On the existence of solutions to stochastic mathematical programs with equilibrium constraints

Anton Evgrafov* and Michael Patriksson†

May 14, 2001

Abstract

We generalize Stochastic Mathematical Programs with Equilibrium Constraints (SMPEC) introduced by Patriksson and Wynter [Operations Research Letters, Vol. 25, pp. 159–167, 1999] to allow joint upper-level constraints, and to change continuity assumptions w.r.t. uncertainty parameter assumed before by measurability assumptions. For this problem, we prove the measurability of a lower-level mapping and the existence of solutions. We also discuss algorithmic aspects of the problem, in particular the construction of an inexact penalty function for the SMPEC problem, and touch upon a question of distribution sensitivity. Applications to structural topology optimization and other fields are mentioned.

Keywords: Bilevel programming, equilibrium constraints, stochastic programming, existence of solutions, stochastic Stackelberg game.

1 Introduction

The consideration of uncertain data in engineering and economical hierarchical decision-making processes naturally leads to the formulation of such processes as stochastic mathematical programs with equilibrium constraints (SMPEC) introduced in Ref. 1. We extend the framework presented there to include more general constraints and probabilistic settings, while correcting the error in the proof of the existence of solutions (Ref. 1, Corollary 2.5).

SMPEC generalize deterministic MPEC, or generalized bilevel programming problems (Ref. 2) by explicitly incorporating possible uncertainties in the problem data to obtain robust solutions. For a discussion of possible applications of the model see Ref. 1; applications to structural optimization are discussed in Refs. 3, 4. A special form of SMPEC was formulated in Ref. 5 in a framework of stochastic Stackelberg games. Thus the model has applications in economics as well.

*Department of Mathematics, Chalmers University of Technology, SE-412 96 Gothenburg, Sweden, email: toxa@math.chalmers.se

†Department of Mathematics, Chalmers University of Technology, SE-412 96 Gothenburg, Sweden, email: mipat@math.chalmers.se

Let $(\Omega, \mathfrak{S}, \mathrm{P})$ be a complete probability space. The stochastic MPEC is:

$$\min \ \mathrm{E}_\omega[f(x, \boldsymbol{\xi}(\omega), \omega)] := \int_\Omega f(x, \boldsymbol{\xi}(\omega), \omega)\, \mathrm{P}(d\omega)$$
$$\text{s.t.} \quad \begin{cases} \big(x, \boldsymbol{\xi}(\omega)\big) \in \mathcal{Z}(\omega), & \text{P-a.s.} \\ \boldsymbol{\xi}(\omega) \in \mathcal{S}(x, \omega), & \text{P-a.s.} \end{cases} \qquad (\mathrm{SMPEC} - \Omega)$$

where $\boldsymbol{\xi} : \Omega \to \mathbb{R}^m$ is a random element in $(\Omega, \mathfrak{S}, \mathrm{P})$, $\mathcal{Z} : \Omega \rightrightarrows \mathbb{R}^n \times \mathbb{R}^m$ is a point-to-set mapping representing the upper-level constraints, and $\mathcal{S} : \mathbb{R}^n \times \Omega \rightrightarrows \mathbb{R}^m$ is a set of solutions to a lower-level parametric variational inequality problem:

$$\mathcal{S}(x, \omega) := \{\, \xi \in \mathbb{R}^m \mid -T(x, \xi, \omega) \in N_{\mathcal{Y}(x,\omega)}(\xi) \,\}. \tag{1}$$

The lower-level problem is defined by the mapping $T : \mathbb{R}^n \times \mathbb{R}^m \times \Omega \to \mathbb{R}^m$ and a feasible set mapping $\mathcal{Y} : \mathbb{R}^n \times \Omega \rightrightarrows \mathbb{R}^m$ having closed convex images, and $N_{\mathcal{Y}(x,\omega)} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ denotes the normal cone mapping to the set $\mathcal{Y}(x, \omega)$.

The outline of the paper is as follows. In section 2 the question of feasibility is addressed. The main result is the measurable dependence of the solution set to a variational inequality problem on an uncertainty parameter. Thus it is a generalization of the measurability of the marginal mapping for optimization problems (cf. Lemma III.39 in Ref. 6 and Theorem 8.2.11 in Ref. 7). In section 3, the existence of solutions to $(\mathrm{SMPEC} - \Omega)$ is proved, generalizing Corollary 2.5 in Ref. 1. In section 4 as an example we apply the existence result to a structural optimization problem. Section 5 discusses penalization procedures, generalizing Theorem 9.2.2 in Ref. 8 to $(\mathrm{SMPEC} - \Omega)$ and outlining one possible approach to solve SMPEC.

2 Feasibility

The crucial part of the proof of the existence of solutions to a deterministic MPEC is the closedness of the feasible set (Ref. 2). The typical situation with SMPEC is that for almost any ω the closedness of an "ω-slice" $\mathcal{F}_\omega = \mathcal{Z}(\omega) \cap \mathrm{gr}[x \to \mathcal{S}(x, \omega)]$ of the feasible set can be established using the existing results.

Consider now $x \in \mathbb{R}^n$. Suppose that for almost any ω we obtain a point $(x, \xi(\omega)) \in \mathcal{F}_\omega$. The objective function can be evaluated at $(x, \xi(\cdot))$ *only* if the function $\xi(\omega)$ is $\mathfrak{S}$-measurable. Thus the question arises, whether we can guarantee the existence of some $\mathfrak{S}$-measurable function $\boldsymbol{\xi}$ such that for almost any ω the following two conditions hold: $(x, \boldsymbol{\xi}(\omega)) \in \mathcal{F}_\omega$ (feasible solution) and $f(x, \boldsymbol{\xi}(\omega), \omega) \le f(x, \xi(\omega), \omega)$ ("non-worse" solution).

Our approach to the problem is as follows. We will use the measurability in ω for fixed x of $\mathcal{S}(x, \omega)$ and $\mathcal{Z}_x(\omega) := \{\, \xi \in \mathbb{R}^N \mid (x, \xi) \in \mathcal{Z}(\omega) \,\}$ (cf. Section 2 in Ref. 9, Chapter III in Ref. 6, or Chapter 8 in Ref. 7 for the definition of measurability of set-valued mappings). After that, we can apply the theorem about the measurability of marginal mappings (cf. Lemma III.39 in Ref. 6 or Theorem 8.2.11 in Ref. 7) to give an affirmative answer to the posed question.

We shall presume throughout that $\mathcal{Z}_x(\omega)$ and $\mathcal{Y}(x, \omega)$ are measurable in ω for any $x \in \mathbb{R}^n$. A sufficient condition is, e.g., Theorem 8.2.9 in Ref. 7, cited here for convenience.

Theorem 2.1 (Inverse image). *Consider a complete σ-finite measure space $(\Omega, \mathfrak{S}, \mathrm{P})$, complete separable metric spaces X, Y, measurable set-valued maps $F : \Omega \rightrightarrows X$, $G : \Omega \rightrightarrows Y$ with closed images. Let $g : \Omega \times X \to Y$ be a Carathéodory map. Then, the set-valued map H, defined by $H(\omega) = \{ x \in F(\omega) \mid g(\omega, x) \in G(\omega) \}$ is measurable.*

Remark 2.1. If the mappings $\mathcal{Z}_x(\omega)$, $\mathcal{Y}(x, \omega)$ are defined by inequalities of the type $\{ \xi \in \mathbb{R}^m \mid g_x(\xi, \omega) \leq 0 \}$, where g_x is a Carathéodory mapping, then they are measurable.

The next proposition asserts the measurability of the mapping $\mathcal{S}(x, \cdot)$.

Proposition 2.1 (Measurability of $\mathcal{S}(x, \cdot)$). *Suppose that the mapping $\mathcal{Y}$ is measurable in ω for any fixed x and has closed convex images for any x and almost any ω. Let the mapping T be continuous in y and measurable in ω (i.e. Carathéodory) for any x. Then, the mapping $\mathcal{S}$ is measurable in ω for any x.*

Proof. Fix x and consider the mapping $\widetilde{\mathcal{S}} : \mathbb{R}^n \times \Omega \rightrightarrows \mathbb{R}^m$ given by the normal equation:

$$\widetilde{\mathcal{S}}(x, \omega) := \{\nu \in \mathbb{R}^m \mid T(x, \Pi_{\mathcal{Y}(x,\omega)}(\nu), \omega) + \nu - \Pi_{\mathcal{Y}(x,\omega)}(\nu) = 0\},$$

where $\Pi_{\mathcal{Y}(x,\omega)} : \mathbb{R}^m \to \mathbb{R}^m$ denotes the Euclidean projection operator onto the closed convex set $\mathcal{Y}(x, \omega)$. By Corollary 8.2.13 in Ref. 7, the mapping $\omega \to \Pi_{\mathcal{Y}(x,\omega)}(\nu)$ is measurable for any ν. Since T is Carathéodory in the variables $\xi \times \omega$ and the Euclidean projection is continuous, the resulting mapping $T(x, \Pi_{\mathcal{Y}(x,\omega)}(\nu), \omega)$ is Carathéodory in variables $\nu \times \omega$. Thus we can apply Theorem 2.1 to conclude the measurability of $\widetilde{\mathcal{S}}$ for any ν.

Recalling that $\mathcal{S}(x, \omega) = \Pi_{\mathcal{Y}(x,\omega)}(\widetilde{\mathcal{S}}(x, \omega))$ by Proposition 1.3.3 in Ref. 2, we can apply Theorem 8.2.7 in Ref. 7 about direct image to obtain the measurability of the mapping $\operatorname{cl} \mathcal{S}(x, \cdot)$ for any x. Since T is continuous in ξ and $\mathcal{Y}$ has closed images, the mapping $\mathcal{S}$ has closed images and we are done. □

3 Existence of solutions

Let $\mathcal{X}$ denote the projection of the feasible set of the upper-level problem on the space of x variables: $\mathcal{X} := \{ x \in \mathbb{R}^n \mid \exists\, \xi(\omega) : (x, \xi(\omega)) \in \mathcal{Z}(\omega)$for almost any $\omega \}$. Let also denote by $\mathcal{F}(x, \omega)$ the "x-slice" of the feasible set of (SMPEC $- \Omega$): $\mathcal{F}(x, \omega) := \mathcal{Z}_x(\omega) \cap \mathcal{S}(x, \omega)$. We will say that the function $f : \mathbb{R}^n \times \mathbb{R}^m \times \Omega$ is uniformly weakly coercive w.r.t. to x and the set $\mathcal{X}$ if the set $\{x \in \mathcal{X} \mid f(x, \xi, \omega) \leq c\}$ is bounded for any $c \in \mathbb{R}$.

The approach to the existence proof is close in spirit to that of Theorem 14.60 in Ref. 10 about the interchangeability of integration and optimization. The difficulty is that we have "coupling" variables x which do not allow us to use the pointwise minimization in a straightforward way.

The next theorem generalizes Corollary 2.5 in Ref. 1 in the following ways: we allow joint upper-level constraints $\mathcal{Z}$, do not require any continuity of the involved mappings with respect to ω, and consider an arbitrary probability measure on a complete probability space.

Theorem 3.1 (Existence of solutions). *Suppose that the following assumptions are fulfilled:*

1. *the mappings* $\mathcal{Z}_x(\cdot)$ *and* $\mathcal{S}(x,\cdot)$ *are measurable for any* x*,*

2. *the set* $\mathcal{Z}(\omega)$ *and the mapping* $x \to \mathcal{S}(x,\omega)$ *are closed for almost all* $\omega \in \Omega$*,*

3. *the mapping* $f(x,\xi,\omega)$ *is continuous in* (x,ξ)*, measurable in* ω*, uniformly weakly coercive w.r.t.* x *and the set* $\mathcal{X}$*, and bounded from below by an* $(\mathfrak{S},\mathrm{P})$*-integrable function,*

4. *for any* $x \in \mathcal{X}$ *there is a neighborhood* $U_x \ni x$ *such that the set* $\cup_{\widetilde{x}\in U_x\cap\mathcal{X}} Z_{\widetilde{x}}(\omega)$ *is bounded for almost any* ω*,*

5. *the set* $\mathcal{F}(x_0,\omega)$ *is nonempty for some* $x_0 \in \mathcal{X}$ *and almost any* ω*.*

Then, the problem $(\mathrm{SMPEC}-\Omega)$ *possess at least one optimal solution.*

Proof. Owing to the conditions 1, 5, and the Measurable Selection Theorem (e.g. Refs. 9, 11) there exists a random element $\boldsymbol{\xi}(\omega) \in \mathcal{F}(x_0,\omega)$ for almost all ω, i.e., the problem is feasible. Consider an arbitrary minimizing sequence $\{(x_k,\boldsymbol{\xi}_k)\}$. Uniform weak coercivity in assumption 3 implies that there must be a subsequence of the sequence with a converging x-component. Let us renumber the whole sequence, so that $\bar{x} := \lim_{k\to\infty} x_k$. Consider now a measurable function $\widetilde{f}(\omega) := \liminf_{k\to\infty} f(x_k,\boldsymbol{\xi}_k(\omega),\omega)$. Using the lower boundedness of f (in assumption 3), we get $\mathrm{E}_\omega[\widetilde{f}(\omega)] \le \lim_{k\to\infty} \mathrm{E}_\omega[f(x_k,\boldsymbol{\xi}_k(\omega),\omega)]$.

On the other hand, the uniform local boundedness assumption 4 implies that for almost any ω there is an infinite sequence of indices $k(\omega)$ such that there exists $\widetilde{\xi}(\omega) := \lim_{k(\omega)\to\infty} \boldsymbol{\xi}_{k(\omega)}(\omega)$ and so that $\widetilde{f}(\omega) = \lim_{k(\omega)\to\infty} f(x_{k(\omega)},\boldsymbol{\xi}_{k(\omega)}(\omega),\omega)$. The assumed closedness of the mappings $\mathcal{Z}$ and $\mathcal{S}$ (assumption 2) implies that $\widetilde{\xi}(\omega) \in \mathcal{F}(\bar{x},\omega)$ for almost any ω. Note that the continuity assumptions on f imply that $f(\bar{x},\widetilde{\xi}(\omega),\omega) = \widetilde{f}(\omega)$ almost everywhere.

Consider now the ω-parametric optimization problem in the variables $\xi(\omega)$:

$$
\begin{aligned}
&\min f(\bar{x},\xi(\omega),\omega)\\
&\text{s.t.}\quad \begin{cases} \big(\bar{x},\xi(\omega)\big) \in \mathcal{Z}(\omega), & \text{P-a.s.}\\ \quad\;\; \xi(\omega) \in \mathcal{S}(\bar{x},\omega), & \text{P-a.s.} \end{cases}
\end{aligned}
\tag{2}
$$

We know that the problem has a nonempty, closed and bounded feasible set for almost any ω, that also depends on ω in a measurable way. Thus we can apply Theorem 8.2.11 in Ref. 7 to obtain the existence of a measurable solution $\bar{\boldsymbol{\xi}}(\omega)$ such that $f(\bar{x},\bar{\boldsymbol{\xi}}(\omega),\omega) \le f(\bar{x},\widetilde{\xi}(\omega),\omega)$ owing to the optimality of $\bar{\boldsymbol{\xi}}$ and the feasibility of $\widetilde{\xi}$ for the problem (2).

Thus we have found a feasible solution $(\bar{x},\bar{\boldsymbol{\xi}}(\omega))$ with desirable properties. □

Remark 3.1. For examples of conditions implying the closedness of $x \to \mathcal{S}(x,\omega)$ (assumption 1) we cite assumption (iii) in Ref. 1 (which must hold for almost any ω in addition to the continuity of the mapping T in (x,y)):

(iii) The lower-level constraint set, $\mathcal{Y}(x)$, is of the form $\mathcal{Y}(x) := \{\, \xi \in \mathbb{R}^m \mid g_i(x,\xi) \le 0, \quad i = 1,\dots,k \,\}$, where each function $g_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is continuous on $\mathbb{R}^n \times \mathbb{R}^m$ and convex in ξ for each x. Further, either $g_i(x,\cdot) = g_i(\cdot)$, $i = 1,\dots,k$, that is, $\mathcal{Y}(x) = \mathcal{Y}$, or for each upper-level feasible x there is a $\xi \in \mathbb{R}^m$ such that $g_i(x,\xi) < 0$, $i = 1,\dots,k$.

Another example is the set assumptions of Corollary 3.1 in Ref. 4; this closedness result is stated for a specific stochastic bilevel programming problem arising in contact mechanics.

4 Application to stochastic structural optimization

In this section we apply Theorem 3.1 to show the existence of a truss with a minimal weight under stochastic loads and stress constraints. In the case of discrete measures with finite support the problem was extensively studied in Ref. 4.

The problem formulation is:

$$\begin{aligned}
&\min_{(x,s(\cdot))} \; 1^T x \\
&\text{s.t.} \begin{cases} 0 \leq x, & \\ |s(\omega)| \leq \sigma x, & \text{P-a.s.,} \\ s(\omega) \text{ solves } (\mathcal{C})_x(\omega), & \text{P-a.s.,} \end{cases}
\end{aligned} \tag{$\mathcal{W}$}$$

where the lower-level problem $(\mathcal{C})_x(\omega)$ is:

$$\begin{aligned}
&\min_s \mathcal{E}(x,s) := \frac{1}{2}\sum_{i=1}^n \frac{s_i^2}{Ex_i} \\
&\text{s.t.} \sum_{i=1}^n B_i^T s_i = F(\omega).
\end{aligned} \tag{$(\mathcal{C})_x(\omega)$}$$

The upper-level (*design*) variable x_i represents a volume of material allocated at the bar i ($x_i = 0$ represents structural void), the lower-level (*state*) variable $s_i(\cdot)$ represents a force in the bar i multiplied by the bar length, $E > 0$ is the Young's modulus of the structure material, $\sigma > 0$ is the maximal allowable stress, $F : \Omega \to \mathbb{R}^k$ is a stochastic load, B_i, $i = 1, \ldots, n$ are the kinematic transformation matrices of size $1 \times k$, and $\mathcal{E} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is an extended real-valued functional, representing the elastic energy of the structure. The problem $(\mathcal{C})_x(\omega)$ is the mechanical *principle of minimum of complementary energy.* Thus making the identifications $\mathcal{Z} := \{\, (x,s) \in \mathbb{R}^n \times \mathbb{R}^n \mid 0 \leq x, |s| \leq \sigma x \,\}$ and $\mathcal{S}(x,\omega) := \{\, s \in \mathbb{R}^n \mid s \text{ solves } (\mathcal{C})_x(\omega) \,\}$ we can see that the problem $(\mathcal{W})$ perfectly fits into the framework of $(\text{SMPEC} - \Omega)$.

Proposition 4.1. *Let $F : \Omega \to \mathbb{R}^k$ be measurable. Suppose that the problem $(\mathcal{W})$ has a feasible point $(x, s(\omega))$ such that* $\mathrm{P}(\mathcal{E}(x,s(\omega)) < \infty) = 1$. *Then it possess at least one optimal solution.*

Proof. Obviously, assumptions 3–5 of Theorem 3.1 are fulfilled. Furthermore, assumption 1 holds (it is trivial for $\mathcal{Z}_x$ and it is an immediate consequence of Lemma III.39 in Ref. 6 for $\mathcal{S}(x,\cdot)$). The set $\mathcal{Z}$ is closed. Thus it remains to show the closedness of $x \to \mathcal{S}(x,\omega)$ for any ω to verify assumption 2 and conclude the existence of solutions.

The required property follows from Corollary 3.1 in Ref. 4 under the additional assumption of boundedness of energy functional $\mathcal{E}(x,s)$. Theorem 4.3 in Ref. 4 implies that one can add the *redundant* (such that no optimal solution can violate it) constraint $\mathcal{E}(x,s) \leq \nu$ to the problem $(\mathcal{W})$. Since the function $\mathcal{E}$ is l.s.c. (Ref. 12, p. 83), the set $\widetilde{\mathcal{Z}} := \{ (x,s) \in \mathcal{Z} \mid \mathcal{E}(x,s) \leq \nu \}$ is closed.

We finish the proof by the application of Theorem 3.1. □

5 Inexact penalization

Compared with one-level problems, bilevel optimization algorithms are much less straightforward to develop owing to the non-convex nature of the problem and its absence of constraint qualifications for nonlinear programming (Ref. 2). One approach is to move the equilibrium constraint as a penalty into the objective function. For examples of penalty functions leading to algorithmic solutions to MPEC, see Refs. 2, 13–15 and references therein. In particular, exact penalties are of great importance, since they lead to exact solutions while they do not require the penalty parameter to tend to infinity (Ref. 16). One cannot however expect to be able to construct an exact penalty for SMPEC problems, given an exact penalty for each ω, as the following simple example shows. The reason is again the presence of the "coupling" upper-level variables.

Example 5.1. *Let* $(\Omega, \mathfrak{S}, \mathrm{P}) = ([0,1], \bar{\mathcal{B}}, \lambda)$, *where* λ *is a Lebesgue measure, and* $\bar{\mathcal{B}}$ *is a* σ*-algebra of Lebesgue measurable sets on* $[0,1]$. *Let* $\mathcal{Z}(\omega) = [0,\omega] \times \{ 0 \}$, $f(x,\xi,\omega) = (x-1/2)^2$, $\mathcal{Y}(x,\omega) = \{ 0 \}$, $T(x,\xi,\omega) = 0$, *For any* $\omega \in [0,1]$ *an exact penalty for the "fixed-*ω*" problem is, for example,* $G(x,\xi,\omega) = \max\{ x - \omega, 0 \}$.

Nevertheless, since

$$\int_0^1 [(x-1/2)^2 + \mu \max\{ x-\omega, 0 \}] \, \lambda(d\omega) = (x-1/2)^2 + \mu \frac{x^2}{2},$$

the minimizing sequence is $x_\mu = 1/(\mu+2) \to 0$ *as* $\mu \to \infty$, *and thus it does not reach the optimal (actually, the only feasible) point of the given SMPEC,* $x^* = 0$, *for any finite value of* μ.

In the following theorem we show that, given a penalty function for almost any ω, we can construct an inexact penalty function for SMPEC. It generalizes Theorem 9.2.2 in Ref. 8. Note that we do not necessarily have compact sequences for the lower-level variables, so we do not necessarily have convergence for these variables. In the case of discrete measures supported by finite sets, the theorem reduces to Theorem 9.2.2 in Ref. 8.

We will write $\mathrm{val}(\mathcal{P})$ for the optimal value of the optimization problem $(\mathcal{P})$.

Theorem 5.1. *Suppose that the assumptions of Theorem 3.1 are satisfied, so that there is an optimal solution to* $(\mathrm{SMPEC}-\Omega)$. *Let also* $G(x,\xi,\omega)$ *be non-negative, continuous in* (x,ξ) *for almost any* ω, *and measurable in* ω *for any* $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^m$ *function such that* $\mathcal{S}(x,\omega) = \{ \xi \mid (x,\xi) \in \mathcal{Z}(\omega), G(x,\xi,\omega) = 0 \}$. *Then the penalized problem:*

$$\begin{aligned} \min \ & \mathrm{E}_\omega[f(x,\boldsymbol{\xi}(\omega),\omega) + \mu G(x,\boldsymbol{\xi}(\omega),\omega)] \\ \text{s.t.} \ & \big(x,\boldsymbol{\xi}(\omega)\big) \in \mathcal{Z}(\omega), \quad \mathrm{P}\text{-}a.s. \end{aligned} \qquad (\mathrm{SMPEC}-\Omega)_\mu$$

has an optimal solution for any $\mu \geq 0$ *and*

$$\sup_{\mu \geq 0} \operatorname{val}(\mathrm{SMPEC} - \Omega)_\mu = \lim_{\mu \to \infty} \operatorname{val}(\mathrm{SMPEC} - \Omega)_\mu = \operatorname{val}(\mathrm{SMPEC} - \Omega).$$

Furthermore, any limit point of the upper-level optimal solutions $\{x_\mu\}$ *to* (SMPEC$-$ $\Omega)_\mu$ *(and there is at least one) is an upper-level optimal solution to* (SMPEC $-$ Ω)*.*

Proof. For any $\mu \geq 0$ the problem $(\mathrm{SMPEC} - \Omega)_\mu$ satisfies the assumptions of Theorem 3.1 (where we can put $\mathcal{S}_\mu(x, \omega) = \{\, \xi \in \mathbb{R}^m \mid (x, \xi) \in \mathcal{Z}(\omega) \,\}$), and thus possess a solution $(x_\mu, \boldsymbol{\xi}_\mu(\cdot))$.

Following the proofs of Lemma 9.2.1 and Theorem 9.2.2 in Ref. 8, we get:

$$\begin{aligned} \operatorname{val}((\mathrm{SMPEC} - \Omega)) &\geq \sup_{\mu \geq 0} \operatorname{val}(\mathrm{SMPEC} - \Omega)_\mu \\ &= \lim_{\mu \to \infty} \operatorname{val}(\mathrm{SMPEC} - \Omega)_\mu \\ &= \lim_{k \to \infty} \mathrm{E}_\omega[f(x_{\mu_k}, \boldsymbol{\xi}_{\mu_k}(\omega), \omega)] \end{aligned} \tag{3}$$

for some $\mu_k \to \infty$.

By the uniform coercivity (assumption 3 of Theorem 3.1) of f in x, and by the properties of G as a penalty function, the sequence $\{x_{\mu_k}\}$ is bounded. Switching to a subsequence if necessary, we may assume that $\lim_{k \to \infty} x_{\mu_k} = \widetilde{x}$. Owing to the lower boundedness of f (assumption 3 of Theorem 3.1) we have that $\lim_{k \to \infty} \mathrm{E}_\omega[f(x_{\mu_k}, y_{\mu_k}(\omega), \omega)] \geq \mathrm{E}_\omega[\liminf_{k \to \infty} f(x_{\mu_k}, y_{\mu_k}(\omega), \omega)]$. By the boundedness of the feasible set (assumption 4 of Theorem 3.1) for almost any ω, there is a sequence $k(\omega)$ such that $\boldsymbol{\xi}_{\mu_{k(\omega)}}(\omega) \to \widetilde{\xi}(\omega)$ and $\liminf_{k \to \infty} f(x_{\mu_k}, \boldsymbol{\xi}_{\mu_k}(\omega), \omega) = \lim_{k(\omega) \to \infty} f(x_{\mu_{k(\omega)}}, \boldsymbol{\xi}_{\mu_{k(\omega)}}(\omega), \omega) \geq f(\widetilde{x}, \widetilde{\xi}(\omega), \omega)$, for P-almost any ω. Owing to the closedness (assumption 2 of Theorem 3.1) of $\mathcal{Z}$, $(\widetilde{x}, \widetilde{\xi}(\omega)) \in \mathcal{Z}(\omega)$, P-a.s.

Following the proof of Theorem 9.2.2 in Ref. 8 we get:

$$0 = \lim_{k \to \infty} \mathrm{E}_\omega[G(x_{\mu_k}, \boldsymbol{\xi}_{\mu_k}(\omega), \omega)] \geq \mathrm{E}_\omega[\liminf_{k \to \infty} G(x_{\mu_k}, \boldsymbol{\xi}_{\mu_k}(\omega), \omega)],$$

and, $\liminf_{k \to \infty} G(x_{\mu_k}, \boldsymbol{\xi}_{\mu_k}(\omega), \omega) \geq G(\widetilde{x}, \widetilde{\xi}(\omega), \omega) \geq 0$ for P-almost any ω by the continuity and non-negativity of G, thus showing that $\tilde{\xi}(\omega) \in \mathcal{F}(x, \omega)$ for P-almost any ω. Considering the parametric optimization problem (2) we can find a measurable function $\widetilde{\boldsymbol{\xi}}(\cdot) \in \mathcal{F}(x, \cdot)$ such that $f(\widetilde{x}, \widetilde{\xi}(\omega), \omega) \geq f(\widetilde{x}, \widetilde{\boldsymbol{\xi}}(\omega), \omega)$ P-a.s., thus showing that

$$\sup_{\mu \geq 0} \operatorname{val}(\mathrm{SMPEC} - \Omega)_\mu \geq \mathrm{E}_\omega[f(\widetilde{x}, \widetilde{\boldsymbol{\xi}}(\omega), \omega)] \geq \operatorname{val}(\mathrm{SMPEC} - \Omega).$$

Together with (3) this proves the claim. □

6 Concluding remarks

The case of discrete measures was considered in Ref. 1 where some algorithms were proposed. In order to solve a general SMPEC problem it is natural to approximate it with a sequence of simpler problems involving only discrete probability measures. Such a discretization procedure could be applied either to the original problem (SMPEC $-$ Ω) or to the penalized one $(\text{SMPEC} - \Omega)_\mu$. The hope is that solutions to these discrete problems would converge to a solution to the original problem, and this question is related to the stability of stochastic optimization problems with respect to small changes in probability measure. The question of stability of bilevel programming problems is not so well investigated in the literature even in the deterministic case. For existing results we mention Refs. 17–20.

Existing results about the stability of optimization problems with respect to changes in the probability measure usually presumes the existence of a constraint qualification (Ref. 21), which are by no means satisfied by SMPEC problems, or they are posed in the spaces of continuous functions (Refs. 22–24), which also is not the case for a general SMPEC. (To apply the latter results we need to assume the uniqueness of solutions to a lower-level problem and the continuity of solutions with respect to ω.)

One can also view the lower-level problem as a variational inequality problem (VIP) in a Banach space X, under the additional assumptions that $\boldsymbol{\xi}(\cdot) \in X$ and $T(x, \boldsymbol{\xi}(\cdot), \cdot) \in X^*$, but the results one could obtain with such an identification are too limited in our setup. For example, neither the pseudo-monotonicity property (typically assumed in the existence of solutions results) nor the strong monotonicity property (usually assumed in the stability of solutions results (Refs. 25–28)) of the resulting VIP do not follow from the corresponding property of the operator $T(x, \cdot, \omega)$ holding for almost any ω.

Despite all these difficulties it is possible to show the convergence of some discretization schemes under additional assumptions, for the specific cases of SMPEC discussed in Ref. 4 in application to structural optimization in contact mechanics. Furthermore, assuming the continuity of the problem's data with respect to ω, it is possible to analyze a distribution sensitivity for such stochastic structural optimization models; this is the topic of ongoing research.

References

1. PATRIKSSON, M., and WYNTER, L., *Stochastic Mathematical Programs with Equilibrium Constraints*, Operations Research Letters. Vol. 25, No. 4, pp. 159–167, 1999.

2. LUO, Z.-Q., PANG, J.-S., and RALPH, D., *Mathematical Programs with Equilibrium Constraints*, Cambridge University Press, Cambridge, 1996.

3. CHRISTIANSEN, S., PATRIKSSON, M., and WYNTER, L., *Stochastic Bilevel Programming in Structural Optimization*, Technical Report, PRiSM, Université de Versilles, Versailles, France, 1999. To appear in Structural Optimization.

4. PATRIKSSON, M. and PETERSSON, J., *Existence and Continuity of Optimal Solutions to Some Structural Topology Optimization Problems Including Unilateral Constraints and Stochastic Loads*, Technical Report, Department of Mathematics, Chalmers University of Technology, Göteborg University,

SE-412 96 Göteborg, Sweden, 2000. Under review for Zeitschrift für Angewandte Mathematik und Mechanik.

5. LUH, P. B., CHANG, T. S., and NING, T., *Pricing Problems with a Continuum of Customers as Stochastic Stackelberg Games*, Journal of Optimization Theory and Applications, Vol. 55, No. 1, pp. 119–131, 1987.

6. CASTAING, C., and VALADIER, M., *Convex Analysis and Measurable Multifunctions*, Springer-Verlag, Berlin, Lecture Notes in Mathematics, Vol. 580, 1977.

7. AUBIN, J.-P., and FRANKOWSKA, H., *Set-Valued Analysis*, Systems & Control: Foundations & Applications, Vol. 2, Birkhäuser Boston Inc, Boston, 1990.

8. BAZARAA, M. S., SHERALI, H. D., and SHETTY, C. M., *Nonlinear Programming*, second edition, John Wiley & Sons, Inc, New York, 1993.

9. HIMMELBERG, C. J. *Measurable Relations*, Fundamenta Mathematicae, Vol. 87, pp. 53–72, 1975.

10. ROCKAFELLAR, R. T. and WETS, R. J.-B., *Variational Analysis*, Grudlehren der Matematischen Wissenschaften, Vol. 317, Springer-Verlag, Berlin, 1998.

11. AUMANN, R. J., *Measurable Utility and the Measurable Choice Theorem*, La Décision, Agrégation et Dynamique des Ordres de Préférence (Actes Colloq. Internat., Aix-en-Provence, 1967), Éditions du Centre Nat. Recherche Sci., Paris, Vol. 2, pp. 15–26, 1969.

12. ROCKAFELLAR, R. T., *Convex Analysis*, Princeton University Press, Princeton, 1970.

13. PANG, J.-S., *Error Bounds in Mathematical Programming*, Papers from the International Workshop on Error Bounds held at City University of Hong Kong, Kowloon, December 8–14, 1998, Edited by Zhi-Quan Luo and Jong-Shi Pang, Mathematical Programming, Vol. 79, No. 1–3, Serie B, pp. 299–332, 1997.

14. YE, J. J., ZHU, D. L., and ZHU, Q. J., *Exact Penalization and Necessary Optimality Conditions for Generalized Bilevel Programming Problems*, SIAM Journal on Optimization, Vol. 7, No. 2, pp. 481–507, 1997.

15. SCHOLTES, S., and STÖHR, M., *Exact Penalization of Mathematical Programs with Equilibrium Constraints*, SIAM Journal on Control and Optimizaiton, Vol. 37, No. 2, pp. 617–652, 1999.

16. BURKE, J. V., *An Exact Penalization Viewpoint of Constrained Optimization*, SIAM Journal on Control and Optimization, Vol. 29, No. 4, pp. 968–998, 1991.

17. LIGNOLA, M. B., and MORGAN, J., *Topological Existence and Stability for Stackelberg Problems*, Journal Optimization Theory and Applications, Vol. 84, No. 1, pp. 145–169, 1995.

18. WANG, S.-Y., WANG, Q., and COLADAS URIA, L., *A Stability Theorem in Nonlinear Bilevel Programming*, Qüestiió (2), Vol. 20, No. 2, pp. 215–222, 1996.

19. JIA, F., YANG, F., and WANG, S., *Sensitivity Analysis in Bilevel Linear Programming*, Systems Science and Mathematical Sciences, Vol. 11, No. 4, pp. 359–366, 1998.

20. LUCET, Y. and YE, J. J., *Sensitivity Analysis for Optimization Problems with Variational Inequality Constraints*, CECM Preprint Server, July, 2000.

21. LEPP, R., *Approximations to Stochastic Programs with Complete Recourse*, SIAM Journal on Control and Optimization, Vol. 28, No. 2, pp. 382–394, 1990.

22. ROBINSON, S. M., and WETS, R. J.-B., *Stability in Two-Stage Stochastic Programming*, SIAM Journal on Control and Optimization, Vol. 25, No. 6, pp. 1409–1416, 1987.

23. KALL, P., *On Approximations and Stability in Stochastic Programming*, Parametric Optimization and Related Topics, Mathematical Research, Vol. 35, Akademie-Verlag, Berlin, pp. 387–407, 1987.

24. RÖMISCH, W., and SCHULTZ, R., *Distribution Sensitivity in Stochastic Programming*, Mathematical Programming, Vol. 50, No. 2, pp. 197–226, 1991.

25. DING, X. P., *General Algorithm of Solutions for Nonlinear Variational Inequalities in Banach Space*, Computers and Mathematics with Applications, Vol. 34, No. 9, pp. 131–137, 1997.

26. WATSON, P. J., *Variational Inequalities in Nonreflexive Banach Spaces*, Applied Mathematics Letters, Vol. 10, No. 2, pp. 45–48, 1997.

27. LEVY, A. B., *Sensitivity of solutions to variational inequalities on Banach spaces*, SIAM Journal on Control and Optimization, Vol. 38, No. 1, pp. 50–60, 1999.

28. DING, X. P., *General Algorithm for Nonlinear Variational Inequalities in Reflexive Banach Spaces*, Mathematical Analysis and Applications, Hadronic Press, Palm Harbor, Florida, pp. 75–90, 2000.