A BRIEF INTRODUCTION TO TOPOLOGICAL DYNAMICS

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1. Topological ergodicity and minimality

Let G be a topological group.

Definition 1.1 (G-space)**.** A Hausdorff space X is called a *G-space* if it is endowed with a *jointly continuous* action of G by homeomorphisms, which we write as

$$(g,x) \mapsto gx, \quad \text{for } g \in G \text{ and } x \in X.$$

We say that

- X is *topologically ergodic* if for all non-empty open sets $U, V \subset X$,
$$gU \cap V \neq \emptyset, \quad \text{for some } g \in G.$$
- X is *minimal* if there are no non-empty proper closed G-invariant subsets of X.

Clearly every minimal G-space is topologically ergodic.

Proposition 1.2 (Fundamental Theorem in Topological Dynamics)**.** *Every* compact *G-space X admits at least one closed G-invariant subset such that the restriction of the G-action to this subset is minimal.*

Proof. Zorn's Lemma. □

1.1. Basic notation.

Given $A \subset G$ and $B \subset X$ we define the *action set* of A and B by

$$AB = \bigcup_{a \in A} aB \subset X.$$

Given $x \in X$, we define the *return set to B* by

$$B_x = \{g \in G : gx \in B\} \subset G.$$

We note that

$$AB_x = (AB)_x \quad \text{and} \quad B_{gx} = B_x g^{-1}, \quad \text{for all } g \in G,\, A \subset G,\, x \in X \text{ and } B \subset X.$$

1.2. Topological ergodicity and existence of dense orbits.

Let X be a G-space and let ν be a Borel probability measure on X. We say that ν is *G-invariant* if

$$\nu(gB) = \nu(B), \quad \text{for all } g \in G \text{ and Borel sets } B \subset X,$$

and *(measurably) ergodic* if it is G-invariant and $\nu(B) = 1$ for every G-invariant Borel set $B \subset X$ with positive ν-measure. Recall that the *support of ν*, here denoted by $\operatorname{supp}(\mu)$, is defined as the set of $x \in X$ such that $\nu(U) > 0$ for every open neighborhood U of x. One readily checks that $\operatorname{supp}(\nu) \subset X$ is always closed and if ν is G-invariant, then $\operatorname{supp}(\nu)$ is G-invariant.

Proposition 1.3. *Suppose that X is a G-space and there exists an ergodic G-invariant Borel probability measure ν on X with full support. Then X is topologically ergodic.*

Proof. Since ν has full support, $\nu(U) > 0$ for every non-empty open set $U \subset X$, and thus we have $\nu(GU) = 1$ and $\nu(GU \cap V) = \nu(V) > 0$ for every non-empty open set $V \subset X$ which shows that X is topologically ergodic. □

Recall that a *Baire space* is a Hausdorff space with the property that every countable intersection of open dense sets is non-empty. Baire's Category Theorem asserts that every completely metrizable space is Baire.

Proposition 1.4 (Topological Zero-One Law)**.** *Suppose that X is a second countable Baire G-space and define the set*

$$X_o := \{x \in X : X = \overline{Gx}\} \subset X.$$

Then,

$$X \text{ is topologically ergodic} \iff X_o \text{ is a dense } G_\delta\text{-set.}$$

Proof. Let (U_n) be a countable basis for the topology on X, and note that $X_o = \bigcap_n GU_n$, which shows that X_o is always a G_δ-set.

- If X is topologically ergodic, then GU_n is dense for every n, and thus X_o is dense, by our assumption that X is a Baire space.
- If X_o is dense, then GU_n must be dense for every n, and thus $GU_n \cap V \neq \emptyset$ for every open set $V \subset X$. Since every non-empty open set U contains at least one open set of the form U_n, we conclude that X is topologically ergodic.

□

Remark 1.5. There are examples of *compact*, but not second countable, topologically ergodic G-spaces without any points with dense orbits. For instance, let G be any countable group and suppose that Y is a compact G-space with an *ergodic* G-invariant Borel probability measure ν without atoms. Let X denote the Gelfand spectrum of the C*-algebra $L^\infty(Y,\nu)$, and view X as a compact G-space under the dual action equipped with a G-invariant Borel probability measure μ of full support, which is defined as the push-forward of ν under the Gelfand map. Since $L^\infty(Y,\nu)$ is not norm separable, X is not second countable by Urysohn's Lemma. We leave it as an exercise to prove that μ is non-atomic and

$$\mu(B^o) = \mu(\overline{B}), \quad \text{for all Borel sets } B \subset X.$$

In particular, $\mu(\overline{Gx}) = \mu(Gx) = 0$, since G is countable and μ is non-atomic. Since μ is ergodic and has full support on X, we conclude that X is topologically ergodic by Proposition 1.3.

In the presence of an ergodic measure, the set of points with dense orbits is also "measurably" large.

Proposition 1.6 (Zero-One Law)**.** *Suppose that*

- *X is second countable.*
- *ν is G-invariant and ergodic.*

Then,

$$\operatorname{supp}(\nu) = \overline{Gx}, \quad \text{for } \nu\text{-a.e. } x \in X.$$

Proof. Since $\operatorname{supp}(\nu)$ is again a second countable G-space, we may assume that μ has full support. Let (U_n) be a countable basis for the topology on X and note that

$$\bigcap_n GU_n = \{x \in X : \overline{Gx} = X\}.$$

Since ν has full support and is ergodic, we have $\nu(GU_n) = 1$ for every n and thus the intersection above is ν-conull. □

1.3. **Existence of ergodic measures.** Let X be a compact G-space and let $\mathscr{P}(X)$ denote the convex set of Borel probability measures on X. We write $\mathscr{P}_G(X)$ for the (possibly empty) convex set of G-invariant elements in $\mathscr{P}(X)$. We note that $\mathscr{P}_G(X)$ is weak*-compact once viewed as a subset of $C(X)^*$. An element $\nu \in \mathscr{P}_G(X)$ is *extreme* if it cannot be written on the form

$$\nu = p\mu + q\eta, \quad \text{for some } \mu \neq \eta \in \mathscr{P}_G(X) \text{ and } 0 < p, q < 1 \text{ such that } p + q = 1.$$

By Krein-Milman's Theorem, the set $\mathscr{P}_G(X)^{\text{ext}}$ of extreme points in $\mathscr{P}_G(X)$ is always non-empty. If X is compact and second countable, then $\mathscr{P}_G(X)^{\text{ext}}$ is a G_δ-set and thus Borel measurable. If the second countability assumption is dropped, then $\mathscr{P}_G(X)^{\text{ext}}$ need not be Borel measurable.

We now connect extreme points to ergodic measures.

Proposition 1.7 (Ergodicity vs. Extremality)**.** *If X is a compact G-space and $\mathscr{P}_G(X) \neq \emptyset$, then*

$$\mathscr{P}_G(X)^{ext} = \mathscr{P}_G^{erg}(X).$$

In particular,

- $\mathscr{P}_G^{erg}(X) \neq \emptyset$.
- *if $\mathscr{P}_G(X) = \{\nu\}$, then ν is ergodic.*

Proof. Suppose that $\nu \in \mathscr{P}_G(X)$ is ergodic, but not an extreme point in $\mathscr{P}_G(X)$. Then we can write $\nu = p\mu + q\eta$ for two distinct G-invariant Borel probability measures μ and η on X for some real numbers $0 < p, q < 1$ such that $p + q = 1$. This forces both μ and η to be absolutely continuous with respect to ν, and their Radon-Nikodym derivatives with respect to ν are G-invariant. Since ν is ergodic, this further forces them to be ν-essentially equal to one, and thus $\mu = \eta$, which contradicts our assumption.

Suppose that $\nu \in \mathscr{P}_G(X)$ is not ergodic and let $B \subset X$ be a G-invariant Borel set such that $0 < \nu(B) < 1$ and define the G-invariant Borel probability measures μ and η on X by

$$\mu(A) = \frac{\nu(A \cap B)}{\nu(B)} \quad \text{and} \quad \eta(A) = \frac{\nu(A \cap B^c)}{\nu(B^c)}, \quad \text{for Borel sets } A \subset X.$$

Since $\nu = \nu(B)\mu + (1 - \nu(B))\eta$ is a non-trivial decomposition of ν in $\mathscr{P}_G(X)$ we conclude that ν is not an extreme point in $\mathscr{P}_G(X)$. □

2. Amenable groups

We shall now confine our attention to those topological groups for which ergodic measures exist for *any* compact G-space.

Definition 2.1 (Amenable group)**.** A topological group G is *amenable* if $\mathscr{P}_G(X)$ is non-empty for every *compact* G-space X.

Recall that a topological group is *solvable* if there exist *closed* subgroups $G_i < G$, $i = 1, \dots, n$, such that

$$\{e\} = G_o = \lhd G_1 \lhd G_2 \dots \lhd G_n = G$$

and G_{i+1}/G_i is abelian for every i. We mention without proof:

Theorem 2.2 (Markov-Kakutani)**.** *Every solvable group is amenable.*

Theorem 2.3. *Suppose that G is a* locally compact *group which contains a closed free subgroup on two generators. Then G is not amenable.*

Remark 2.4. If the assumption that G is locally compact is dropped, then the last theorem does not necessarily hold. For instance, suppose that G is the free group on two generator and let $U(\ell^2(G))$ denote the group of unitary operators on $\ell^2(G)$ equipped with the strong operator topology. Then the embedding of G into $U(\ell^2(G))$ via the left regular representation is discrete, and thus the image of G is closed, but it can be shown that $U(\ell^2(G))$ is amenable.

Let G be a topological group and let E be a Banach space equipped with a jointly continuous isometric action of G. Then G also acts jointly continuously on E^* by

$$g\lambda(u) = \lambda(g^{-1}u), \quad \text{for } u \in E \text{ and } \lambda \in E^*.$$

If $K \subset E^*$ is a weak*-compact G-invariant convex subset, then we refer to K as an *affine G-space*. We denote by K^G the (possibly empty) set of G-fixed elements. One example to have in mind is when X is a compact G-space and the regular representation of G on $C(X)$. In this case,

$$E = C(X) \quad \text{and} \quad K = \mathscr{P}(X) \quad \text{and} \quad K^G = \mathscr{P}_G(X).$$

Proposition 2.5. *A topological group G is amenable if and only if $K^G \neq \emptyset$ for every non-empty affine G-space K.*

Proof. The "only if"-direction follows from taking

$$E = C(X) \quad \text{and} \quad K = \mathscr{P}(X) \subset E^*,$$

where X is a compact G-space. For the "if"-direction, if G is amenable, then $\mathscr{P}_G(K)$ is non-empty. Pick $\eta \in \mathscr{P}_G(K)$, and define its *barycenter* by

$$x = \int_K y \, d\eta(y) \in \mathscr{P}(X).$$

Since η is G-invariant, we see that $x \in K^G$. □

Proposition 2.6 (Unique ergodicity vs. minimality)**.** *Suppose that*

- *G is an amenable group.*
- *X is a compact G-space and $\mathscr{P}_G(X) = \{\nu\}$.*

Then $Y := \mathrm{supp}(\nu) \subset X$ is a minimal G-space.

Proof. Suppose that $Z \subset Y$ is a non-empty proper closed G-invariant subset. Since G is amenable, there exists $\mu \in \mathscr{P}_G(Z) \subset \mathscr{P}_G(Y)$ such that $\mu(Z) = 1$. However, by assumption, $\mu = \nu$, and thus we have $Z = \mathrm{supp}(\nu) = Y$, which is a contradiction. □

Proposition 2.7. *Suppose that*

- *G is an amenable group.*
- *X and Y are compact G-spaces.*
- *$\beta : X \to Y$ continuous surjective G-map.*

Then the induced map $\beta_* : \mathscr{P}_G(X) \to \mathscr{P}_G(Y)$ is surjective.

Proof. Fix $\nu \in \mathscr{P}_G(Y)$. We wish to prove that the set

$$K = \left\{\mu \in \mathscr{P}(X) : \beta_*\mu = \nu\right\} \subset \mathscr{P}(X)$$

is non-empty. Since K is obviously a closed G-invariant and convex subset of $C(X)^*$, amenability of G will imply that K^G is non-empty, which is what we would like to prove.

Let $V := \beta^* C(Y) \subset C(X)$, which is a G-invariant sub-space, and we may view ν as a positive linear functional on V. By Hahn-Banach's Theorem, there is $\mu \in C(X)^*$ which extends ν with the same norm. Hence,

$$\|f\|_\infty - \mu(f) = \mu(\|f\|_\infty - f) \leq \| \|f\|_\infty - f\|_\infty \leq \|f\|_\infty,$$

which shows that μ is non-negative, and thus $\mu \in K$. □

Remark 2.8. If the assumption that G is amenable is dropped, then the last proposition need no longer be true. There are examples of compact G-spaces X and Y, where G is the free group on two generators, and a continuous G-map $\beta : X \to Y$ such that $\mathscr{P}_G(X)$ is empty, while $\mathscr{P}_G(Y)$ is not.

3. Ergodic Theorems

Let G be a locally compact group with right Haar measure m. Suppose that $(\mathscr{H},\pi)$ is a strongly continuous unitary representation of G on a Hilbert space $\mathscr{H}$. Given a sequence $(F_t) \subset G$ of pre-compact subsets with positive m-measures, we define the operators

$$A_t v = \frac{1}{m(F_t)} \int_{F_t} \pi(s) v \, dm(s), \quad \text{for } v \in \mathscr{H}.$$

For instance, if X is a G-space and ν is a G-invariant Borel probability measure on X, then

$$\pi(g) f(x) = f(g^{-1}x), \quad \text{for } f \in L^2(X,\nu) \text{ and } g \in G, \tag{3.1}$$

defines a unitary representation on the Hilbert space $L^2(X,\nu)$.

We say that $(F_t) \subset G$ is

- a *norm-good sequence* for $(\mathscr{H},\pi)$ if the limits
$$A_\infty v := \lim_t A_t v,$$
exist in the norm topology on $\mathscr{H}$ for all $v \in \mathscr{H}$. If it is norm-good for *every* unitary representation of G, then we simply say that (F_t) is *norm-good*.
- a *pointwise good sequence* for (X,ν) if the limits
$$A_\infty f(x) := \lim_t A_t f(x),$$
exist for ν-almost every $x \in X$ for all $f \in L^2(X,\nu)$, where X and π are as in (3.1). If it is pointwise good for *every* (X,ν), then we simply say that (F_t) is *pointwise good*.
- an *invariant sequence* if it is norm-good and $A_\infty v$ is $\pi(G)$-invariant for every $v \in \mathscr{H}$ and unitary representation $(\mathscr{H},\pi)$.

It is not hard to show that if (F_t) is pointwise good, then it is automatically norm-good.

We state without proofs:

Proposition 3.1. *Suppose that G is amenable. Then,*

- *There exist a sequence $(F_t) \subset G$ of compact subsets with positive m-measures such that*
$$\lim_t \frac{m(F_t K \Delta F_t)}{m(F_t)} = 0, \quad \textit{for every } \text{compact } \textit{subset } K \subset G. \tag{3.2}$$
- *Every sequence (F_t) in G for which* (3.2) *holds is norm-good and invariant, and there is a sub-sequence (F'_t) such that for some constant C, we have*
$$m\Big(\bigcup_{s<t} F'_s F'^{-1}_t\Big) \le m(F'_t), \quad \textit{for all } t. \tag{3.3}$$
- *If a sequence $(F_t) \subset G$ satisfies both* (3.2) *and* (3.3) *then it is pointwise good.*

In particular, suppose that

- X is a G-space.
- ν is an ergodic measure.
- $B \subset X$ is a Borel set with positive ν-measure

- $(F_t) \subset G$ satisfies (3.2) and (3.3).

Then,

$$\lim_t \frac{m(\{g \in G : gx \in B\})}{m(F_t)} = \nu(B), \quad \text{for } \nu\text{-almost every } x \in X.$$

4. The Duality Principle

Proposition 4.1. *Let G be a topological group and $L < G$ a dense subgroup. Then,*

$$L \curvearrowright G \text{ is minimal} \iff L \text{ is dense},$$

where L is assumed to act on G by left translations.

Proof. Fix $g \in G$. Its L-orbit in G is Lg. If $U \subset G$ is a non-empty open set, then

$$Lg \cap U = (L \cap Ug^{-1})g \neq \emptyset,$$

since L is dense and Ug^{-1} is open. If L is not dense, then the L-orbit of e never meets the non-empty open set $U = \overline{L}^c \subset G$. ☐

The following result can be used to establish minimality in certain cases.

Proposition 4.2 (Duality principle)**.** *Suppose that*

- *G is a topological group.*
- *$\Lambda, \Delta < G$ are closed subgroups.*

Then,

$$\Lambda \curvearrowright G/\Delta \text{ is minimal} \iff \Delta \curvearrowright \Lambda\backslash G \text{ is minimal}.$$

Proof. We note that both assertions are equivalent to saying that the double sets $\Lambda x \Delta \subset G$ are dense for all $x \in G$. ☐

In particular, suppose that L and H are topological groups and $\Gamma < L \times H$ a closed subgroup. If we apply the previous proposition to

$$G = L \times H \quad \text{and} \quad \Lambda = L \times \{e\} \quad \text{and} \quad \Delta = \Gamma,$$

together with the observation that $\Delta \curvearrowright G/\Lambda$ is the same action as $\mathrm{proj}_H(\Gamma) \curvearrowright H$, then we conclude:

Corollary 4.3. *Suppose that*

- *L and H are topological groups.*
- *$\Gamma < L \times H$ is a closed subgroup.*

Then,

$$L \curvearrowright L \times H/\Gamma \text{ is minimal} \iff H = \overline{\mathrm{proj}_H(\Gamma)}.$$

5. Gottschalk-Hedlund's Theorem

Let G be a topological group.

Definition 5.1 (Syndetic sets)**.** A set $V \subset G$ is *left syndetic* if there exists a finite set $F \subset G$ such that $FV = G$, and we say that V is *right syndetic* if V^{-1} is left syndetic.

If G is abelian, then every left syndetic set is of course right syndetic. This property almost characterizes abelian groups as the following proposition shows.

Proposition 5.2. *Suppose that G has a conjugacy class which is* not *pre-compact. Then there exists a left syndetic subset $C \subset G$ which is* not *right syndetic. If G is σ-compact, then C can be chosen to be σ-compact as well.*

Proof. We fix

- an increasing exhaustion (K_α) of G by compact subsets.
- an element $x \in G$ with a non-compact conjugacy class.

If G is σ-compact, then the exhaustion can be indexed by a countable set.

We shall prove that there exists a net (t_α) in G such that the set

$$T := \bigcup_\alpha t_\alpha K_\alpha \subset G$$

satisfies $xT \cap T = \emptyset$. Once such a net has been constructed, we observe that the set $C := T^c \subset G$ has the properties:

$$\{e,x\}C = G \quad \text{and} \quad (CF)^c = \bigcap_{f \in F} Tf \neq \emptyset, \quad \text{for every finite set } F \subset G,$$

since every finite (compact) subset of G is eventually properly contained in (K_α), which shows that C is a left, but not right, syndetic subset of G.

We shall now inductively construct the net (t_α). For simplicity, we assume that that the directed set can be chosen to be $(\mathbb{N}, \leq)$. Let $t_o = e$ and suppose that we have defined (t_k) for all $k < n$. Since the conjugacy class is *not* pre-compact, but each K_n is compact, we can find $t_n \in G$ such that

$$xt_nK_n \cap \Big(\bigcup_{k=1}^n t_kK_k\Big) = \emptyset \quad \text{and} \quad x^{-1}t_nK_n \cap \Big(\bigcup_{k=1}^n t_kK_k\Big) = \emptyset.$$

This implies that

$$xt_nK_n \cap t_mK_m = \emptyset, \quad \text{for all } m,n \geq 1,$$

and thus $xT \cap T = \emptyset$. $\square$

The following result of Gottschalk and Hedlund provides a dynamical construction of *left* syndetic sets.

Proposition 5.3 (Gottschalk-Hedlund's Theorem)**.** *Suppose that*

- *X is a* compact *G-space.*
- *$x_o \in X$ is an arbitrary point with a dense G-orbit.*

Then,

$$X \textit{ is minimal} \iff U_{x_o} \subset G \textit{ is left syndetic for every open neighborhood } U \textit{ of } x_o.$$

Proof. Suppose that X is minimal. For every non-empty open set, we have $GU = X$. By compactness, we can find a finite set $F \subset G$ such that $FU = G$, and thus $FU_{x_o} = (FU)_{x_o} = G$.

Suppose that X is not minimal and let $Y \subset X$ be non-empty proper closed G-invariant subset of X. Let $U \subset Y^c$ be a non-empty open neighborhood of x_o whose closure V is a proper subset of Y^c. Suppose that there exists a finite set $F \subset G$ such that $FU_{x_o} = (FU)_{x_o} = G$. Since x_o has a dense G-orbit, we conclude $FU \subset X$ is dense, and thus $FV = X$, which contradicts our assumption that V is a *proper* subset of Y^c. Hence $U_{x_o} \subset G$ is not left syndetic. $\square$

6. SAT*-SPACES

Definition 6.1. Let G be a topological group and X a G-space. Suppose that there exists a Borel probability measure ν on X with the properties that if $B \subset X$ is a Borel set, and

- if $\nu(B) = 0$, then $\nu(gB) = 0$ for all $g \in G$.
- if $\nu(B) > 0$ and $\varepsilon > 0$, then there exists $g \in G$ such that $\nu(gB) > 1 - \varepsilon$.

If for every $g \in G$, the Radon-Nikodym derivative $\frac{dg\nu}{d\nu}$ is ν-essentially bounded, then we refer to the pair (X, ν) as a *SAT*-space*.

We state without proof:

Proposition 6.2. *Suppose that*

- (X,ν) *is a SAT*-space.*
- $B \subset X$ *is a Borel set which is not* ν*-conull.*

Then there exists a ν*-conull subset* $X' \subset X$ *such that* $B_x \subset G$ *is* not *right syndetic for any* $x \in X'$.

We can combine the proposition above with Gottschalk-Hedlund's Theorem and conclude:

Corollary 6.3. *Suppose that*

- X *is a compact and second countable minimal* G*-space*
- $\nu \in \mathscr{P}(X)$ *is a SAT* Borel probability measure.*

Then there exists a ν*-conull subset* $X' \subset X$ *with the property that for any non-empty open set* $U \subset X$ *which is not* ν*-conull, we have*

$$U_x \subset G \text{ is left syndetic, but not right syndetic, for every } x \in X'.$$

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