{K_s})},$$

so it follows that

$$\pm \mu_{IJ} dV = \mu \wedge i^{p^2} dz_L \wedge d\bar{z}_K = \sum_{a \in \mathcal{Z}_4^p} \epsilon_a \mu \wedge \gamma_a,$$

where

$$\gamma_a = \bigwedge\nolimits_{s=1}^{p} \frac{i}{4} (dz_{L_s} + i^{a_s} dz_{K_s}) \wedge \overline{(dz_{L_s} + i^{a_s} dz_{K_s})}$$

and $\epsilon_a = \pm 1, \pm i$. Thus μ_{IJ} is a linear combination of locally finite positive measure and hence a complex measure. Moreover,

$$|\mu_{IJ}| dV \leq \sum_a \mu \wedge \gamma_q = \gamma \wedge \bigwedge\nolimits_1^p \sum_{a_s \in \mathcal{Z}_4} \frac{i}{4} (dz_j + i^{a_s} dz_k) \wedge \overline{(dz_j + i^{a_s} dz_k)} =$$

$$\mu \wedge \bigwedge\nolimits_1^p (i dz_{I_s} \wedge d\bar{z}_{I_s} + i dz_{K_s} \wedge d\bar{z}_{K_s}) \leq C \sum_{|J|=p}' \mu_{JJ} dV,$$

where C is just a combinatorial constant. Finally, μ it is real for the same reason as for smooth positive forms. □

Theorem 2.2 (Lelong). *Let Z be a variety of codimension p in a complex manifold X. Then*

$$\phi \mapsto \int_{Z_{reg}} \phi$$

defines a positive closed (p, p)-current $[Z]$.

The simplest and most elegant way is to use Hironaka's theorem. There exists a $k = (n-p)$-dimensional smooth complex manifold $\tilde{Z}$ and a proper mapping $\pi\colon \tilde{Z} \to Z$, such that, if $Y = Z \setminus Z_{reg}$ and $\tilde{Y} = \pi^{-1}Y$, then $\pi\colon \tilde{Z} \setminus \tilde{Y} \to Z \setminus Y$ is a biholomorphism. This is called a resolution of singularities or a desingularization, and is a very deep result which rendered Hironaka a Fields medal. Now since π is proper and $\tilde{Y}$ is a null set in $\tilde{Z}$,

$$\int_{Z_{reg}} \phi = \int_{\tilde{Z}\setminus\tilde{Y}} \pi^*\phi = \int_{\tilde{Z}} \pi^*\phi$$

exists, and

$$\Big|\int_{Z_{reg}} \phi\Big| \le C \sup_Z |\phi|.$$

Moreover,

$$\int_{Z_{reg}} d\psi = \int_{\tilde{Z}} d\pi^*\psi = 0,$$

so that $d[Z] = 0$. The positivity follows from the positivity in the smooth case. Thus the theorem is proved.

[Maste veta har att $\pi^*\phi$ ar glatt i $\tilde{Z}$ om ϕ glatt i omg till Z i X. Kolla upp !!! samt vad det betyder att π holo dvs holo struktur pa Z]

Example 2.1. $Z = \{z^2 - w^2 = 0,\ \tilde{Z} = \mathbb{C},\ \pi(t) = t^3, t^2)$. □

However, it is instructive to consider an elementary proof.

Proof. Since the statement is local we can assume that Z is in $\mathbb{C}^n$. We first prove that Z_{reg} has locally finite area. Fix a point $0 \in Z$. After a rotation we may assume that there is a small polydisk with center 0 such that the projection of Z onto each coordinate plane of codimension p is a finite covering outside some hypersurface in the coordinate plane. Let us call coordinates z and $w = (w_1, \ldots, w_p)$ and $\pi\colon \Delta' \times \Delta'' \subset \mathbb{C}^{n-p}_z \times \mathbb{C}^p_w \to \mathbb{C}^p_z$. Since Z has a finite number of sheets above $\Delta' \setminus Y$, the total area of the projection is finite. Hence the total area is finite in view of ???? in Ch 1.

Hence we have in particular that

$$\Big|\int_{Z_{reg}} \phi\Big| \le C \sup |\phi|$$

and hence $[Z]$ is a well-defined current. If a_j $(1,0)$ smooth, then $ia_1 \wedge \bar{a}_1 \wedge \ldots \wedge ia_{n-1} \wedge \bar{a}_{n-p}$ is a positive form on Z_{reg} and so $[Z]$ is a positive current.

It remains to prove that $[Z]$ is closed. With notation as above, choose an ϵ-neighborhood of Y_ϵ of Y in Δ'. We then have

$$[Z].d\psi = \lim \int_{Z\setminus\pi^{-1}(Y_\epsilon)} d\psi = \int_{Z\cap\partial\pi^{-1}(Y_\epsilon)} \psi.$$

Notice that z are local coordinates on Z at $\partial\pi^{-1}(Y_\epsilon)$. Just a finite number of integrals over ∂Y_ϵ so it is enough to see that it can be chosen so that the area of ∂Y_ϵ tends to zero. However, first consider the regular part of Y. Since it has finite real codim 2 area in Δ' we can cover it by say balls so that the total area of the boundaries are like ϵ. Next, take the components of Y of (complex) codim 2. Again locally finite codim 4 real area etc. balbala ??????????????????? See [**?**] Propsition ??? for details $\square$

We now consider two results that indicate that positive closed currents should be considered as geometric objects, generalizing varieties.

THEOREM 2.3. *Let T be a (k,k)-current such that both T and dT have order* 0 *(i,e, have measure coefficients). If T has support on a variety Z of codimension $p > k$, then $T = 0$.*

LEMMA 2.4. *If T is any current such that $dw_j \wedge T = 0$, then $T = T' \wedge dw_j$, where T' contains no ocurrence of dw_j.*

PROOF. We can write $T = T' \wedge dw_j + \gamma$, where T' and γ have no occurrence of dw_j. Now, $0 = T \wedge dw_j = \gamma \wedge dw_j$, which implies that $\gamma = 0$. $\square$

PROOF. Locally on Z_{reg} we can choose coordinates (z, w) so that $V = \{w = 0\}$. Since T has order zero $w_j T = \bar{w}_j T = 0$. Moreover, since dT has order zero, $(d(w_jT) = dw_j \wedge T + w_j dT = dw_j \wedge T$ so that $dw_j \wedge T = 0$ and similarly $d\bar{w}_j \wedge T = 0$. By repeated use of the lemma it follows that

$$T = T' \wedge dw \wedge_1 \wedge d\bar{w}_1 \wedge \ldots \wedge dw_p \wedge d\bar{w}_p,$$

and since $k < p$, this implies that $T' = 0$. Thus $T = 0$ on Z_{reg}. Thus T has support on $Y = Z \setminus Z_{reg}$. By induction it follows that $T = 0$. $\square$

COROLLARY 2.5. *If T is a positive closed (k,k) current with support on a variety V of codimension $p > k$, then $T = 0$.*

THEOREM 2.6. *Let T be a closed (p,p)-current of order zero, with support on a variety Z of codimension p. Then*

$$T = \sum_j \alpha_j [Z]$$

PROOF. First assume that $p = 0$. Then $dT = 0$ implies that T is a locally constant function. For the general case first consider Z_{reg} and assume again that $Z = \{w = 0\}$. As in the previous proof we find that $T = \tilde{T} dw \wedge d\bar{w}$, and since T has order zero, $\tilde{T}$ is a measure on $\{w = 0\}$. Now, $0 = dT = d_z\tilde{T} \wedge dw \wedge d\bar{w}$ implies that $\tilde{T}$ is locally constant. Since the regular part of each connected component Z_j of Z is connected, it follows that T is $\alpha_j[Z_j]$ there. Finally, $T - \sum_j \alpha_j[Z_j]$ is then a closed $(1,1)$-current of order zero with support on $Z \setminus Z_{reg}$ so it must vanish according to Theorem 2.3. $\square$

2.1. Positive $(1,1)$-currents and psh functions. Let us introduce

$$d^c = \frac{i}{2\pi}(\bar{\partial} - \partial);$$

notice that

$$dd^c = \frac{i}{\pi}\partial\bar{\partial}.$$

THEOREM 2.7. *Assume that $\mu = \sum \mu_{jk} i dz_j \wedge d\bar{z}_k$ is a $(1,1)$-current in X.*
(i) μ is positive
(ii) $\sum_{jk} \mu_{jk} h_j \bar{h}_k \geq 0$, for all $h \in \mathbb{C}^n$
(iii) locally $\mu = i\partial\bar{\partial}u$ for some psh function u.

PROOF. $\square$

THEOREM 2.8 (Poincare-Lelong's equation). *Let f be holomorphic and Z its zero set with irrdeucible components Z_j. Then*

$$dd^c \log|f| = \sum_j \alpha_j[Z_j], \tag{2.1}$$

where α_j is a positive integer, the order of f at Z_j.

PROOF. We first consider the one-variable case. It follows from Proposition 1.1 that

$$dd^c \log|z^m| = m\bar{\partial}\frac{1}{z} \wedge dz/2\pi i = [0].$$

From Theorem 2.7 we know that $dd^c \log|f|$ is a closed positive $(1,1)$-current, and in view of Theorem 2.6 we thus have a representation (2.1), so we just have to see that α_j are positive integers. Locally on $Z_{j,reg}$

we may assume that it is $\{w = 0\}$. Then we can write $f = w^\alpha g(z, w)$, where $g \neq 0$. Since $dd^c \log |g|$ is pluriharmonic therefore,

$$dd^c \log |f| = \alpha dd^c \log |w| = \alpha[0] \otimes 1 = \alpha[w = 0] = \alpha[Z_j].$$

Thus (ii) holds. The remaining verifications are left as exercises. $\square$

It is worthwhile to notice that $\partial \log |f|^2 = \partial f/f$ is locally integrable even at singular points, and that the equality holds in the current sense. To see this we may assume that f is a (nonvanishing times a) Weierstrass polynomial with respect to each variable. Thus $f(z', w) = w^r + a_{r-1}(z') + \cdots$. Outside a ablabla
thenough to see that balbala

Monge-Ampere uppskattning, labla Nirenberg ablabl uppskattning, sid 169 i Dem

2.2. Resolutions of singularities. As a consequence of Hironaka's theorem we have the following extremely useful result.

THEOREM 2.9 (Hironaka). *Let Y be a subvariety of a complex manifold X. Then for any point there is neighborhood $\mathcal{U}$ and a proper holomorphic mapping $\pi\colon \tilde{\mathcal{U}} \to \mathcal{U}$ that is a biholomorphism $\tilde{U} \setminus \tilde{Y} \to \mathcal{U} \setminus Y$, and such that $\tilde{Y}$ has normal crossings in $\tilde{\mathcal{U}}$.*

Let $f = (f_1, \ldots, f_m)$ be holomorphic functions on X and let $Y = \{f_1 f_2 \cdots f_m = 0\}$. Then for any point there is neighborhood $\mathcal{U}$ and a proper holomorphic mapping $\pi\colon \tilde{\mathcal{U}} \to \mathcal{U}$ that is a biholomorphism $\tilde{U} \setminus \tilde{Y} \to \mathcal{U} \setminus Y$, and such that $\tilde{Y}$ has normal crossings in $\tilde{\mathcal{U}}$. This means that locally in $\tilde{U}$ one can choose coordinates τ such that $\tilde{Y}$ is the zero set of a monomial $\mu = \tau_1^{\alpha_1} \cdots \tau_n^{\alpha_n}$. However, since then the zero set of the function $\pi^* f_j$ is contained in the set where the monimial μ vanishes, it follows that π^* is itself a monomial times a non-vanishing function.

Toric resolution, balbalbala $(\mathbb{C}^*)^n$ etc etc etc etc balblabalbal.

THEOREM 2.10. *If f_j are monomials in $\mathbb{C}^n$, there is a toric resolution $\tilde{\pi}\colon \tilde{X} \to \mathbb{C}^n$ such that locally in $\tilde{X}$, one of the monomials divides all the other ones.*

Given a tuple $f = (f_1, \ldots, f_m)$ after two resolution we may assume that locally $f = f_0(f_1', \ldots, f_m') = f_0 f'$, where f_0 is a monomial and f' is a non-vanishing tuple of holomorphic functions.

PROPOSITION 2.11. *Assume that* $\pi\colon \tilde{X} \to X$ *is a resolution and*

$$\tilde{X} \setminus \tilde{Y} \simeq X \setminus Y,$$

Assume that γ *is a smooth form in* $X \setminus Y$ *and that* $\omega = d\gamma$. *Assume that* $\tilde{\gamma} = \pi^*\gamma$ *is locally integrable in* $\tilde{X}$. *Then*
(i) γ *is locally integrable in* X
(ii) if

$$d\tilde{\gamma} = T + \tilde{\omega}\mathbf{1}_{\tilde{X}\setminus\tilde{Y}},$$

then

$$d\gamma = \pi_* T + \omega\mathbf{1}_{X\setminus Y}.$$

(iii) if $T \geq 0$ *then* $\pi_* T \geq 0$, *if* $dT = 0$ *then* $d\pi_* T = 0$, *if* T *has order zero, then* π_* *has order zero.*

PROOF. For test forms $\xi \in \mathcal{D}(X)$, let

$$\Lambda\xi = \int_{\tilde{X}\setminus\tilde{Y}} \pi^* f \wedge \pi^*\xi,$$

which is welldefined since $\pi^* f$ id locally integrable and π is proper so that $\pi^*\xi$ is a test form in $\tilde{X}$. If $\operatorname{supp}\xi \subset K$, then

$$|\Lambda\xi| \leq \int_{\tilde{X}\cap\pi^{-1}K} |\pi^* f| \sup_{\pi^{-1}K} |\pi^*\xi| \leq C_K \sup_K |\xi|.$$

It follows that Λ defines a locally finite measure m in $\tilde{X}$, and since

$$\Lambda\xi = \int f \wedge \xi$$

if $\xi \in \mathcal{D}(X \setminus Y)$ it follows that f is the restriction of m to $X \setminus Y$ and hence locally integrable. So (i) is proved.

To see (ii), notice that

$$\pm\, df.\xi = \int_X f \wedge d\xi = \int_{X\setminus Y} f \wedge d\xi = \int_{\tilde{X}\setminus\tilde{Y}} \pi^* f \wedge d\pi^*\xi =$$
$$\pm\, T.\pi^*\xi + \int_{\tilde{X}\setminus\tilde{Y}} \pi^*(df) \wedge \pi^*\xi = \pi_* T.\xi + \int_{X\setminus Y} df \wedge \xi.$$

The proofs of the remaining statements are left as an exercise. $\square$

Notice that similar statements hold with d replaced by d^c, ∂, $\bar{\partial}$, or dd^c.

2.3. King's formula. As an application of this technique we can give a quite expedient proof of King's formula which is a generalization of the Poincare-Lelong formula.

PROPOSITION 2.12. *Assume that f is a tuple of holomorphic functions with common zero set Z, and that $|f|$ is Hermitian any norm. Then*

$$(dd^c \log|f|)^k, \quad d\log|f| \wedge (dd^c \log|f|)^k, \quad \log|f|(dd^c \log|f|)^k,$$

considered as forms in $X \setminus Z$ are locally integrable in X. Moreover, if Z has codimension p, then for each $k < p$,

$$dd^c[\log|f|(dd^c \log|f|)^{k-1}] = (dd^c \log|f|)^k \tag{2.2}$$

in the current sense, and also

$$d^c[\log|f|(dd^c \log|f|)^{p-1}] = d^c \log|f| \wedge (dd^c \log|f|)^{p-1}$$

in the current sense.

Here $|f|^2 = \sum_{jk} h_{jk} f_j \bar{f}_k$ where h_{jk} is a Hermitian matrix that very well may depend on z.

PROOF. We first prove that $\omega = (dd^c \log|f|)^k$ is locally integrable. Given a point x there is a neighborhood $\mathcal{U}$ such that we have a resolution $\tilde{\mathcal{U}} \to \mathcal{U}$ in which $\pi^* f_j$ are monomials. In view of Proposition 2.11 it is enough to prove that $\pi^*\omega$ is locally integrable in $\tilde{U}$. On the other hand, to this end, again by Proposition 2.11 it is enough to prove that the pullback to another resolution, as in Theorem **??**, is locally integrable. However, locally in this resolution $\tilde{f} = f_0 f'$ with $f' \neq 0$ and f_0 is a monomial. Thus we have that

$$dd^c \log|\tilde{f}| = dd^c \log|f_0| + dd^c \log|f'| = dd^c \log|f'|$$

outside $\tilde{Y}$ (in fact outside $\tilde{Z}$, where $Z = \{f = 0\}$, since f_0 is plutiharmoniic there. Thus $(dd^c \log|f|)^k$ smooth, and thus in particular locally integrable. Moreover, if

$$f_0 = \tau_1^{a_1} \cdots \tau_n^{a_n}, \quad a_j \geq 0,$$

then

$$\log|f_0| = \sum_j a_j \log|\tau_j|, \quad \frac{df_0}{f_0} = \sum_j a_j \frac{d\tau_j}{\tau_j}$$

which are locally integrable. It follows that the pullback of the other forms are locally integrable times snooth and hence locally integrable.

For the secon statement, we notice that

$$dd^c[\log|f| \wedge (dd^c \log|f'|)^{k-1}] = d[d^c \log|f| \wedge (dd^c \log|f'|)^{k-1}] =$$
$$[f_0 = 0] \wedge (dd^c \log|f'|)^{k-1} + (dd^c \log|f'|)^k$$

in the resolution. Thus the restriction of $dd^c[\log|f| \wedge (dd^c \log|f|)^{k-1}]$ to $Z \subset Y$ is the push-forward of $[f_0 = 0] \wedge (dd^c \log|f'|)^{k-1}$. However, this is a closed (k,k)-form of order zero with support on Z so it vanishes if $k < p$ and (**??**) below holds if $k = p$. □

Clearly (2.2) holds outside Z. When $k \geq p$ it turns out that some residues occur at Z. We define

$$(dd^c \log|f|)^p = dd^c[\log|f|(dd^c \log|f|)^{p-1}]$$

Write $(dd^c \log|f|)^p \mathbf{1}_{X \setminus Z}$ to emphasize not consider any possible residues at Z.

THEOREM 2.13 (King's formula). *Let f be a holomorphic tuple with zero set Z and let Z_j be the irreducible components of minimal codimension p. Then*

$$(dd^c \log|f|)^p = \sum_j \alpha_j [Z_j] + (dd^c \log|f|)^p \mathbf{1}_{X \setminus Z}, \tag{2.3}$$

where α_j are positive integers, the so-called Hilbert-Samuel multiplicities.

The Hilbert-Samuel multiplicity of f at a regular point of Z is defined as follows: For generic choices of $[a_j] \in \mathbb{P}^{m-1}$ the restriction of f to a generic p-plane through the point is a mapping $\mathbb{C}^p \to \mathbb{C}^p$ with isolated zero at the point. The number α is the degree of that mapping. We first consider the special case when $p = m = n$.

PROPOSITION 2.14. *If $f = (f_1, \dots, f_n)$ has an isolated zero at $z = 0$, then using the trivial metric*

$$dd^c[\log|f|(dd^c \log|f|)^{n-1}] = d[d^c \log|f| \wedge (dd^c \log|f|)^{n-1}] = \alpha[0],$$

where α is the degree of the mapping f.

PROOF. First notice that

$$(dd^c \log|f|)^n = \bar\partial\Big[\frac{1}{(2\pi i)^n}\frac{\partial|f|^2}{|f|^2} \wedge \big(\bar\partial\frac{\partial|f|^2}{|f|^2}\big)^{n-1}\Big] = \bar\partial f^* B,$$

where

$$\frac{1}{(2\pi i)^n}\frac{\partial|z|^2}{|z|^2} \wedge \Big(\bar\partial\frac{\partial|z|^2}{|z|^2}\Big)^{n-1} = B(z),$$

is the Bochner-Martinelli kernel. If $\bar\partial f^*B = \alpha[0]$, then bu Stokes' theorem

$$\alpha = \int_{\partial D} f^*B$$

so it is enough to make sure that this integral is the degree of the mapping f. For w close to 0, let $f_w = f - w$. By Sard's theorem 0 is a regular value for f_w for almost all w. For such a value, 0 has d preimages $\zeta^1, \ldots, \zeta^d$, and f_w is a biholomorhic in a neighborhood of each of them. Therefore the integral by Stokes' theorem is equal to

$$\sum_1^d \int_{|\zeta-\zeta^j|=\epsilon} f^*B.$$

However, since biholomorphism each of them equal to 1 since

$$\int_{\partial D} B = 1.$$

Since moreover the integral is obviously continuous in w and integer valued almost everywhere it follows that it is a constant integer in a neighborhood of 0; this value is by definition the degree at $w = 0$, and so the proposition is proved. $\square$

LEMMA 2.15. *Assume that f and g are holomorphic tuples such that $|f| \sim |g|$ and $Z = \{f = 0\}$ has codimension p. Then*

$$(dd^c \log |f|)^p \mathbf{1}_Z = (dd^c \log |g|)^p \mathbf{1}_Z.$$

PROOF. We may assume $p < n$, because otherwise we can just add a dummy variable z_{n+1}. Consider a fixed irreducible component Z' of Z of codimension p. On the regular part Z'_{reg} let us fix two distinct points a, b and define a real smooth function $t(z)$ with values between 0 and 1 such that it is identically 1 in a neighborhood of a and 1 in a neighborhood of b. Then $u = t|f| + (1-t)|g|$ is a nonnegative function. In an appropriate resolution where $f = f_0 f'$ and $g = g_0 g'$ and f_0 and g_0 are (nonvanishing holo times) monomal it follows that $f_0 = c g_0$ with $c \neq 0$, so

$$u = |f_0|(t|f'| + (1-t)|g'|).$$

Precisely as before it follows that

$$(dd^c \log u)^p \mathbf{1}_{Z'}$$

is a closed (p,p)-current of order 0, and hence it must be $\alpha[Z']$ for some constant α. However it is equal to ??? at a and ?? at b so klart! $\square$

LEMMA 2.16. *If f is a holomorphic m-tuple with isolated zero at 0 (detta inte nodvandigt men gor beviset lattare ngt), then for generic choices of $[a] = ([a_1], \dots, [a_n]) \in (\mathbb{P}^{m-1})^n$, $j = 1, \dots, n$, the n-tuple $a \cdot f = (a_1 \cdot f, \dots, a_n \cdot f)$ satisfies $|a \cdot f| \sim |f|$.*

Se Demailly!! Ch. 8 Them 10.3

PROOF OF KING'S FORMULA. Since the set of possible choices of a in the preceding lemma is connected, the degree α of the mapping $a \cdot f$ must be independent of a, and by Lemmas 2.15 2.16 and Proposition 2.14, it follows that

$$(dd^c \log |f|)^n \mathbf{1}_{\{0\}} = \alpha[0].$$

This proves King in case $p = n$. Finally, if $p < n$ and point on Z_{reg} and p-plane through point, that intersects Z_{reg} transversally, then may assume Z_{reg} is $\{w = 0\}$ and the plane is $\{z = c\}$. We know that

$$\alpha[w = 0] = dd^c[\log |f|(dd^c \log |f|)^{p-1}]\mathbf{1}_Z.$$

For degree reasons can replace each d, d^c with d_w, d^c_w. However, then the equality just means the degree of the mapping f restricted to the plane $z = c$, with parameter c. In particular α must be an integer. $\square$

REMARK 2.1. Let J be any ideal sheaf in X with zero set Z of minimal codimension p. Then by a partition of unity we can find a real function u such that locally $u \sim \log |f|$, where f is a tuple that defines J and $|\cdot|$ is any metric. It follows from the proof above that

$$(dd^c u)^p \mathbf{1}|_Z = \sum_j \alpha_j [Z_j].$$

$\square$

Let

$$\mathcal{A}^f_{k,\lambda} = \bar\partial |f|^{2\lambda} \wedge \frac{\partial |f|^2 \wedge (\bar\partial \partial |f|^2)^{k-1}}{(2\pi i)^p |f|^{2k}}.$$

LEMMA 2.17. *For the trivial metric we have that*

$$dd^c(\log |f|(dd^c \log |f|)^{p-1})\mathbf{1}_Z = \lim_{\lambda \to 0^+} \mathcal{A}^f_{p,\lambda}.$$

PROOF. Since $\log |f|(dd^c \log |f| \mathbf{1}_{X \setminus Z})^{p-1} \mathbf{1}_{X \setminus Z}$ is locally integrable in X, and $\lambda \mapsto (|f|^{2\lambda} - 1)$ is increasing for $\lambda > 0$ we have by dominated convergence that

$$\int \log |f|(dd^c \log |f|)^{p-1} \wedge dd^c \phi =$$

$$\lim_{\lambda \to 0^+} \int \frac{1}{2\lambda} \big(|f|^{2\lambda} - 1\big)(dd^c \log |f|)^{p-1} \wedge dd^c \phi.$$

The current $(dd^c \log|f|)^{p-1}\mathbf{1}_{X\setminus Z}$ is closed in the current sense according to (**??**), and an integration by parts therefore gives

$$\lim_{\lambda\to 0^+}\int \bar\partial|f|^{2\lambda}\wedge\frac{\partial|f|^2}{2\pi i|f|^2}\wedge\Big(\bar\partial\frac{\partial|f|^2}{2\pi i|f|^2}\Big)^{p-1}\wedge\phi+ \tag{2.4}$$
$$\lim_{\lambda\to 0^+}\int |f|^{2\lambda}\Big(\bar\partial\frac{\partial|f|^2}{2\pi i|f|^2}\Big)^{p}\wedge\phi,$$

which proves the lemma since the second term in (2.4) is precisely

$$\int (dd^c\log|f|)^p\mathbf{1}_{X\setminus Z}\wedge\phi$$

(the finiteness of the limit is ensured by King's formula). □

Namn Rouche's sats har.

3. Coleff-Herrera currents

We will first consider a class of residue currents that are called Coleff-Herrera currents, or *local residual currents*.

DEFINITION 1. Let V be an analytic variety in X of pure codimension p. A (r,p)-current μ with support on V is a Coleff-Herrera current on V, $\mu\in \mathrm{C}H_V$, if it is $\bar\partial$-closed,

$$\bar I_V\mu=0, \tag{3.1}$$

and has the following property: For any holomorphic function h that does not vanish identically on any irreducible component of V, we require that $\lambda\mapsto|h|^{2\lambda}\mu$, a priori defined for λ with $\mathrm{Re}\,\lambda$ large, has an analytic continuation to $\mathrm{Re}\,\lambda>-\epsilon$ and the value at $\lambda=0$ coincides with μ.

It is easy to see that $\mathrm{C}H_V$ is a sheaf of $\mathcal{O}$-modules, and a subsheaf of the sheaf $\mathcal{C}_V^{0,p}$ of $(0,p)$-currents with support on V and of $\mathcal{C}^{0,p}$.

The property (3.1) just means that $\bar h\mu=0$ for any holomorphic h that vanishes on V. The last property is called the *standard extension property*, SEP, and means that μ is determined by its values on $V\setminus Y$ for any hypersurface Y not containing any irreducible component of V. In fact, if $\mu\in \mathrm{C}H_V$ has support on $V\subset Y$, then $|h|^{2\lambda}\mu$ must vanish if $\mathrm{Re}\,\lambda$ is large enough, and by the uniqueness of analytic continuation thus $\mu=0$. Given the other conditions, it turns out that SEP is automatically fulfilled on V_{reg} so the interesting case is when the zero set Y of h contains the singular locus of V.

The SEP can also be expressed in the following way: If h is a holomorphic function that does not vanish identically on any irreducible component of V, and $\chi(t)$ is a smooth cutoff function on $\mathbb{R}$ that is 0 for $t < 1$ and 1 for $t > 2$, then

$$\lim_{\epsilon} \chi(|h|/\epsilon)\mu = \mu. \tag{3.2}$$

Classically SEP is formulated with the characteristic function for $\{t \geq 1\}$ rather than a smooth approximand. However, in that case it is not obvious that $\chi(|h|/\epsilon)\mu$ has a meaning; this requires an additional argument. The reason for our seemingly technical choice of definition of SEP is merely practical. In Subsection 3.11 we will prove that any of these definitions of SEP can be used to define the sheaf $\mathrm{C}H_Z$.

This sheaf CH_V of currents is natural for several reasons. It is well-known that the so-called local (moderate) cohomology sheaves $\mathcal{H}^k_{[V]}(\mathcal{O})$ vanish for $k < p$ whereas $\mathcal{H}^p_{[V]}(\mathcal{O})$ is isomorphic the sheaf $\mathrm{C}H_V$; i.e., each class is represented by a unique Coleff-Herrera current. Another reason is that there is a close connection between Coleff-Herrera currents and Noetherian differential operators. Moreover, Coleff-Herrera currents (of bidegree (n, p)) are natural generalizations of Abelian differentials.

3.1. The monomial case. Let t be coordinates in $\mathbb{C}^n$. Let α be a smooth form. We know that each expression

$$\big[\alpha \wedge \frac{1}{t_1^{m_1}} \cdots \frac{1}{t_k^{m_k}} \bar\partial \frac{1}{t_{k+1}^{m_{k+1}}} \wedge \ldots \wedge \bar\partial \frac{1}{t_p^{m_p}}\big] \tag{3.3}$$

has a well-defined meaning as a current, since it is just the tensor product of one-variable currents, and that its action on a test form ϕ can be computed as the value at $\lambda = 0$ of the function

$$\lambda \mapsto \int \frac{|v_1 t_1^{a_1}|^{2\lambda} \cdots |v_k t_k^{a_k}|^{2\lambda} \bar\partial |v_{k+1} t_{k+1}^{a_{k+1}}|^{2\lambda} \wedge \bar\partial |v_p t_p^{a_p}|^{2\lambda} \wedge \alpha \wedge \phi}{w_1^{m_1} \cdots t_p^{m_p}},$$

where v_j are any nonvanishing functions. It follows from Corollary 1.3 that

$$\big[\bar\partial \frac{1}{t_p^{\alpha_p}} \wedge \ldots \wedge \bar\partial \frac{1}{t_1^{\alpha_1}}\big] \wedge \frac{dt_1^{\alpha_1} \wedge \ldots \wedge dt_p^{\alpha_p}}{(2\pi i)^p} =$$
$$\alpha_1 [t_1 = 0] \otimes \ldots \otimes \alpha_p [t_p = 0] = \alpha_1 \cdots \alpha_p [t_1 = \cdots = t_p = 0].$$

In the sequel we will skip the brackets in (3.3). It is clear that (3.3) iss commuting in the principal value factors and anti-commuting in the residue factors.

If α has compact support we say that (3.3) is an *elementary current.*

It is readily shown that if τ is elementary as in (3.3), then

$$\bar\partial\tau = \sum_1^k \alpha\wedge\frac{1}{t_1^{m_1}}\cdots\frac{1}{t_{j-1}^{m_{j-1}}}\frac{1}{t_{j+1}^{m_{j+1}}}\cdots\frac{1}{t_k^{m_k}}\bar\partial\frac{1}{t_j^{m_j}}\wedge\bar\partial\frac{1}{t_{k+1}^{m_{k+1}}}\wedge\ldots\wedge\bar\partial\frac{1}{t_p^{m_p}}+$$
$$\bar\partial\alpha\wedge\frac{1}{t_1^{m_1}}\cdots\frac{1}{t_k^{m_k}}\bar\partial\frac{1}{t_{k+1}^{m_{k+1}}}\wedge\ldots\wedge\bar\partial\frac{1}{t_p^{m_p}}$$

and thus a finite sum of elementary currents. In the same way for $\partial\tau$.

Notice that if α is of the form $\bar t_\nu\alpha'$ for $\nu\le k$, then τ vanishes.

3.2. Pseudomeromorphic currents. We now introduce a class of currents that we call *pseudomeromorphic.* They share some important properties with the space of normal currents; a main feature is that a pseudomeromorphic current μ must vanish if its support is too small compared to its degree. We shall also see that a pseudomeromorphic current admits a reasonable restriction $\mu\mathbf{1}_Z$ to any subvariety Z. It turns out that many interesting currents that appear in the multi-variable residue theory are indeed pseudomeromorphic.

Let $\mathcal{U}$ be an open set in a complex manifold X and let $\pi^1\colon\mathcal{U}_1\to\pi^1\mathcal{U}_1\subset\mathcal{U}$, $\pi^{k+1}\colon\mathcal{U}_{k+1}\to\pi^{k+1}\mathcal{U}_{k+1}\subset\mathcal{U}_k$, $k\le N$, be a finite sequence of modifications, and assume that τ is an elementary current in $\mathcal{U}_N$. Then

$$\pi^1_*\cdots\pi^\ell_*\tau \tag{3.4}$$

is a current in $\mathcal{U}$ with compact support. We define $\mathcal{PM}(\mathcal{U})$ as the space of locally finite sums of currents of this type. For simplicity we write π_* for $\pi^1_*\cdots\pi^\ell_*$. Then each $T\in\mathcal{PM}(\mathcal{U})$ can (locally in $\mathcal{U}$) be written as

$$T=\sum\pi_*\tau_\ell$$

where π denote possibly different sequences of modifications and τ_ℓ are elementary. To be precise we should keep indices even on π_* but for simplicity we suppress balbla.

It is easy to see that this defines a sheaf $\mathcal{PM}$ over X. Recall that if τ is an elementary current, then $\bar\partial\tau$ is a finite sum of elementary currents. Since moreover $\bar\partial$ commutes with push-forwards it follows that $\mathcal{PM}$ is closed under $\bar\partial$. In the same way it is closed under ∂. Moreover, if ξ is a smooth form, then

$$\xi\wedge\pi^1_*\cdots\pi^\nu_*\tau=\pi^1_*\cdots\pi^\ell_*((\pi^1)^*\cdots(\pi^\nu)^*\xi\wedge\tau)$$

so $\mathcal{PM}$ is closed under multiplication with smooth forms.

Proposition 3.1. *Assume that $T \in \mathcal{PM}$ has support on the variety V and that h is a holomorphic function that vanishes on V. Then $\bar{h}T = d\bar{h} \wedge T = 0$.*

Proof. Consider an elementary residue current τ in the coordinates t. Locally in $\mathcal{U}_\nu$ we can make take a resolution of singularities $\pi^{\nu+1}\colon \mathcal{U}_{\nu+1} \to \pi^{\nu+1}\mathcal{U}_{\nu+1} \subset \mathcal{U}_\nu$ such that each $(\pi^{\nu+1})^* t_j$ as well as $(\pi^{\nu+1})^*(\pi^\nu)^* \cdots (\pi^1)^* h$ are monomials times non-vanishing. It follows that we have a representation (3.4) of T such that the pullback of h is a monomial times nonvanishing for each ℓ.

Now

$$|h|^{2\lambda}\pi^1_* \cdots \pi^\nu_* \tau = \pi^1_* \cdots \pi^\nu_* |(\pi^\nu)^* \cdots (\pi^1)^* h|^{2\lambda}\tau).$$

Notice that the analytic continuation to $\lambda = 0$ of

$$|(\pi^\nu)^* \cdots (\pi^1)^* h|^{2\lambda}\tau)$$

exists and is equal to τ if none of the coordinates in the residue factors of τ is a factor in $\pi^* h$ and zero if at least one of these coordinates is a factor in $\pi^* h$. We thus have a decomposition

$$T = \sum_\ell \pi_* \tau'_\ell + \sum_\ell \pi_* \tau''_\ell$$

where the first sum contains precisely those τ_ℓ that have a residue coordinate as one of the factors in $\pi^* h$.

Since h vanishes on the support of T it follows that $|h|^{2\lambda} T = 0$ for $\operatorname{Re}\lambda >> 0$. It follows that

$$0 = |h|^{2\lambda} T|_{\lambda=0} = \sum_\ell \pi_* \tau'',$$

and hence

$$T = \sum \pi_* \tau'.$$

Since $\bar{\pi}^* h \tau'_\ell = d\bar{\pi}^* h \wedge \tau'_\ell = 0$, the proposition follows. □

Precisely as in the proof of Theorem 2.3 we get

Theorem 3.2. *If $T \in \mathcal{PM}$ has bidegree (k, p) and has support on a variety V with $codim V > p$, then $T = 0$.*

In fact, locally $Z_{reg} = \{w_1, \ldots, w_{p+1} = 0\}$ and by ???? we have that $d\bar{w}_j \wedge T = 0$ for $j = 1, \ldots, p+1$. Since T has bidegree $(0, p)$ it follows that T must vanish. Thus the support must be contained in $Z \setminus Z_{reg}$ which has codimension at least $p + 2$, and by finite induction we conclude that $T = 0$.

We have the following analogue of Theorem 2.6.

THEOREM 3.3. *Assume that V is a subvariety with pure codimension p. If μ is is pseudomeromorphic of bidegree $(*, p)$ with support on V and $\bar\partial\mu = 0$ then μ is a Coleff-Herrera current on Z.*

We will see later on that each Coleff-Herrera current is pseudomeromorphic.

PROOF. We alrady know from Proposition 3.1 that μ is annihilated by $\bar I_V$ so we just have to check the SEP. Let h be a function not vanishing identically on any irreducible component of Z. As in the proof of Proposition 3.1 one verifies that $|h|^{2\lambda}\mu|_{\lambda=0}$ exists and is pseudomeromorphic. Then

$$T = \mu_f - |h|^{2\lambda}\mu^f|_{\lambda=0}$$

must have its support contained in $Y = Z \cap \{h = 0\}$ which has codimension $p+1$ and so it must vanish, i.e., the SEP holds. $\square$

3.3. Coleff-Herrera-Passare products. It is comparatively easy (Malgrange) to prove existence of a current U such that $fU = 1$. Obs clear that $\psi \in (f)$ if and only if $\psi\bar\partial U = 0$.

A specific choice of such a current is the principal value current

$$\Big[\frac{1}{f}\Big].\xi = \lim_\epsilon \int_{|f|>\epsilon} \frac{\xi}{f}.$$

However, the existence of this current is highly nontrivial, and for the general case there is no known argument that does not involve Hironaka's theorem. (kolla Mazzilli!!) It follows by Stokes' theorem that

$$\bar\partial\Big[\frac{1}{f}\Big].\xi = \lim_\epsilon \int_{|f|=\epsilon} \frac{\xi}{f}.$$

Let $\chi(t)$ be the characteristic function for the interval $[1, \infty)$. Then

$$\bar\partial\Big[\frac{1}{f}\Big] = \lim_{\epsilon\to 0} \frac{\bar\partial\chi(|f|/\epsilon)}{f}.$$

Therefore if we have $f_1, \ldots, f_m$ it is natural to try to define

$$\Big[\bar\partial\frac{1}{f_1} \wedge \ldots \wedge \bar\partial\frac{1}{f_p}\Big]$$

as

$$\lim_{\epsilon\to 0} \frac{\bar\partial\chi(|f_1|/\epsilon)}{f_1} \wedge \ldots \wedge \frac{\bar\partial\chi(|f_m|/\epsilon)}{f_m},$$

that is,

$$\Big[\bar\partial\frac{1}{f_1} \wedge \ldots \wedge \bar\partial\frac{1}{f_p}\Big].\xi = \lim_{\epsilon\to 0} \int_{|f_j|=\epsilon_j} \frac{\xi}{f_1 \cdots f_p}. \tag{3.5}$$

This was done by Coleff and Herrera in (year) bala. However, to make sure that the limit exists they assumed that $\epsilon_{j+1}/\epsilon_j$ tends to zero fast, for all j.

They also showed that it is anticommutative in the factors of f defines a complete intersection. Passare showed that OK outside some sectors; later on counterexamples, Passare-Tsikh even for $m = 2$ and complete intersection. JEB proved non-existence is "generic". However Samuelsson recently showed that if let χ be a smooth approximand then the unrestricted limit exists, at least for $m \leq 3$. Here to begin with we will stick to the analytic continuation definition instead. For the equivalence, see Subsection 3.11 below??. It is therefore natural with the following definition.

THEOREM 3.4. *For a general holomorphic mapping $f = (f_1, \ldots, f_m)$ one can define a current*

$$\tau = \Big[\frac{1}{f_{p+1}\cdots f_m}\bar\partial\frac{1}{f_1}\wedge\cdots\wedge\bar\partial\frac{1}{f_p}\Big] \tag{3.6}$$

as the value at $\lambda = 0$ of

$$\tau^\lambda = \frac{|f_{p+1}\cdots f_m|^{2\lambda}\bar\partial|f_1|^{2\lambda}\wedge\ldots\wedge\bar\partial|f_p|^{2\lambda}}{f_1\cdots f_m} \tag{3.7}$$

(a priori a well-defined current if $\operatorname{Re}\lambda >> 0$) and we have the following properties:

(o) τ is a pseudomeromorphic current.

(i) τ is anti-commuting in the indices $1, \ldots, p$ and commuting in the other ones.

(ii) The formal Leibniz' rule holds, i.e.,

$$\bar\partial\tau = \sum_{j=p+1}^{m}\Big[\frac{1}{f_{p+1}\cdots\hat f_j\cdots f_m}\bar\partial\frac{1}{f_j}\wedge\bar\partial\frac{1}{f_1}\wedge\cdots\wedge\bar\partial\frac{1}{f_p}\Big].$$

(iii) If f_{p+1} is nonvanishing, then

$$f_{p+1}\tau = \Big[\frac{1}{f_{p+2}\cdots f_m}\bar\partial\frac{1}{f_1}\wedge\cdots\wedge\bar\partial\frac{1}{f_p}\Big]$$

and if f_1 is non-vanishing, then $\tau = 0$.

(iv) The support of τ is contained in $Y = \{f_1 = \cdots = f_p = 0\}$.

(v) If h is holomorphic, then $|h|^{2\lambda}\tau$ has an analytic continuation to $\lambda = 0$.

(vi) If h is holomorphic and vanishing on Y, then $\bar h\tau = 0$ and $d\bar h\wedge\tau = 0$.

EXAMPLE 3.1. If s is a complex variable then a simple computation reveals that

$$|s|^{2b}\bar\partial|s|^{2a}=\frac{a}{a+b}\bar\partial|s|^{2a+2b}. \tag{3.8}$$

Moreover, we notice that

$$\lambda\mapsto|s|^{2\lambda}\big[\frac{1}{s^m}\big]$$

a priori defined for $\operatorname{Re}\lambda>>0$, has an analytic current-valued continuation to $\operatorname{Re}\lambda>-1$, and the value at $\lambda=0$ is precisely $[1/s^m]$. In fact, by Prop ??? (??), for $\operatorname{Re}\lambda$ large,

$$|s|^{2\lambda}\big[\frac{1}{s^m}\big]=|s|^{1\lambda-2m}\bar s^m s^m\big[\frac{1}{s^m}\big]=|s|^{1\lambda-2m}\bar s^m=|s|^{2\lambda}/s^m.$$

□

PROOF. We can assume that X is so small that we have a resolution $\pi\colon\tilde X\to X$ such that locally in $\tilde X$, $\pi^*f_j=u_j\mu_j$, where μ_j are monomials and u_j are nonvanishing. For $\operatorname{Re}\lambda>>0$ with a suitable partition of unity ρ_j on $\tilde X$, since $\pi^{-1}(\operatorname{supp}\phi)$ is compact, we have for $\operatorname{Re}\lambda>>0$ that

$$\int_X\tau^\lambda\wedge\phi=\int_{\tilde X}\pi^*\tau^\lambda\wedge\pi^*\phi=\sum_j\int\pi^*\tau^\lambda\rho_j\wedge\pi^*\phi,$$

where

$$\int\pi^*\tau^\lambda\rho_j\wedge\pi^*\phi= \\ \int\frac{|u_{p+1}\mu_{p+1}\cdots u_m\mu_m|^{2\lambda}\bar\partial|u_1\mu_1|^{2\lambda}\wedge\ldots\wedge\bar\partial|u_p\mu_p|^{2\lambda}\wedge\alpha\wedge\pi^*\phi}{\mu_1\ldots\mu_m}, \tag{3.9}$$

and α is smooth with compact support. By repeated use of (3.8) and the observations in Subsection 3.1, it follows that (3.9) has an analytic continuation to $\operatorname{Re}\lambda>-\epsilon$. Thus (3.7) has a current valued analytic continuation to $\operatorname{Re}\lambda>-\epsilon$. Moreover, it follows that the value at $\lambda=0$ of (3.9) coincides with the value at $\lambda=0$ of

$$\int\frac{|\mu_{p+1}\cdots\mu_m|^{2\lambda}\bar\partial|\mu_1|^{2\lambda}\wedge\ldots\wedge\bar\partial|\mu_p|^{2\lambda}\wedge\alpha\wedge\pi^*\phi}{\mu_1\ldots\mu_m}.$$

Expanding each expression $\bar\partial|\mu_i|^{2\lambda}$ and using ablbala we can conclude that in fact τ is indeed pseudomeromorphic.

The properties (i) and (ii) are immediate from the definition. Now notice that if v is non-vanishing, then $\lambda\mapsto v^\lambda\tau^\lambda$ and $\bar\partial v\lambda\wedge\tau$ are both holomorphic for $\operatorname{Re}\lambda \geq -\epsilon$ and the values at $\lambda = 0$ are τ and 0, respectively. In fact,

$$v^{\lambda'}\tau^{\lambda}.\phi = \tau^{\lambda}.v^{\lambda'}\phi$$

is holomorphic for $\lambda' \in \mathbb{C}$ and $\operatorname{Re}\lambda > -\epsilon$, and the value at $\lambda' = \lambda = 0$ is equal to $\tau.\phi$. In the same way, ????? Taking $v = |f_{p+1}|^2$ we now get the first statement in (iii). The second one follows in a similar way.

It follows from (iii) that τ vanishes where $f_j \neq 0$, for $j = 1, \ldots, p$ and hence (iv) holds.

Properties (v) and (vi) follows immediately from Proposition ??? and its proof since τ is pseudomeromorphic.

$\square$

EXAMPLE 3.2. Let $f_2(z) = z_1^2$ and $f_1(z) = z_1 z_2$. Then, for instance, we have that

$$z_1 z_2\Big[\frac{1}{z_1 z_2}\bar\partial\frac{1}{z_1^2}\Big] = \frac{z_1 z_2 |z_1 z_2|^{2\lambda}\bar\partial|z_1|^{4\lambda}}{z_1 z_2 z_1^2}\Big|_{\lambda=0} =$$
$$= \frac{2}{3}\frac{|z_2|^{2\lambda}\bar\partial|z_1|^{6\lambda}}{z_1^2}\Big|_{\lambda=0} = \frac{2}{3}\bar\partial\frac{1}{z_1^2}.$$

$\square$

The point here is that f_1 and f_2 have a common factor. However, in the complete intersection case this phenomenon never occurs; it is indeed possible to cancel any denominator by multiplication.

THEOREM 3.5. *Assume that f is a complete intersection. Then*

$$f_{p+1}\left[\frac{1}{f_{p+1}\cdots f_m}\bar\partial\frac{1}{f_1}\wedge\cdots\wedge\bar\partial\frac{1}{f_p}\right] = \left[\frac{1}{f_{p+2}\cdots f_m}\bar\partial\frac{1}{f_1}\wedge\cdots\wedge\bar\partial\frac{1}{f_p}\right] \tag{3.10}$$

and

$$f_p\left[\frac{1}{f_{p+1}\cdots f_m}\bar\partial\frac{1}{f_1}\wedge\cdots\wedge\bar\partial\frac{1}{f_p}\right] = 0. \tag{3.11}$$

PROOF. Let T be the difference between the left and right hand sides in (3.10). From Theorem 3.4 follows that T has its support on the set $Z = \{f_1 = \cdots = f_p = f_{p+1} = 0\}$, and by the assumption of a complete intersection, Z has codimension $p+1$. Since T is pseudomeromorphic of bidegree $(0, p)$ it follows that it must vanish and therefore (3.10) holds.

By Leibniz' rule,

$$\Big[\frac{1}{f_{p+1}\cdots f_m}\bar\partial\frac{1}{f_1}\wedge\ldots\wedge\bar\partial\frac{1}{f_p}\Big]=\bar\partial\Big[\frac{1}{f_pf_{p+1}\cdots f_m}\bar\partial\frac{1}{f_1}\wedge\ldots\wedge\bar\partial\frac{1}{f_{p-1}}\Big]-\Big[\frac{1}{f_p}\bar\partial\frac{1}{f_{p+1}\cdots f_m}\wedge\bar\partial\frac{1}{f_1}\wedge\ldots\wedge\bar\partial\frac{1}{f_{p-1}}\Big].$$

Since $\bar\partial$ commutes with multiplication with f_{k+1}, (3.10) and a new application of Leibniz' rule gives (3.11). □

THEOREM 3.6. *Let f be a complete intersection. Then the Coleff-Herrera product*

$$\mu^f=\Big[\bar\partial\frac{1}{f_p}\wedge\cdots\wedge\bar\partial\frac{1}{f_1}\Big]$$

is indeed a Coleff-Herrera current on $Z=\{f_1=\ldots=f_p=0\}$.

PROOF. It follows from Theorem 3.4 that μ^f has its support on Z and that it is $\bar\partial$-closed, and since it is pseudomeromorphic it follows from balblaa that it is Coleff-Herrera. □

EXAMPLE 3.3. Let Z be a variety of pure codimension p, then $[Z]$ is in $\mathcal{C}H_Z$. In fact, we only have to check SEP. If Z is smooth, it is clear that $|h|^{2\lambda}[Z]$ has an analytic cont and $=[Z]$ at $\lambda=0$. In general use $\pi\colon\tilde Z\to Z$. Also see $\xi\mapsto[Z].Q\xi$ where Q holo dif operator is in $\mathcal{C}H_Z$. Will see that essentially all $\mathcal{C}H_Z$ of this form □

REMARK 3.1 (Resolutions and dimension of subvarieties). In a resolution $\pi\colon\tilde X\to X$, the inverse image $\tilde Y$ of a variety Y in X is (usually) a hypersurface in $\tilde X$ so any assumption about big codimension, e.g., an assumption about complete intersection, will necessarily be destroyed. However, it will be reflected on the pullback of a test form in the following way. Any smooth $(0,q)$-form ψ can locally be written $\psi=\sum_\nu\psi_\nu\bar\omega_\nu$, where ω_ν are holomorphic $(0,q)$-forms and ψ_ν are smooth. Now assume that the complex dimension of Y is smaller than q, so that (the pullback of) ψ vanishes of Y for degree reasons. Moreover, assune that s is a local coordinate function in $\tilde X$ such that $\{s=0\}\subset\tilde Y$. Then $\pi^*\omega_\nu$ is holomorphic and vanishes on the hyperplane $\{s=0\}$ and therefore it is a sum of terms, each of which is either divisible by s or by ds. It follows that $\tilde\psi$ is a sum of terms each of which is a smooth form times $\bar s$ or a smooth form times $d\bar s$. □

EXAMPLE 3.4. Let $X=\mathbb{C}^2_{z,w}$ and $Y=\{0\}$ and let $\tilde X$ be the blow-up at 0, and assume that $z=s,\ w=st$, so that $\tilde Y=\{s=0\}$. Then $\pi^*d\bar w=\bar{dt}+\bar s d\bar t$, so both kind of terms may appear. □

3.4. The duality theorem. Let $f = f_1, \dots, f_p$ be a complete intersection on some complex manifold X and let μ^f be the associated Coleff-Herrera product. It follows from Theorem 3.5 that the ideal sheaf (f) generated by f is contained in the annihilator sheaf $ann\,\mu^f$. We shall now prove that we in fact have equality (first proved independently by Passare and Dickenstein-Sessa in 1985). This also follows from Theorem 3.5 in combination with the exactness of the sheaf complex

$$0 \to \mathcal{O} \to \mathcal{C}^{0,0} \xrightarrow{\bar\partial} \mathcal{C}^{0,1} \xrightarrow{\bar\partial} \tag{3.12}$$

and some homological algebra. However, to make the argument conceptually ??? it is convenient to consider f as a section of a (trivial) holomorphic vector bundle.

Thus E be a trivial vector bundle of rank p with global frame $e_1, \dots, e_p$ and let e_j^* be its dual frame for the dual bundle E^*. If we consider $f = \sum_j f_j e_j^*$ as a section of the dual bundle, it induces a mapping δ_f on the exterior algebra ΛE. We will also consider differential forms and currents with values in Λ. For instance $\mathcal{E}_{0,k}(\Lambda^\ell E)$ is the sheaf of smooth $(0,k)$-forms with values in Λ^ℓ which we consider as a subsheaf of the sheaf of the bundle $\Lambda(E \oplus T^*(X))$. For the reader not familiar with vector bundle formalism, a section of $\mathcal{E}_{0,k}(\Lambda^\ell E)$ is just a formal expression

$$v = \sum_{|I|=\ell}{}' f_I \wedge e_I$$

where f_I are smooth $(0,k)$-forms, and with the convention that $d\bar z_j \wedge e_j + -e_j \wedge d\bar z_j$ etc. In the same way we have the sheaf $\mathcal{C}^{0,k}(\Lambda^\ell E)$ etc. Notice that both $\bar\partial$ and δ_f acts as anti-derivations on these sheaves, i.e.,

$$\bar\partial(f \wedge g) = \bar\partial f \wedge g + (-1)^{\deg f} f \wedge \bar\partial g$$

if at least one of f and g is smooth, and similarly for δ_f. Moreover, it is straight forward to check that

$$\delta_f \bar\partial = -\bar\partial \delta_f. \tag{3.13}$$

This means that we have a so-called double complex, and the operator on the total complex is

$$\nabla_f = \delta_f - \bar\partial.$$

It is also an anti-derivation, and because of (3.13) we have that

$$\nabla_f^2 = 0. \tag{3.14}$$

If $\mathcal{L}^k = \oplus_j \mathcal{E}_{0,j-k}(\Lambda^j E)$, then $\nabla_f \colon \mathcal{L}^k \to \mathcal{L}^{k+1}$. In view of the exactness of (3.12), the double complex $\mathcal{C}^{0,k}(\Lambda^\ell E)$ is exact in the k-direction except at $k=0$ where we have the cohomology sheaf $\mathcal{O}(\Lambda^\ell)$. Notice that a holomorphic function ϕ is in the ideal J if and only if $\delta_f \psi = \phi$ has a solution $\psi \in \mathcal{O}(\Lambda^1 E)$. By standard homological algebra it follows that the natural mapping

$$\operatorname{Ker}_{\delta_f} \mathcal{O}(\Lambda^{p-k} E)/\delta_f \mathcal{O}(\Lambda^{p-k+1} E) \to \operatorname{Ker}_{\nabla_f} \mathcal{L}^k / \nabla_f \mathcal{L}^{k-1}$$

is an isomorphism. We can just as well replace $\mathcal{E}$ with $\mathcal{C}$. In particular, the case $k=0$ has the following two useful implications.

Lemma 3.7. *If there is a current solution V to $\nabla_f V = \phi$, then locally ϕ belongs to the ideal $J = (f)$. If $\mu \wedge e \in \mathcal{C}^{0,p}(\Lambda^p E)$ and $\nabla_f(\mu \wedge e) = 0$, then there is a function $\psi \in \mathcal{O}$, unique in $\mathcal{O}/J$, such that $\nabla_f V = \psi - \mu$ has a solution.*

For the reader's convenience we supply a direct proof.

Proof. Let $V = V_1 + \ldots + V_p$, where $V_k \in \mathcal{C}^{0,k-1}(\Lambda^k E)$. Then $\bar\partial V_p = 0$ and since (3.12) is exact locally we can solve $\bar\partial W_p = V_p$. Now, $\bar\partial[V_{p-1} + \delta_f W_p] = \bar\partial V_{p-1} - \delta_f \bar\partial W_p = \bar\partial V_{p-1} - \delta_f V_p = 0$ and so we can solve $\bar\partial W_{p-1} = V_{p-1} + \delta_f W_p$. Continuing in this way we finally get that $\psi = V_1 + \delta_f W_2$ is a holomorphic solution to $\delta_f \psi = \phi$. The second statement is proved in a similar way. $\square$

Consider the current

$$V = \Big[\frac{1}{f_1}\Big] e_1 + \Big[\frac{1}{f_2}\bar\partial\frac{1}{f_1}\Big] \wedge e_1 \wedge e_2 + \Big[\frac{1}{f_3}\bar\partial\frac{1}{f_2} \wedge \bar\partial\frac{1}{f_1}\Big] \wedge e_1 \wedge e_2 \wedge e_3 + \cdots \tag{3.15}$$

A a simple computation, using Theorem 3.5, yields that

$$\nabla_f V = 1 - \mu^f \wedge e,$$

where $e = e_1 \wedge \ldots \wedge e_p$.

Proposition 3.8. *Let f be a complete intersection and assume that $\nabla_f U = 1 - \mu \wedge e$. Then $ann\, \mu = J$.*

Proof. If $\phi \in ann\, \mu$, then $\nabla_f U\phi = \phi - \phi\mu \wedge e = \phi$ and hence $\phi \in J$ by Lemma 3.7. Conversely, if $\phi \in J$, then there is a holomorphic ψ such that $\phi = \delta_f \psi = \nabla_f \psi$ and hence $\phi\mu = \nabla_f \psi \wedge \mu = \nabla_f(\psi \wedge \mu) = 0$. $\square$

Theorem 3.9 (Duality theorem). *If f is a complete intersection, then $ann\, \mu^f = (f)$.*

REMARK 3.2. Assume that X is a Stein manifold. Then the complex of global sections induced by (3.12) is exact, and hence the duality result holds globally on X, i.e., a global holomorphic function ϕ is in the ideal generated by $f_1, \ldots, f_m$ in the ring $\mathcal{O}(X)$ if and only if $\phi\mu^f = 0$. $\square$

3.5. A uniqueness result. We shall now see that μ^f is the unique current in $\mathrm{C}H_Z$ such that $\mu^f \wedge e$ is ∇_f-cohomologous to 1, but first we need a simple but important lemma.

LEMMA 3.10. *If μ is in CH_Z and for each neighborhood ω of Z we have locally current W with support in ω such that $\bar\partial W = \mu$, then $\mu = 0$.*

The proof will also provide a description of μ locally on Z_{reg}. Later on we will see that a similar description holds even across the singular part.

PROOF. In fact, locally on Z_{reg} can choose coordinates (z, w) such that $Z = \{w = 0\}$. Since $\bar w_j \mu = 0$ and $\bar\partial\mu = 0$ it follows that $d\bar w_j \wedge \mu = 0$, $j = 1, \ldots, p$, and hence $\mu = \mu_0 d\bar w_1 \wedge \ldots \wedge d\bar w_p$. We claim that locally

$$\mu = \sum_{|\alpha|<M} a_\alpha(z)\bar\partial\frac{1}{w_1^{\alpha_1+1}} \wedge \ldots \wedge \bar\partial\frac{1}{w_p^{\alpha_p+1}}, \tag{3.16}$$

where $a_\alpha(z)$ are holomorphic functions. In fact, let a_α be the push-forwards of $w^\alpha dw \wedge \mu$ under the projection $(z, w) \mapsto z$, i.e., if $\psi(z)dz \wedge d\bar z$ is a test form, then

$$a_\alpha.\psi(z)dz \wedge d\bar z = \mu.w^\alpha dw \wedge \psi(z)dz \wedge d\bar z. \tag{3.17}$$

To prove that (3.16) holds, it is enough to check the equality on test forms of the type $\xi(z, w)dw \wedge d\bar z \wedge dz$. Applying the right hand side of (3.16) to the test form we get

$$\int_z \sum_{|\alpha|<M} a_\alpha(z)\frac{\partial_w^\alpha \xi(z, 0)}{\alpha!} dz \wedge d\bar z. \tag{3.18}$$

On the other hand, by Taylor's formula,

$$\xi(z, w) = \sum_{|\alpha|<M} \frac{\partial_w^\alpha \xi(z, 0)}{\alpha!} w^\alpha + \cdots + \mathcal{O}(|w|^M),$$

where $\cdots$ denote terms containing some factor $\bar w_j$, and if we apply μ to $\xi dw \wedge d\bar z\; dz$ we again get (3.18) in view of (3.17). Since $\bar\partial\mu = 0$, it follows that $\bar\partial a_\alpha = 0$ and so a_α are holomorphic.

Notice that by the ????? we have

$$\bar\partial\frac{1}{w_1^{\alpha_1+1}}\wedge\ldots\wedge\bar\partial\frac{1}{w_p^{\alpha_p+1}}\wedge dw_1^{\alpha_1+1}\wedge\ldots\wedge dw_p^{\alpha_p+1}/(2\pi)^p=[Z].$$

We have (for $|\beta|=M$) that

$$\bar\partial(\gamma\wedge dw^\beta)=a_\beta\beta_1\cdots\beta_p[V].$$

However, if ν is the component of $\gamma\wedge dw^\beta$ of bidegree $(p,p-1)$ in w, then we must have

$$d_w\nu=\bar\partial_w\nu=a_\beta\beta_1\cdots\beta_p[V].$$

Integrating with respect to w we now get that $a_\beta(z)=0$. By finite induction we can conclude that $\mu=0$. Thus μ vanishes on Z_{reg} and by the standard extension property follows that $\mu=0$. $\square$

THEOREM 3.11. *Let f be a complete intersection. If $\nabla_f U=1-\mu\wedge e$ and $\mu\in CH_Z$, then μ is equal to the CH product μ^f.*

It can of course also be stated: If $\nabla_f W=\tau\wedge e$ and $\tau\in \mathrm{C}H_Z$, then $\tau=0$.

PROOF. Let ω be any neighborhood of Z and take a cutoff function χ that is 1 in a neighborhood of Z and with support on ω. Let $\sigma=\sum_j(\bar f_j/|f|^2)e_j$. Then

$$u=\frac{\sigma}{\nabla_f\sigma}$$

is smooth outside Z and clearly $\nabla_f u=1$ there. Thus

$$g=\chi-\bar\partial\chi\wedge u$$

is a smooth form in ω and $\nabla_f g=0$. Moreover, $g_0=1$ in a neighborhood of Z. Therefore,

$$\nabla_f[g\wedge(U-V)]=g_0(\mu^f-\mu)\wedge e=(\mu^f-\mu)\wedge e,$$

and therefore we have a solution to $\bar\partial W=\mu^f-\mu$ with support in ω. In view of Lemma 3.10 we have that $\mu=\mu^f$. $\square$

PROPOSITION 3.12. *Let $J=(f)$ be a complete intersection ideal and assume that $\mu\in CH_Z$ and that $J\mu=0$. Then there is a holomorphic function ψ, unique in $\mathcal{O}/J$, such that*

$$\mu=\psi\mu^f.$$

PROOF. From the assumptions follows that $\nabla_f\mu=0$. In view of Lemma 3.7 we have U such that $\nabla_f U=\psi-\mu\wedge e$. Now take V such that $\nabla_f V=1-\mu^f\wedge e$. Then $\nabla_f(\psi V)=\psi-\psi\mu^f\wedge e$. It now follows from Theorem 3.11 that $\mu=\psi\mu^f$. $\square$

3.6. Local structure of Coleff-Herrera currents.

Proposition 3.13. *Suppose that Z has pure codimension p and Z' is a union of irreducible components of Z. Then $CH_{Z'}$ is precisely the currents in CH_Z that have support on Z'.*

Proof. Clearly $\mathrm{C}H_{Z'} \subset \mathrm{C}H_Z$. Conversely, if $\mu \in \mathrm{C}H_Z$ has support on Z and $\psi \in I_{Z'}$ we have to check that $\bar{\psi}\mu = 0$. Let ϕ vanish on $Z \setminus Z'$ but not identically on Z'. Then $\phi\psi$ vanishes on Z and hence $\bar{\phi}\bar{\psi}\mu = 0$. It follows that $\bar{\psi}\mu = 0$ has support on $Z' \cap \{\phi = 0\}$, and by SEP it follows that it vanishes identically. □

allting r pa groddniva !!

Proposition 3.14. *Let Z be a union of irreducible components of Z'. Then CH_Z coincides with the set of currents μ in $CH_{Z'}$ such that μ has support on Z.*

Proof. Clearly $\mathrm{C}H_Z \subset \mathrm{C}H_{Z'}$. We must check that if $\mu \in \mathrm{C}H_{Z'}$ has support on Z and ψ vanishes on Z then $\bar{\psi}\mu = 0$. However, $\bar{\psi}\mu$ clearly has support on Z, and moreover, it must vanish on Z_{reg} so it therefore has support on $Z' \setminus Z'_{reg}$. Take h that vanishes on this set but not identically on any irreducible component of Z'. Then $|h|^{2\lambda}\bar{\psi}\mu$ vanishes for $\mathrm{Re}\,\lambda >> 0$, and hence at $\lambda = 0$, but on the other hand, by SEP the value at 0 is equal to $\bar{h}\mu$. □

Any Z of pure codim p is subset of a complete intersection Z_f; this follows from structure and balbala.

It now follows that any CH-current locally is a holomorphic function times a Coleff-Herrera product, as was first noticed by Björk.

Theorem 3.15 (Local structure theorem). *Let Z be any variety of pure codimension p. Any $\mu \in CH_Z$ is locally of the form $\psi\mu^g$ where g is complete intersection.*

Proof. Any Z of pure codim p is subset of a complete intersection Z_f; this follows from structure and balbala. For sufficiently large k, $g = (f_1^k, \dots, f_p^k)$ will annihilate $\mu \in \mathrm{C}H_Z$ and hence $\mu = \psi\mu^g$ according to Proposition 3.12. □

Example 3.5. It follows that there is a holomorphic $(p, 0)$-form A such that

$$A \wedge \mu^g = [Z].$$

□

3.7. Exactness of the Koszul complex. Let now ϕ be a section of $\mathcal{O}(\Lambda^k E)$ such that $\delta_f \phi = 0$ and assume that $k \geq 1$. If we let V be a solution of $\nabla V = 1 - \mu^f \wedge e$, then clearly

$$\nabla V \wedge \phi = (1 - \mu^f \wedge e) \wedge \phi = \phi,$$

for simple degree reasons, and by (a simple variant of) Lemma 3.7 we get a holomorphic solution of $\delta_f \psi = \phi$. Thus we have

PROPOSITION 3.16. *If f is a complete intersection then the sheaf complex*

$$0 \to \mathcal{O}(\Lambda^p E) \to \mathcal{O}(\Lambda^{p-1} E) \to \ldots \to \mathcal{O}(\Lambda^1 E) \to \mathcal{O} \to \mathcal{O}/J \to 0$$

is exact.

3.8. Bochner-Martinelli type residues. Kan ses som medelvarden av olika CH !!

Let $f = \sum f_j e_j^*$. To begin with we do not assume that f is a complete intersection. We now choose a Hermitian metric on E; if we consider E as a vector bundle over X the metric may very well vary with the point z. If we fix a basis $e_1, \ldots, e_p$ of E, (holomorphic frame if we use the vector bundle point of view) we can take the trivial metric with respect to this frame, i.e., balbla.

Let

$$\sigma = \sum_j \sigma_j e_j$$

be the pointwise minimal solution to $f\sigma = 1$ in $X \setminus Z$. If the metric on E^* is given by the Hermitian positively definite matrix h_{jk}, so that

$$|f|^2 = \sum_{jk} f_j \bar{f}_k h_{jk},$$

then it is easily checked that

$$\sigma_j = \sum_k \frac{\bar{f}_k h_{jk}}{|f|^2}.$$

In $X \setminus Z$ we define

$$u = \frac{\sigma}{\nabla_f \sigma} = \sigma + \sigma \wedge \bar{\partial}\sigma + \ldots \sigma \wedge (\bar{\partial}\sigma)^{p-1}.$$

It follows immediately that

$$\nabla_f u = 1$$

in $X \setminus Z$.

THEOREM 3.17. *The function* $\lambda \mapsto |f|^{2\lambda}u$ *has a current valued analytic continuation to* $\mathrm{Re}\,\lambda > -\epsilon$. *The value at* $\lambda = 0$,

$$U = |f|^{2\lambda}u|_{\lambda=0},$$

is a current extension of u *across* Z *and*

$$\nabla_f U = 1 - R,$$

where

$$R = \bar\partial |f|^{2\lambda} \wedge u|_{\lambda=0}$$

is a current with support on Z. *It is annihilated by* $\bar h$ *and* $d\bar h$ *for* $h \in I_Z$. *If codim*$Z = p'$, *then*

$$R = R_{p'} + \cdots + R_p.$$

If $|\phi| \le C|f|^k$, *then* $\phi R_k = 0$.

PROOF. If $f = f_0 f' = f_0(f_1', \dots, f_p')$, where $f' \neq 0$, then

$$\sigma = \frac{1}{f_0}\sigma'$$

where σ' is smooth across Z. In fact, $\bar f_k = \bar f_0 \bar f_k'$ and $|f|^2 = |f_0|^2|f'|^2$ so that $\sigma_j = \sum_k \bar f_k' h_{jk}/f_0$. Thus

$$u_k = \sigma \wedge (\bar\partial\sigma)^{k-1} = \frac{\alpha}{f_0^k}, \tag{3.19}$$

where α is smooth.

Both the definition and the statement is clearly local and therefore we can assume that the bundle E is trivial in $\mathcal{U} \subset X$. Using a Hironaka resolution $\pi\colon \tilde{\mathcal{U}} \to \mathcal{U}$ followed by suitable toric resolutions $\pi^j\colon \tilde{\mathcal{U}}^j \to \pi^j(\tilde{\mathcal{U}}_j) \subset \tilde{\mathcal{U}}$ we have

$$\int_{\mathcal{U}} |f|^{2\lambda}u_k \wedge \phi = \int_{\tilde{\mathcal{U}}} \sum_j |\pi^* f|^{2\lambda}(\pi^* u_k)\rho_j \wedge \pi^*\phi =$$
$$\sum_{jk} \int_{\tilde{\mathcal{U}}_j} |(\pi^j)^*\pi^* f|(\pi^j)^*\pi^* u_k \wedge (\pi^j)^*\pi^*\phi.$$

If we for each j choose a suitable partition of unity ρ_{jk} we have a local coordinate system t in a neighborhood of the support of each ρ_{jk} such that $(\pi^j)^*\pi^* f = f_0 f'$ there, where f_0 is a monomial in t and f' is non-vanishing. In view of (3.19) each terms is like

$$\int |f_0|^{2\lambda}|f'|^{2\lambda}\frac{\alpha_j}{f_0^k} \wedge (\pi^j)^*\pi^*\phi,$$

and thus the proposed analytic continuation exists.

there, where f_0 is a monomial in t and f' is non-vanishing. In view of ??? it is now clear that the analytic continuation exists and moreover, that

$$\int_{\mathcal{U}} |f|^{2\lambda} u_k \wedge \phi|_{\lambda=0} = \sum_\ell \tau_\ell \wedge (\pi^j)^* \pi^* \phi,$$

where each τ_ℓ has the form

$$\tau = \frac{\alpha}{t_1^{a_1} \cdots t_r^{a_r}}$$

in suitable local coordinates t, where α has compact support. Thus we have that

$$U_k = \sum_\ell \pi_* \pi_*^\ell \tau_\ell. \tag{3.20}$$

Moreover,

$$\tau = \bar\partial \frac{1}{t_1^{a_1}} \wedge \frac{\alpha}{t_2^{a_2} \cdots t_r^{a_r}}$$

Since

$$\nabla(|f|^{2\lambda} u) = |f|^{2\lambda} - \bar\partial |f|^{2\lambda} \wedge u \tag{3.21}$$

and clearly $|f|^{2\lambda}$ has a continuation to $\operatorname{Re}\lambda > -\epsilon$ which is 1 for $\lambda = 0$, the desired continuation of the last term follows, and if we define the currents U and R^f as the values of the corresponding terms at $\lambda = 0$, then (**??**) follows from (3.21). In particular, it follows that R^f has support on Y.

Follows that R_k is in $\mathcal{PM}$ and hence since support on Y must vanish if balbalbal

Thus we have that

$$R_k = \sum_\ell \pi_* \pi_*^\ell \tau_\ell. \tag{3.22}$$

$$\tau = \bar\partial \frac{1}{t_1^{a_1}} \wedge \frac{\alpha}{t_2^{a_2} \cdots t_r^{a_r}}$$

It now follows that

Finally; if $|h| \le C|f|^k$, then in a resolution where $\pi^* f = f_0 f'$ and $\pi^* h$, where f_0 and $\pi^* h$ are monomials, we must have that f_0^k divides h and hence blabla vanishes. $\square$

Corollary 3.18. *If $\phi \in \mathcal{O}$ and $\phi R = 0$, then $\phi \in (f)$.*

The algebraic meaning and generalizations will be discussed in ????

THEOREM 3.19 (Briançon-Skoda). *Suppose that* $f = (f_1, \dots, f_m)$ *and* ϕ *are germs at* 0 *such that* $|\phi| \le |f|^{\min(m,n)}$. *Then* $\phi \in (f)$.

Notice that if f is a complete intersection then $R = R_p$ and $\bar\partial R = 0$. Moreover, it is easily checked precisely as for the CH product, that R_p has SEP. In view of Theorem ??? we thus have

THEOREM 3.20. *If* f *is a complete intersection then*

$$R = R_p = \mu^f \wedge e_1 \dots \wedge e_p,$$

where μ^f *is the CH product*

$$\mu^f = \bar\partial \frac{1}{f_p} \wedge \dots \wedge \bar\partial \frac{1}{f_1}.$$

We can now use the invariance of the current R to get

PROPOSITION 3.21 (Transformation law). *If* g *is a holomorphic invertible* $p \times p$ *matrix and* $f' = gf$, *then*

$$\bar\partial \frac{1}{f'_p} \wedge \dots \wedge \bar\partial \frac{1}{f'_1} = \det g \bar\partial \frac{1}{f_p} \wedge \dots \wedge \bar\partial \frac{1}{f_1}.$$

PROOF. In fact, let e' the frame such that $e^* f = (e')^* gf$. Then $e' = eg^T$ and hence, since by the invariance

$$\mu^f \wedge e_1 \wedge \dots \wedge e_p = \mu^{gf} \wedge e'_1 \wedge \dots \wedge e'_p$$

and $e'_1 \wedge \dots \wedge e'_p = \det g^T e_1 \wedge \dots \wedge e_p$, the proposition follows. □

Now fix a frame e_j and choose the trivial metric. (Kommer att kika pa invarianta versioner langre fram)

3.9. Factorization of Lelong currents.

THEOREM 3.22. *Let* $f = f_1 e_1^* + \dots + f_p e_p^*$ *be a holomorphic and* $df = df_1 \wedge e_1^* + \cdots df_p \wedge e_p^*$, *and* R *wrt trivial metric. Then*

$$R_k \cdot (df)_k / (2\pi i)^k = (dd^c \log |f|)^k \mathbf{1}_Z,$$

where $\cdot$ *denote the natural contraction between* $\Lambda^k E$ *and* $\Lambda^k E^*$.

COROLLARY 3.23. *If* $codim Z = p$, *then*

$$R_p \cdot (df)_p / (2\pi i)^p = \sum_j \alpha_j [Z_j^p]$$

If f is a complete intersection, then $R = \mu^f \wedge e_1 \dots \wedge e_p$ and $(df)_p = df_1 \wedge \dots \wedge df_p \wedge e_p \wedge \dots \wedge e_1$ and by ??? we therefore have

COROLLARY 3.24. *If f is complete intersection, then*

$$\bar\partial\frac{1}{f_p}\wedge\ldots\wedge\bar\partial\frac{1}{f_1}\wedge\frac{df_1\wedge\ldots\wedge df_p}{(2\pi i)^p}=\sum_j\alpha_j[Z_j].$$

PROOF. Obs that R_k is the value at $\lambda=0$ of

$$\bar\partial|f|^{2\lambda}\wedge u_k dbar|f|^{2\lambda}\wedge\frac{\bar f\cdot e\wedge(d\bar f\cdot e)^{k-1}}{|f|^{2k}}$$

Notice that the contraction means that each occurrence of e_j in u_k is replaced by df_j. Since $\sigma=\sum_j\bar f_j e_j$ etc we have that it is the value at $\lambda=0$ of

$$\bar\partial|f|^{2\lambda}\wedge\frac{\partial|f|^2}{|f|^2}\wedge\Big(\bar\partial\frac{\partial|f|^2}{2\pi i|f|^2}\Big)^{k-1}.$$

However, this is precisely $\mathcal{A}^f_{k,\lambda}$ so the statement follows from Lemma ??? $\square$

3.10. Restrictions of pseudomeromorphic currents. Let $T\in\mathcal{PM}$ and Z any subvariety. If h is a tuple

3.11. Various definitions of the CH product. Obs that all the abstrct stuff about CH_Z only uses SEP to ensure that μ is determined by its values on Z_{reg}. Hence we can just as well start with the BM current R^f and prove that any abtract CH current of the form αR^f.

THEOREM 3.25. *Let μ be BM-current. Then*

$$\lim\chi(|h|/\epsilon)\mu=\mu$$

if either χ is a smooth cutoff or h is any such that h,f complete intersection, or χ charact function, and $h=0$ contains $Z\setminus Z_{reg}$.

PROPOSITION 3.26. *Let χ be a fixed function as above. The class of $\bar\partial$-closed $(0,p)$-currents μ with support on Z that are annihilated by $\bar I_Z$ and satisfy (**??**) coincides with our class CH_Z.*

SKETCH OF PROOF. Let f be a complete intersection such that Z is a union of irreducible components of Z^f, and let μ^f_{BM} be the Bochner-Martinelli residue current. We first show that this current satisfies (**??**). In fact, μ^f_{BM} is the push-forward of simple current of the form (**??**) above.

We may also assume where $h=\tau_{i_1}^{b_1}\cdots\tau_{i_k}^{b_k}$. However, by a standard argument the push-forward of this term will vanish if τ_1 is one of τ_{i_ℓ}.

Therefore, the existence of the product in (**??**) and the equality follow from the simple observation that

$$\int_{|s_1,\dots,s_\mu} \chi(|s_1^{c_1}\cdots s^{c_\mu}|/\epsilon)\frac{\psi(s)}{s_1^{\gamma_1}\cdots s_\mu^{\gamma_\mu}} \to pv\int_{s_1,\dots,s_\mu} \frac{\psi(s)}{s_1^{\gamma_1}\cdots s_\mu^{\gamma_\mu}}. \tag{3.23}$$

Let temporarily $\mathrm{C}H_Z^{class}$ denote the class of current defined in the proposition. Clearly any current $\alpha\mu^f_{BM}$ where α is holomorphic and vanishing on $Z^f\setminus Z$, belongs to this class. Moreover, it is clear that if a current in this class vanishes on Z_{reg} then it vanishes identically. Therefore, the uniqueness theorem above holds for this class as well, as is easily checked. As before it therefore follows that any current in $\mathrm{C}H_Z^{class}$ can be written $\alpha\mu^f_{BM}$. It follows that $\mathrm{C}H_Z^{class} = \mathrm{C}H_Z$. □

3.12. Noetherian differential operators.

THEOREM 3.27 (Björk). *Let V be a germ of an analytic variety of pure codimension p at $0\in\mathbb{C}^n$. Then there is a neighborhood Ω of 0 such that for each $\mu\in CH_V(E_0^*)$ in Ω, there are holomorphic differential operators $\mathcal{L}_1,\dots,\mathcal{L}_\nu$ in Ω such that for any $\phi\in\mathcal{O}(E_0)$, $\mu\phi=0$ if and only if*

$$\mathcal{L}_1\phi=\cdots=\mathcal{L}_\nu\phi=0 \ \textit{on} \ V. \tag{3.24}$$

PROOF. Let V be a germ of a variety of pure codimension p at $0\in\mathbb{C}^n$. It follows from the local normalization theorem that one can find holomorphic functions $f_1,\dots,f_p$ in an open neighborhood Ω, forming a complete intersection, such that V is a union of irreducible components of $V_f=\{f=0\}$, and such that

$$df_1\wedge\ldots\wedge df_p\neq 0$$

on $V\setminus W$ where W is a hypersurface not containing any component of V_f. By a suitable choice of coordinates $(\zeta,\omega)\in\mathbb{C}^{n-p}\times\mathbb{C}^p$ we may assume that W is the zero set of

$$h=\det\frac{\partial f}{\partial\omega}.$$

Let

$$z=\zeta,\quad w=f(\zeta,\omega).$$

Since

$$\frac{d(z,w)}{(\zeta,\omega)}=\det\begin{bmatrix} I & 0\\ \partial f/\partial\zeta & \partial f/\partial\omega\end{bmatrix}=\det\frac{\partial f}{\partial\omega}$$

so locally outside W (z,w) is a local holo coordinate system. From ???? we know that there is an M such that

$$\mu=A\Big[\bar\partial\frac{1}{f_1^M}\wedge\ldots\bar\partial\frac{1}{f_p^M}\Big]$$

locally in $\Omega \setminus W$ so that

$$\mu.\xi = \int_{w=0} \sum_{\alpha<M} \frac{\partial^{M-\alpha} A(z,0)}{\partial w^{M-\alpha}} c_\alpha \frac{\partial^\alpha \xi}{\partial w^\alpha}$$

It follows that

$$\phi\mu.\xi = \int_{w=0} \sum_{\ell \le M} Q_\ell \phi \frac{\partial^\ell}{\partial w^\ell} \xi,$$

where

$$Q_\ell = \sum_{\ell \le \alpha \le M} \frac{\partial^{M-\alpha} A}{\partial w^{M-\alpha}} \frac{\partial^{\alpha-\ell}}{\partial w^{\alpha-\ell}}$$

If this holds for all $\xi \in \mathcal{D}(\Omega)$ we must have ?? that $Q_\ell \phi = 0$ on $Z \cap \Omega$ for all $\ell \le M$.

Now we notice that

$$\begin{bmatrix} \frac{\partial \zeta}{\partial z} & \frac{\partial \zeta}{\partial w} \\ \frac{\partial \omega}{\partial z} & \frac{\partial \omega}{\partial w} \end{bmatrix} = \begin{bmatrix} I & 0 \\ \frac{\partial f}{\partial \zeta} & \frac{\partial f}{\partial \omega} \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ -\big(\frac{\partial f}{\partial \omega}\big)^{-1} \frac{\partial f}{\partial \zeta} & \big(\frac{\partial f}{\partial \omega}\big)^{-1} \end{bmatrix}$$

and hence

$$\frac{\partial \omega}{\partial w} = \big(\frac{\partial f}{\partial \omega}\big)^{-1} = \frac{1}{h} \gamma,$$

where γ is holomorphic. It follows that

$$\frac{\partial}{\partial w_j} = \sum_k \frac{\partial \omega_k}{\partial w_j} \frac{\partial}{\partial \omega_k} = \frac{1}{h} \sum_k \gamma_{jk} \frac{\partial}{\partial \omega_k}. \tag{3.25}$$

It follows from (3.25) that Q_ℓ are (semi-)global differential operators of the form $Q_\ell = \mathcal{L}_\ell / h^N$, where $\mathcal{L}_\ell$ are holomorphic. Now, $\phi\mu = 0$ iff $\phi\mu = 0$ on $X \setminus W$ by the SEP, and this holds iff $\mathcal{L}_\ell \phi = 0$ on $Z \setminus W$ which by continuity holds iff $\mathcal{L}_\ell \phi = 0$ on Z. □

It follows from the proof that if

$$Q = \sim_{\alpha \le M} \frac{\partial^{M-\alpha} A}{\partial w^{M-\alpha}} c_\alpha \frac{\partial^\alpha}{\partial w^\alpha}$$

then $Q = \mathcal{L}/h^N$ for some N and hence

$$\mu.\xi = \int_Z \frac{1}{h^N} \mathcal{L} \xi \tag{3.26}$$

for ξ with support outeside W. We now define

$$\mu.\xi = \int_Z \frac{1}{h^N} \mathcal{L} \xi = \mu.\xi = \int_Z |h|^{2\lambda} \frac{1}{h^N} \mathcal{L} \xi |_{\lambda=0}$$

for general ξ.

Proposition 3.28. *Then* (3.26) *holds across* W.

Proof. The existence of the anal cont as before. Let the rigt hand side τ. Then $T = \mu - \tau$ has support on $Z \cap W$. Thus $|h|^{2\lambda}(\mu - \tau) = 0$ for $\mathrm{Re}\,\lambda >> 0$ so enough to see that

$$|h|^{2\lambda}\tau|_{\lambda=0} = \tau.$$

However

$$\int_Z \frac{1}{h^N}\mathcal{L}(|h|^{2\lambda}\xi) = \int_Z |h|^{2\lambda}\frac{1}{h^N}\mathcal{L}\xi + \cdots,$$

so must see that $\cdots$ vanishes when $\lambda = 0$. Take for instance $\mathcal{L} = \partial/\partial\omega$, so we get a term like

$$\lambda \int_Z |h|^{2\lambda}\frac{1}{h^{N+1}}\frac{\partial h}{\partial \omega}\mathcal{L}\xi$$

which certainly vanishes when $\lambda = 0$. $\square$

3.13. Vanishing of Coleff-Herrera currents. If $\mu \in \mathrm{C}H_Z(X)$ and X Stein, then we can solve $\bar\partial V = \mu$ in X.

Theorem 3.29. *Assume that X is Stein and $Z \subset X$ has pure codimension p. If $\mu \in CH_Z(X)$ and $\bar\partial V = 0$ in X the following are equivalent:*

(i) $\mu = 0$.

(ii) For all $\psi \in \mathcal{D}_{n,n-p}(X \setminus Z)$ such that $\bar\partial\psi = 0$ in some nbh of Z we have that

$$\int V \wedge \psi = 0.$$

(iii) There is a solution to $\bar\partial w = V$ in $X \setminus Z$.

(iv) For each neighborhood ω of Z there is a solution to $\bar\partial w = V$ in $X \setminus \omega$.

Proof. It is easy to check that (i) implies all the other conditions. Assume that (ii) holds. Since $\mu = \alpha\mu^f$ by the structure theorem it follows from the strong duality principle, [**?**] and [**?**], that α belongs to the ideal (f) and so $\mu = 0$. A direct argument is obtained by mimicking the proof of Lemma 3.10 above: Locally on $Z_{reg} = \{w = 0\}$ we have (3.16), and by choosing $\xi(z,w) = \psi(z)\chi(w)dw^\beta \wedge sz \wedge d\bar z$ for a suitable cutoff function χ and test functions ψ, we can conclude successively from (ii) that all the coefficients a_α vanish, so that $\mu = 0$ there. Hence $\mu = 0$ globally by the SEP. Another possible way to proceed is to use a local $\bar\partial$-formula to see that (ii) implies a (iv) locally.

Clearly (iii) implies (iv). Finally assume that (iv) holds. Given ξ in (ii) we can choose ω such that $\bar\partial\xi$ vanishes in a neighborhood of $\bar\omega$. Then

$$\int V \wedge \bar\partial\xi = \int d(w \wedge \bar\partial\xi) = 0$$

by Stokes' theorem, so (ii) holds. Alternatively, given $\omega \supset Z$ choose $\omega' \subset\subset \omega$ and a solution to $\bar\partial w = V$ in $X \setminus \omega'$. If we extend w arbitrarily across ω' the form $U = V - \bar\partial w$ is a solution to $\bar\partial U = \mu$ with support in ω. In view of Lemma 3.10 thus $\mu = 0$. $\square$

Notice that V defines a Dolbeault cohomology class ω^μ in $X\setminus$ that only depends on μ, and that conditions (ii)-(iv) are statements about this class.

For an interesting application, fix a current $\mu \in \mathrm{C}H_Z$. Then the theorem gives several equivalent ways to express that a given $\phi \in \mathcal{O}$ belongs to the annihilator ideal of μ. In the case when $\mu = \mu^f$ for a complete intersection f, one gets back the equivalent formulations from [**?**] and [**?**].

Remark 3.3. If μ is an arbitrary $(0,p)$-current with support on Z and $\bar\partial V = \mu$ we get an analogous theorem if condition (i) is replaced by:
(i)' $\mu = \bar\partial\gamma$ for some γ with support on Z.
This follows from the Dickenstien-Sessa decomposition $\mu = \mu_{CH} + \bar\partial\gamma$, where μ_{CH} is in $\mathrm{C}H_Z$. See [**?**] for the case Z is a complete intersection and [**?**] for the general case. $\square$

3.14. Local cohomology. Let J be an ideal sheaf in X of pure codimension p and let Z be its zero set. We let $\mathcal{C}^{0,k}_J$ denote the sheaf och $(0,k)$-current such that $J\mu = 0$ and $\mathcal{C}^{0,k}_Z$ the sheaf of $(0,k)$-current with support on V. If the current μ is annihilated by J, then it is clear that the support of μ must be contained in Z, i.e., $\mathcal{C}^{0,k}_J \subset \mathcal{C}^{0,k}_Z$. Moreover, if μ has support on Z, then locally it has finite order, and hence it must be annihilated by some power of J, see,e.g., Hö ????, so we have

$$\mathcal{C}^{0,k}_V = \cup_\ell \mathcal{C}^{0,k}_{J^\ell}.$$

We introduce the local cohomology sheaves sheaves

$$\mathcal{H}^k_J = \frac{\mathrm{Ker}\,_{\bar\partial}\mathcal{C}^{0,k}_J}{\bar\partial\mathcal{C}^{0,k-1}_J}$$

and

$$\mathcal{H}^k_Z = \frac{\mathrm{Ker}\,_{\bar\partial}\mathcal{C}^{0,k}_Z}{\bar\partial\mathcal{C}^{0,k-1}_Z}.$$

Let $\mathrm{C}H_J$ be the subsheaf of $\mathrm{C}H_V$ of currents that are annihilated by J. We have the following basic result.

Theorem 3.30. *Let J be an ideal sheaf in X of pure codimension p and let Z be its zero set. Then we have*

(a) $\mathcal{H}_Z^k = \mathcal{H}_J^k = 0$ *for* $k < p$.
(b) $CH_Z \simeq \mathcal{H}_Z^p$ *and* $CH_J \simeq \mathcal{H}_J^p$.

We will only consider the case when J is a complete intersection here, and postpone the general case until Ch ???.

The basic ingredient in the proof is the stalkwise injectivity of the sheaf $\mathcal{C}$, due to Malgrange. More precisely: Take a current $\nu \in \mathcal{C}(\Lambda^k E)$ and suppose that $\delta_f \nu = 0$. If $k = p$ this precisely means that $\nu = \mu \wedge e$ where $\mu \in \mathcal{C}_J$. It is the more suprising that

THEOREM 3.31 (Malgrange). *If f is a complete intersection, $\nu \in \mathcal{C}(\Lambda^k E)$, $k < p$, and $\delta_f \nu = 0$, then locally one can find $\gamma \in \mathcal{C}(\Lambda^{k+1} E)$ such that $\delta_f \gamma = \nu$.*

We provide a direct proof here by means of integral formulas.

PROOF. □

PROOF OF DS1. Again consider our double sheaf complex $\mathcal{C}^{0,k}(\Lambda^\ell E)$. Since we know now that it is locally (i.e., stalkwise) exact also in ℓ-direction, except at $\ell = p$, we get canonical isomorphisms

$$\frac{\operatorname{Ker}_{\delta_f} \mathcal{O}(\Lambda^k E)}{\delta_f \mathcal{O}(\Lambda^{k+1} E)} \simeq \mathcal{H}_J^{p-k} \tag{3.27}$$

such that ψ corresponds to ν if and only if $\nabla_f V = \psi - \nu$. In view of Proposition 3.16 the first part of (a) follows. The second part follows by choosing high powers of J.

Now assume that $k = p$. We now have

$$\mathcal{O}/J \to CH_J \to \mathcal{H}_J^p \tag{3.28}$$

where the first mapping is $\psi \mapsto \psi \mu^f \wedge e$. The composed mapping then coincides with the isomorphism in (3.27) since $\psi - \psi\mu^f \wedge e = \nabla_f V \psi$. However, the second mapping in (3.28) is injective by Lemma 3.10, and hence both mappings are isomorphisms. □

4. Exercises

EXERCISE 31. Show that the functions of λ in Lemma 1.4 are meromorphic in the whole plane with simple poles at $-1, -2, -3, \ldots$.