ZAMM · Z. angew. Math. Mech. **00** (2002) 0, 1–15

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Stochastic Structural Topology Optimization: Existence of Solutions and Sensitivity Analyses

We consider structural topology optimization problems including unilateral constraints arising from, for example, non-penetration conditions in contact mechanics or non-compression conditions for elastic ropes. To construct more realistic models and to hedge off possible failures or an inefficient behaviour of optimal structures, we allow parameters (for example, loads) defining the problem to be stochastic. The resulting nonsmooth stochastic optimization problem is an instance of stochastic mathematical programs with equilibrium constraints (SMPEC), or stochastic bilevel programs. The existence as well as the continuity of optimal solutions with respect to the lower bounds on the design variables are established. The question of continuity of the optimal solutions with respect to small changes in the probability measure is analysed. For a subclass of the problems considered the answer is affirmative, thus establishing the robustness of optimal solutions.

Keywords: Bilevel programming, stochastic programming, robust optimization, ε-perturbation, stress constraints

1 Introduction and notation

Does the introduction of multiple load cases into a topology optimization problem always lead to robust optimal designs? The large number of publications aiming to achieve robust solutions by optimizing for several (in some cases the continuum) load cases suggests that the answer should be positive.

The answer of course depends on the definition of "robustness" and the type of optimization problem under consideration. The reason for considering several load cases is to incorporate the *uncertain* nature of the loads into the model, while the desired property of a robust design is to change continuously as a model of reality (loading conditions, material properties, etc.) changes. To thoroughly answer the posed question it is necessary to measure the closeness of two models of (uncertain) reality.

In this paper we consider two of the most natural and classic structural topology optimization problems: the finding of a maximally stiff truss under a volume constraint, and the finding of a truss of minimal weight under stress constraints. The uncertainty due to several factors (such as loads unknown in advance, varying material properties, manufacturing errors, etc.) is taken into account. Capturing the uncertainty in the model through the use of probability theory allows us to construct general models, and through the associated probability measure, it is possible to interpret the "continuous change in the model of reality" as a continuous change in a topological space of measures.

To include a wide range of applications we allow mechanical structures to be *unilaterally* constrained, i.e., some parts of the structure might come into unilateral frictionless contact with rigid obstacles, while some other parts might sustain only tensile forces. Practical applications of unilateral contact include such machine elements as joints, hinges, press-fits, and examples of structures with tensile-only members include suspension bridges and cranes.

In addition to extending "classic" structural topology optimization results (existence of optimal designs, convergence of ε-perturbations) to the general stochastic setting, we analyse the continuity of the optimal solutions with respect to changes in the probability measure. The results of this analysis give us explicit information about when the introduction of uncertainty into the structural topology optimization models indeed leads to robust optimal designs.

1.1 Historical overview

The study of the topology optimization of trusses dates back at least as early as the beginning of the previous century [21]. Practice has shown that allowing truss topology to change leads to exceedingly efficient designs. Thus both optimization and mechanical models were considerably generalized in many aspects by many authors during the last thirty years (see, for example, the surveys [29, 27]).

Unfortunately, designs obtained from a topology optimization procedure have a principal drawback. They may be very inefficient or can even fail when loading conditions change slightly. An attempt to maintain the efficiency of topology optimization while hedging off possible failures or an inefficient behaviour has given rise to the field of *robust topology optimization*. Owing to the anticipated fact that the "real" probability model is never known, and the reported high sensitivity of solutions to stochastic structural optimization problems with respect to small changes in probability measure (e.g. [3, pp. 20–22]), many probability-free worst-case ("pessimistic") models of uncertainty have been developed as an alternative to probabilistic ones. In such worst-case models, uncertain parameters are assumed to vary in convex ("convex uncertainty models") or even in polyhedral ("polyhedral uncertainty models") sets. An efficient numerical approach to solve problems of this type is known [4], which however has a considerable drawback: the algorithm can treat uncertainties with respect to loading conditions only. Furthermore, loads are restricted to lie in some small ellipsoid around the "primal loads", a condition that further reduces the generality of the algorithm.

Recently in [8] and [23] stochastic structural topology optimization problems have been formulated and analysed in the case of discrete probability spaces, with emphasis on sensitivity analysis leading to numerical methods. The sensitivity analysis conducted as one part of this paper is an extension of the results in [23] to a more general probabilistic setting.

Although many results in this paper can be viewed as extensions of the results found in [23], there are large differences between the problems considered in the present paper and those in the paper cited. First of all, the optimization problems become infinite-dimensional, which makes even the existence of solutions difficult to establish (cf. [11, 23]); for example, we need to check the measurability of random quantities involved, or to deal with closed and bounded but non-compact sets. Secondly, some local error bounds used in [23] are not valid in the present situation; we will propose a solution to this problem in Subsection 3.2 for a restricted set of mechanical models. At last, but not the least, a completely new but extremely important kind of sensitivity analysis is added in the stochastic case, namely the one concerned with the continuity of the optimal solutions with respect to small changes in the underlying probability measure. Even though some of the results obtained are of the "negative" nature, they contribute to our knowledge and understanding of the difficult problems arising in structural topology optimization.

1.2 Mechanical equilibrium

In this subsection we introduce the notation and mechanical principles necessary to state the problems we are going to analyse.

Given positions of the nodes, the *design* (and topology in particular) of a truss can be described by the following set of *design* variables:

- $x_i \geq 0$, $i = 1, \ldots, m$, representing the volume of material, allocated to the bar i in the structure;
- $X_j \geq 0$, $j = 1, \ldots, r_2$, representing the volume of material, allocated to the cable j.

We introduce two index sets of the present (or active) members in the structure: $\mathcal{I}(x) = \{ i = 1, \ldots, m \mid x_i > 0 \}$ and $\mathcal{J}(X) = \{ j = 1, \ldots, r_2 \mid X_j > 0 \}$; we denote by $\mathcal{I}^c(x)$ (respectively $\mathcal{J}^c(X)$) the complement of $\mathcal{I}(x)$ (resp. $\mathcal{J}(X)$) in $\{ 1, \ldots, m \}$ (resp. $\{ 1, \ldots, r_2 \}$).

For a vector $v \in \mathbb{R}^n$ and an index $I = \{ i_1, \ldots, i_{|I|} \} \subseteq \{ 1, \ldots, n \}$ we denote by v_I the subvector $(v_{i_1}, \ldots, v_{i_{|I|}})^t$.

Given a particular design the equilibrium status of a truss including cables and frictionless unilateral contact conditions can be described by the following set of *state* variables:

- s_i, $i \in \mathcal{I}(x)$, representing the tensile force in the bar i times bars length;
- S_j, $j \in \mathcal{J}(x)$, representing the tensile force in the cable j;
- λ is the vector of dimension r_1 of contact forces.

To simplify the notation we collect all s_i, $i = 1, \ldots, m$ (resp. S_j, $j = 1, \ldots, r_2$) in one vector s of dimension m (resp. S of dimension r_2).

Let $(\Omega, \mathfrak{S}, \mathrm{P})$ be a complete probability space, and let $\omega \in \Omega$. The particular values of state variables at the equilibrium are determined by the principle of minimum complementary energy (in our case it is the (x, X, ω)-

parametric quadratic programming problem):

$$
(\mathcal{C})_{(x,X)}(\omega)\begin{cases}
\min\limits_{(s,S,\lambda)} \mathcal{E}(x,X,s,S,\lambda,\omega) := \dfrac{1}{2}\displaystyle\sum_{i\in\mathcal{I}(x)} \frac{s_i^2}{E(\omega)x_i} + g_1^T(\omega)\lambda \\
\qquad\qquad + \displaystyle\sum_{j\in\mathcal{J}(X)} \Big(\frac{(L_j(\omega)S_j)^2}{2E_c(\omega)X_j} + (g_2(\omega))_j S_j\Big), \\
\text{s.t.} \begin{cases}
C_1^T(\omega)\lambda + \displaystyle\sum_{i\in\mathcal{I}(x)} B_i^T(\omega)s_i + \sum_{j\in\mathcal{J}(X)} S_j\gamma_j(\omega) = f(\omega), \\
\lambda \geq 0, \\
S_{\mathcal{J}(X)} \geq 0, \\
s_{\mathcal{I}^c(x)} = 0, \\
S_{\mathcal{J}^c(X)} = 0,
\end{cases}
\end{cases}
$$

where the functions in the problem have the following meaning from a mechanical point of view:
- $E(\omega)$ and $E_c(\omega)$ are Young's moduli for the structure and cable materials respectively;
- $B_i(\omega)$ is the kinematic transformation matrix for the bar i;
- $\gamma_j(\omega)$ is the unit direction vector of the cable j;
- $(g_2(\omega))_j$ is the initial slack of the cable j;
- $L_j(\omega)$ is the length of the cable j;
- $C_1(\omega)$ is the quasi-orthogonal kinematic transformation matrix for rigid obstacles;
- $g_1(\omega) \geq 0$ is the vector of the initial gaps;
- $f(\omega)$ is the vector of external forces.

For the problem to be tractable we assume that all functions listed above are $\mathfrak{S}$-measurable. We further assume that the matrix C_1 is quasi-orthogonal, that is, that $C_1 C_1^T = I$. This condition is fulfilled if at each node either there is at most one rigid support or multiple supports "act" in directions orthogonal to each other.

Note, that from the quasi-orthogonality of C_1 it follows that λ is uniquely determined by (s,S) and depends continuously on them [23]:

$$
\lambda = C_1(\omega)\Big(f(\omega) - \sum_{i\in\mathcal{I}(x)} B_i^T(\omega)s_i - \sum_{j\in\mathcal{J}(X)} S_j\gamma_j(\omega)\Big). \tag{1.1}
$$

These facts will be often used without backward reference.

Bilevel structural topology optimization problems, where the lower-level problem is $(\mathcal{C})_{(x,X)}(\omega)$, were extensively studied in [23]. We cite three important results from this paper for the reader's convenience and later use.

Proposition 1.1. *Fix* $\omega \in \Omega$.

(i) *[23, Theorem 2.1] Suppose the feasible set of the problem* $(\mathcal{C})_{(x,X)}(\omega)$ *is nonempty for some nonnegative design* (x,X)*. Then, there exists a unique optimal solution to the problem* $(\mathcal{C})_{(x,X)}(\omega)$.

(ii) *[23, Theorem 3.1] Let* $\{(x_k,X_k)\}$ *be a nonnegative sequence of designs, converging to* (x,X)*. Suppose, that* $\{(s_k,S_k,\lambda_k)\}$ *is the corresponding sequence of optimal solutions to* $(\mathcal{C})_{(x_k,X_k)}(\omega)$*, and assume that the sequence of energies is bounded, that is, that* $\mathcal{E}(x_k,X_k,s_k,S_k,\lambda_k,\omega) \leq c < \infty$ *for all* k*. Then, there exists a unique optimal solution* (s,S,λ) *to* $(\mathcal{C})_{(x,X)}(\omega)$*, and* $\lim_{k\to\infty}(s_k,S_k,\lambda_k) = (s,S,\lambda)$.

(iii) *[23, Corollary 3.2] Let* (x,X) *be a nonnegative design for which there exists an optimal solution* (s,S,λ) *to the problem* $(\mathcal{C})_{(x,X)}(\omega)$*. Let* $\{(x_k,X_k)\}$ *be a sequence of nonnegative designs which converges to* (x,X)*, and suppose that* $\{(s_k,S_k,\lambda_k)\}$ *is the corresponding sequence of optimal solutions to* $(\mathcal{C})_{(x_k,X_k)}(\omega)$*. Then,* $\lim_{k\to\infty}(s_k,S_k,\lambda_k) = (s,S,\lambda)$.

We note that if the problem $(\mathcal{C})_{(x,X)}(\omega)$ is feasible for some nonnegative design (x,X), then it is feasible for any design $(\widetilde{x},\widetilde{X}) \geq (x,X)$. Furthermore, the feasible set of $(\mathcal{C})_{(\widetilde{x},\widetilde{X})}(\omega)$ includes that of $(\mathcal{C})_{(x,X)}(\omega)$.

At last, we note that the applicability of the problem $(\mathcal{C})_{(x,X)}(\omega)$ is not bounded to truss structures; finite element discretizations of variable thickness sheets in unilateral frictionless contact can be analysed in the same way (cf. Subsection 2.1 and Section 5 in [23]).

1.3 General stochastic minimum compliance problem

We are now ready to state the first problem considered in this paper — the general stochastic minimum compliance problem:

$$(\mathcal{P}_1)\begin{cases} \min\limits_{(x,X,s(\cdot),S(\cdot))} c^f(x,X,s(\cdot),S(\cdot),\lambda(\cdot)) := \int_\Omega \mathcal{E}(x,X,s(\omega),S(\omega),\lambda(\omega),\omega)\,\mathrm{P}(d\omega) \\ \text{s.t.} \begin{cases} \underline{x} \le x \le \overline{x}, & 1_m^T x \le v, \\ \underline{X} \le X \le \overline{X}, & 1_{r_2}^T X \le V, \\ (s(\omega),S(\omega),\lambda(\omega)) \text{ solves } (\mathcal{C})_{(x,X)}(\omega), \text{ P-a.s.,} \end{cases} \end{cases}$$

where v and V are the limits on the amount of cable and structure material correspondingly. In this problem we minimize the *average* value of compliance for multiple load cases.

In topology optimization we set lower bounds $\underline{x} = 0$ and $\underline{X} = 0$.

1.4 Stochastic stress constrained minimum weight problem

The formal problem formulation is as follows:

$$(\mathcal{P}_2)\begin{cases} \min\limits_{(x,X,s(\cdot),S(\cdot))} w(x,X) := \rho_1 1_m^T x + \rho_2 1_{r_2}^T X \\ \text{s.t.} \begin{cases} \underline{x} \le x \le \overline{x}, \\ \underline{X} \le X \le \overline{X}, \\ |s_i(\omega)| \le \overline{\sigma}_1 x_i, & i = 1,\dots,m, & \text{P-a.s.,} \\ L_j S_j(\omega) \le \overline{\sigma}_2 X_j, & j = 1,\dots,r_2, & \text{P-a.s.,} \\ (s(\omega),S(\omega)) \text{ solves } (\mathcal{C})_{(x,X)}(\omega), & & \text{P-a.s.,} \end{cases} \end{cases}$$

where $\overline{\sigma}_1$ and $\overline{\sigma}_2$ are the maximal allowable effective stresses in, and ρ_1 and ρ_2 the densities of the structure and the cable materials, respectively. In this problem we require stress constraints to hold for *almost all* load cases (that is, we allow them to be violated *with probability zero*).

In topology optimization we set lower bounds $\underline{x} = 0$ and $\underline{X} = 0$.

1.5 Outline

The outline of the remaining part of the paper is as follows. In Section 2 the existence of solutions to the problems stated is proved. Section 3 is dedicated to the analysis of the continuity of the optimal solutions with respect to changes in the lower bound on the design variables. The stability of the optimal solutions with respect to small changes in the probability measure is the topic of Section 4. Proofs of some auxiliary results can be found in the Appendix.

2 Existence of solutions

In this section we establish the existence of optimal designs for problems $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ under reasonable assumptions about the underlying mechanical model. The results depend on the closedness of the feasible set, which is typically the main issue when the existence of optimal solutions to MPEC is in question [18, Example 1.1.2].

To prove the existence of optimal designs we need an auxiliary result, which asserts the measurability of solutions to $(\mathcal{C})_{(x,X)}(\omega)$ as functions of ω. In particular, the measurability of the solutions together with the lower semi-continuity of the energy functional imply that we can integrate the energy unless it is "too large".

Proposition 2.1. *Suppose the measurability assumptions stated in Section 1 hold. Suppose further that for almost any ω the feasible set of the problem $(\mathcal{C})_{(x,X)}(\omega)$ is nonempty. Then there exists a unique (up to changes on sets of probability zero) triple of functions $(s(\cdot),S(\cdot),\lambda(\cdot))$ almost everywhere solving the parametric problem $(\mathcal{C})_{(x,X)}(\cdot)$. In addition, these functions are $\mathfrak{S}$-measurable.*

The following result is a generalization of [23, Theorem 3.1 and Corollary 3.2] to a stochastic setting.

Proposition 2.2. *Let a sequence of nonnegative designs* $\{(x_k, X_k)\}$ *converge to* $(\bar{x}, \bar{X})$. *Suppose that* $(s_k(\cdot), S_k(\cdot), \lambda_k(\cdot))$ *solves* $(\mathcal{C})_{(x_k,X_k)}(\cdot)$, *and that the sequence of energy expectations is bounded:*

$$\int_\Omega \mathcal{E}(x_k, X_k, s_k(\omega), S_k(\omega), \lambda_k(\omega), \omega)\, \mathrm{P}(d\omega) \le C < \infty.$$

Then, there exists a solution $(\bar{s}(\cdot), \bar{S}(\cdot), \bar{\lambda}(\cdot))$ *to the problem* $(\mathcal{C})_{(\bar{x},\bar{X})}(\cdot)$, *and* $\{(s_k(\cdot), S_k(\cdot), \lambda_k(\cdot))\}$ *almost sure converges to* $(\bar{s}(\cdot), \bar{S}(\cdot), \bar{\lambda}(\cdot))$.

Theorems 2.3 and 2.4 show the existence of optimal solutions to problems $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$.

Theorem 2.3. (Existence of solutions to $(\mathcal{P}_1)$) *Suppose that for some feasible point* $(x_0, X_0, s(\cdot), S(\cdot))$ *in the problem* $(\mathcal{P}_1)$ *we have* $c^f(x, X, s(\cdot), S(\cdot), \lambda(\cdot)) < \infty$. *Then, there exists at least one optimal solution to* $(\mathcal{P}_1)$.

Proof: Consider an arbitrary minimizing sequence $\{(x_k, X_k, s_k(\cdot), S_k(\cdot))\}$ for the problem $(\mathcal{P}_1)$ together with the corresponding sequence of contact forces $\{\lambda_k(\cdot)\}$. Since the the feasible design space is bounded, without any loss of generality we may assume that the sequence $\{(x_k, X_k)\}$ converges to a limit (x^*, X^*) satisfying the design constraints. Proposition 2.2 for such a sequence implies that the sequence of state variables $\{(s_k(\cdot), S_k(\cdot), \lambda_k(\cdot))\}$ almost sure converges to a limit $(s^*(\cdot), S^*(\cdot), \lambda^*(\cdot))$ solving $(\mathcal{C})_{(x^*,X^*)}(\cdot)$, thus establishing the feasibility of the limit in $(\mathcal{P}_1)$. Furthermore, owing to the l.s.c. property of the energy functional for each ω and Fatou's Lemma, the following inequality holds:

$$0 \le c^f(x^*, X^*, s^*(\cdot), S^*(\cdot), \lambda^*(\cdot)) \le \liminf_{k\to\infty} c^f(x_k, X_k, s_k(\cdot), S_k(\cdot), \lambda_k(\cdot)),$$

whence $(x^*, X^*, s^*(\cdot), S^*(\cdot))$ is an optimal solution to $(\mathcal{P}_1)$. $\square$

The next theorem generalizes Proposition 3.1 in [11], which asserts the existence of solutions to the somewhat simplified version of the problem $(\mathcal{P}_2)$.

Theorem 2.4. (Existence of solutions to $(\mathcal{P}_2)$) *Suppose that the following assumptions are satisfied:*

(*i*) *the feasible set of the problem* $(\mathcal{P}_2)$ *is nonempty;*

(*ii*) $\mathrm{P}(E(\cdot) \ge c) = \mathrm{P}(E_c(\cdot) \ge c) = 1$ *for some constant* $c > 0$;

(*iii*) *the functions* $L_j(\cdot)$, $g_1(\cdot)$, $g_2(\cdot)$, $C_1(\cdot)$, $B_i(\cdot)$ *and* $f(\cdot)$ *are essentially bounded.*

Then, there exists at least one optimal solution to the problem $(\mathcal{P}_2)$.

Proof: Assumption (*iii*) together with the stress constraints and equation (1.1) implies the essential upper-boundedness of the term $g_1^T(\omega)\lambda(\omega)$ on the feasible set by some constant $C < \infty$. Following the proof of Theorem 4.3 in [23] we can show the existence of an upper bound on the energy for some strictly positive design $(\widehat{x}, \widehat{X}) \le (\overline{x}, \overline{X})$:

$$\begin{aligned}
\mathcal{E}(x, X, s(\omega), S(\omega), \lambda(\omega), \omega) &\le \frac{1}{2}\sum_{i=1}^m \frac{\widehat{x}_i(\overline{\sigma}_1)^2}{E(\omega)} \\
&\quad + \sum_{j=1}^{r_2} \max_{\underline{X}_j \le X_j \le \widehat{X}_j} \Big\{0, \Big(\frac{X_j(L_j(\omega)\overline{\sigma}_2)^2}{2E_c(\omega)} + (g_2(\omega))_j \overline{\sigma}_2 X_j\Big)\Big\} + C \\
&=: v(\omega) + C, \qquad \text{P-a.s.},
\end{aligned}$$

such that no optimal design can violate it. Assumptions (*ii*) and (*iii*) imply that $\mathrm{P}(v(\cdot) < \infty) = 1$. We then add a redundant constraint $\mathcal{E}(x, X, s(\omega), S(\omega), \lambda(\omega), \omega) \le v(\omega) + C$ to our problem and use Proposition 1.1(*ii*) to obtain the closedness of the feasible set for almost any ω. Now Theorem 3.1 in [11] asserting the existence of solutions to SMPEC can be applied, and we are done. $\square$

Remark 2.5. Assumption (*ii*) does not allow the cable and structure material to break with a positive probability, because in this case we usually cannot expect the existence of a mechanical equilibrium with probability 1. Assumption (*iii*) is satisfied by most mechanical models; the only questionable assumption is the boundedness of the loads $f(\cdot)$. Our interpretation of this assumption is that since we work in the framework of linear elasticity we cannot consider unbounded loads.

3 Convergence of ε-perturbations

The so-called ε-perturbation of structural topology optimization problems, or approximation with a sequence of *sizing* optimization problems, has become a classic topic. Convergence results of this type allow, at least in principle, to compute the optimal solutions to structural topology optimization problems by solving a sequence of smooth approximating problems. Such approximations do not suffer from many numerical difficulties possessed by $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$, so that efficient solvers are readily available.

For compliance minimization problems a naïve replacement of the lower design bounds $(\underline{x}, \underline{X}) = 0$ with a small positive value $\varepsilon > 0$ tending to zero (whence the name — ε-perturbation) is sufficient. Theorem 3.1 below is an extension of the corresponding result for discrete probability measures (Theorem 4.2 in [23]).

The situation with the stress constrained weight minimization is far more complicated. Sved and Ginos [32] observed that the problem may have singular solutions, which cannot be approximated by the simplistic approach outlined above. The properties of the feasible region were further investigated by Kirsch [16], Cheng and Jiang [7], and Rozvany and Birker [28]. Cheng and Guo [6] proposed a more sophisticated relaxation procedure, where not only lower bounds but also stress constraints were perturbed. They showed the convergence of optimal values of perturbed problems to the optimal value of the original problem, while Petersson [24] showed the convergence of optimal solutions. The ε-relaxation was extended to continuum structures by Duysinx and Bendsøe [9] and Duysinx and Sigmund [10]. Patriksson and Petersson [23] generalized the result for stochastic truss topology optimization problems including unilateral constraints and discrete probability measures. Theorem 3.4 below extends the result of Petersson [24] to the general stochastic setting.

Stolpe and Svanberg [31] demonstrated that singular topologies can occur in multi-load cases even if all other parameters (material properties, stress limits) are kept uniform among the structural members. This implies that in our case singular topologies are quite likely to occur.

3.1 ε-perturbation of $(\mathcal{P}_1)$

Consider the following ε-perturbation of the problem $(\mathcal{P}_1)$:

$$(\mathcal{P}_1^\varepsilon)\begin{cases} \min\limits_{(x,X,s(\cdot),S(\cdot))} c^f(x,X,s(\cdot),S(\cdot),\lambda(\cdot)) \\ \text{s.t.} \begin{cases} \varepsilon 1_m \le x \le \overline{x}, & 1_m^T x \le v, \\ \varepsilon 1_{r_2} \le X \le \overline{X}, & 1_{r_2}^T X \le V, \\ (s(\omega),S(\omega),\lambda(\omega)) \text{ solves } (\mathcal{C})_{(x,X)}(\omega), \text{ P-a.s.} \end{cases} \end{cases}$$

Theorem 3.1. *Suppose that for some $\varepsilon_0 > 0$ there is a solution $(x_0, X_0, s_0(\cdot), S_0(\cdot), \lambda_0(\cdot))$ that is feasible in $(\mathcal{P}_1)$ with $(x_0, X_0) \ge \varepsilon_0 1_{m+r_2}$ and $c^f(x_0, X_0, s_0(\cdot), S_0(\cdot), \lambda_0(\cdot)) < \infty$. For each $\varepsilon_0 \ge \varepsilon > 0$, let $(x_\varepsilon^*, X_\varepsilon^*, s_\varepsilon^*(\cdot), S_\varepsilon^*(\cdot), \lambda_\varepsilon^*(\cdot))$ denote an arbitrary optimal solution to $(\mathcal{P}_1^\varepsilon)$. Then, any limit point of the sequence $\{(x_\varepsilon^*, X_\varepsilon^*, s_\varepsilon^*(\cdot), S_\varepsilon^*(\cdot), \lambda_\varepsilon^*(\cdot))\}$ (and there is at least one) is an optimal solution to $(\mathcal{P}_1^0) = (\mathcal{P}_1)$.*

Proof: According to Theorem 2.3 a solution to $(\mathcal{P}_1^\varepsilon)$ exists for each $\varepsilon_0 \ge \varepsilon \ge 0$. The sequence $\{(x_\varepsilon^*, X_\varepsilon^*, s_\varepsilon^*(\cdot), S_\varepsilon^*(\cdot), \lambda_\varepsilon^*(\cdot))\}$ is feasible to the original problem $(\mathcal{P}_1^0)$. Furthermore, the sequence $\{c^f(x_\varepsilon^*, X_\varepsilon^*, s_\varepsilon^*(\cdot), S_\varepsilon^*(\cdot), \lambda_\varepsilon^*(\cdot))\}$ is non-increasing. Applying Proposition 2.2 we can obtain a feasible solution $(\bar{x}, \bar{X}, \bar{s}(\cdot), \bar{S}(\cdot), \bar{\lambda}(\cdot))$ to $(\mathcal{P}_1)$ such that it is an a.s.-limit of $\{(x_\varepsilon^*, X_\varepsilon^*, s_\varepsilon^*(\cdot), S_\varepsilon^*(\cdot), \lambda_\varepsilon^*(\cdot))\}$.

On the other hand, for any feasible solution $(x, X, s(\cdot), S(\cdot), \lambda(\cdot))$ in $(\mathcal{P}_1)$ with $c^f(x, X, s(\cdot), S(\cdot), \lambda(\cdot)) < \infty$ there is a sequence $\{(x_\varepsilon, X_\varepsilon, s_\varepsilon(\cdot), S_\varepsilon(\cdot), \lambda_\varepsilon(\cdot))\}$ of feasible solutions to $(\mathcal{P}_1^\varepsilon)$ such that $(x_\varepsilon, X_\varepsilon) \to (x, X)$ (cf. Proposition 1.1.2 in [1]). Proposition 1.1(iii) implies that the sequence $\{(s_\varepsilon(\cdot), S_\varepsilon(\cdot), \lambda_\varepsilon(\cdot))\}$ a.s. converges to $(s(\cdot), S(\cdot), \lambda(\cdot))$.

Finally,

$$\begin{aligned} c^f(\bar{x}, \bar{X}, \bar{s}(\cdot), \bar{S}(\cdot), \bar{\lambda}(\cdot)) &\le \liminf_{\varepsilon\to 0} c^f(x_\varepsilon^*, X_\varepsilon^*, s_\varepsilon^*(\cdot), S_\varepsilon^*(\cdot), \lambda_\varepsilon^*(\cdot)) \\ &\le \liminf_{\varepsilon\to 0} c^f(x_\varepsilon, X_\varepsilon, s_\varepsilon(\cdot), S_\varepsilon(\cdot), \lambda_\varepsilon(\cdot)) \\ &\le \lim_{\varepsilon\to 0} c^f(x_\varepsilon, X_\varepsilon, s(\cdot), S(\cdot), \lambda(\cdot)) \\ &= c^f(x, X, s(\cdot), S(\cdot), \lambda(\cdot)), \end{aligned} \tag{3.1}$$

where the inequalities are owing to the l.s.c.-property of $\mathcal{E}$ and Fatou's Lemma, the optimality of $(x_\varepsilon^*, X_\varepsilon^*, s_\varepsilon^*(\cdot), S_\varepsilon^*(\cdot), \lambda_\varepsilon^*(\cdot))$ in $(\mathcal{P}_1^\varepsilon)$, and the optimality of $(s_\varepsilon(\cdot), S_\varepsilon(\cdot), \lambda_\varepsilon(\cdot))$ in $(\mathcal{C})_{(x_\varepsilon,X_\varepsilon)}(\cdot)$, correspondingly. The equality follows from the continuity of c^f with respect to (x, X) (we can move variables (x, X) out of the integrals).

Since the feasible point $(x, X, s(\cdot), S(\cdot))$ was arbitrary, the inequality (3.1) shows the optimality of $(\bar{x}, \bar{X}, \bar{s}(\cdot), \bar{S}(\cdot))$ in $(\mathcal{P}_1^0)$. $\square$

Remark 3.2. For the convergence of optimal values, we note first that the sequence of energies $\{\mathcal{E}(x_\varepsilon^*, X_\varepsilon^*, s_\varepsilon^*(\cdot), S_\varepsilon^*(\cdot), \lambda_\varepsilon(\cdot))\}$ a.s. converges to $\mathcal{E}(\bar{x}, \bar{X}, \bar{s}(\cdot), \bar{S}(\cdot), \bar{\lambda}(\cdot))$, owing to Proposition 1.1, parts *(ii)* and *(iii)*, applied for almost any ω. Thus, to apply the Dominated Convergence Theorem it is only necessary to have an upper bound on the energies. Such an upper bound exists, e.g., in the setting of Section 4.

3.2 ε-perturbation of $(\mathcal{P}_2)$

As we already discussed, the proofs of convergence of ε-perturbations for the problem $(\mathcal{P}_2)$ are much more involved than those for the minimum compliance problem. A typical proof (cf. [24, 23]) makes use of the locally directionally Lipschitz dependence of the state on the design variables, which, in turn, is an implication of particular error bounds for the corresponding lower level problem — the system of linear equations in the case of a truss without unilateral constraints, or the linear complementarity problem in the case of unilaterally constrained truss. In the former case, the famous Hoffman error bound [15] for linear systems is available, and the Lipschitz constant generated by this result in our case can be uniformly estimated with respect to the random outcome ω. In the latter case, the only error bound available is the local one (i.e., one that holds in some small neighbourhood) by Luo and Tseng [19] (see also [18, Theorem 2.3.3]), which can also be viewed as a particular case of the fundamental result of Robinson [25]. Even though the Lipschitz constant can be estimated uniformly with respect to $\omega \in \Omega$ in this situation as well, the neighbourhood in which the error bound is guaranteed to hold may vanish as ω changes. Therefore, this error bound cannot be applied in the case of stochastic optimization with a probability measure having infinite support, as opposed to the discrete case.

Thus, in this subsection we restrict ourselves to a very important special case of a truss without unilateral constraints under stochastic loading, which allows us to formulate the mechanical equilibrium condition as a system of linear equations, parametrized by x. To this end, we first define B as the $m \times n$ matrix created by stacking the matrices B_i on top of each other, and, $D(x)$ as the $m \times m$ diagonal matrix with elements $x_i E$ on a diagonal. Then $s(\omega)$ is a state vector corresponding to a nonnegative design x under loading $f(\omega)$ if and only if for some vector $u(\omega)$ of Lagrange multipliers (nodal displacements) the following system is satisfied [24]:

$$(\mathcal{Q})_x(\omega) \quad \begin{pmatrix} 0 & B^T \\ D(x)B & -I \end{pmatrix} \begin{pmatrix} u(\omega) \\ s(\omega) \end{pmatrix} = \begin{pmatrix} -f(\omega) \\ 0 \end{pmatrix}.$$

Consider now the following ε-perturbation of the problem $(\mathcal{P}_2)$:

$$(\mathcal{P}_2^\varepsilon) \begin{cases} \min\limits_{(x,s(\cdot))} \; w(x) \\ \text{s.t.} \begin{cases} o(\varepsilon)1_m \le x \le \overline{x} + o(\varepsilon)1_m, \\ |s_i(\omega)| \le \overline{\sigma}_1 x_i + \varepsilon, \qquad i = 1, \ldots, m, \quad \text{P-a.s.}, \\ s(\omega) \text{ solves } (\mathcal{C})_x(\omega), \qquad \text{P-a.s.}, \end{cases} \end{cases}$$

where from the function $o : \mathbb{R}_{++} \to \mathbb{R}_{++}$ we only require the properties that $\{o(\varepsilon)/\varepsilon\}$ converges to zero while $\{o(\varepsilon)/\varepsilon^2\}$ is bounded away from zero (e.g., $o(\varepsilon) = \varepsilon^2$ satisfies these requirements).

Before stating the theorem we need a lemma, which asserts the directionally Lipschitz dependence of state variables $s(\omega)$ on design x uniformly in ω.

Lemma 3.3. *Consider a truss without unilateral constraints under essentially bounded stochastic loading $f(\cdot)$. Let $x \ge 0$ be a design for which the problem $(\mathcal{C})_x(\omega)$ has a solution $s(\omega)$ for almost any ω. Let $\Psi > 0$ be arbitrary in $\mathbb{R}^m$ and for $\varepsilon > 0$ set*

$$x_\varepsilon = x + \varepsilon\Psi.$$

Denote by $s_\varepsilon(\omega)$ the corresponding optimal solution to $(\mathcal{C})_{x_\varepsilon}(\omega)$. Then, for some positive constant τ and almost any ω, the inequality

$$\|s_\varepsilon(\omega) - s(\omega)\| \le \tau\varepsilon \tag{3.2}$$

holds for all $\varepsilon > 0$.

Theorem 3.4. *In addition to the the assumptions of Lemma 3.3, suppose that for some* $\varepsilon_0 > 0$ *there is a solution* $(x_0, s_0(\cdot))$ *that is feasible in* $(\mathcal{P}_2)$ *with* $x_0 \geq o(\varepsilon_0)1_m$.

For each $\varepsilon_0 \geq \varepsilon > 0$ *let* $(x^*_\varepsilon, s^*_\varepsilon(\cdot))$ *denote an arbitrary optimal solution to* $(\mathcal{P}^\varepsilon_2)$. *Then, any limit point of the sequence* $\{x^*_\varepsilon\}$ *(and there is at least one) is an optimal solution to* $(\mathcal{P}^0_2) = (\mathcal{P}_2)$. *The sequence of optimal values* $\{w(x^*_\varepsilon, X^*_\varepsilon)\}$ *converges to the optimal value of* $(\mathcal{P}^0_2)$.

Furthermore, for any converging subsequence of the design variables $\{x^*_{\varepsilon_k}\} \to x^*$ *the corresponding sequence of state variables* $\{s^*_{\varepsilon_k}(\cdot)\}$ *a.s. converges to the optimal state* $s^*(\cdot)$, *corresponding to* x^*.

Proof: Using Theorem 2.4 we can verify the existence of optimal solutions to $(\mathcal{P}^\varepsilon_2)$ for each $\varepsilon_0 \geq \varepsilon \geq 0$. Now one can follow the proof of Theorem 4.4 in [23] to conclude that the sequence $\{(x^*_\varepsilon, s^*_\varepsilon(\cdot))\}$ as well as the sequence of energies $\{\mathcal{E}(x^*_\varepsilon, s^*_\varepsilon(\cdot))\}$ is essentially bounded. Thus, there exists a limit point x^* of the design variables. Then, for almost any ω, there is a subsequence $\{\varepsilon_{k(\omega)}\}$ such that $\lim_{k(\omega)\to\infty}(x^*_{\varepsilon_{k(\omega)}}, s^*_{\varepsilon_{k(\omega)}}(\omega)) = (x^*, s^*(\omega))$. Proposition 1.1 (ii) implies that the limit is the solution to the problem $(\mathcal{C})_{x^*}(\omega)$. The continuity of the other constraints implies that $(x^*, s^*(\cdot))$ is feasible in $(\mathcal{P}^0_2)$.

Let $(x, s(\cdot))$ be an arbitrary feasible solution to $(\mathcal{P}_2)$, set $x_\varepsilon = x + o(\varepsilon)1_m$ and let $s_\varepsilon(\cdot)$ solve $(\mathcal{C})_{x_\varepsilon}(\cdot)$. Owing to Lemma 3.3, the following estimation holds:

$$|s_\varepsilon(\cdot)| \leq \tau o(\varepsilon) + |s(\cdot)| \leq \tau o(\varepsilon) + \overline{\sigma}_1[x + (o(\varepsilon) - o(\varepsilon))1_m] = (\tau - \overline{\sigma}_1)o(\varepsilon) + \overline{\sigma}_1 x_\varepsilon \leq \overline{\sigma}_1 x_\varepsilon + \varepsilon,$$

for all ε small enough, where we have used the assumption that $\{o(\varepsilon)/\varepsilon\} \to 0$. Since clearly x_ε satisfies the design constraints, $(x_\varepsilon, s_\varepsilon(\cdot))$ is feasible in $(\mathcal{P}^\varepsilon_2)$. Hence, $w(x^*_\varepsilon) \leq w(x_\varepsilon)$. Letting ε tend to zero in this inequality, we obtain that $w(x^*) \leq w(x)$, whence we may conclude that $(x^*, s^*(\cdot))$ solves $(\mathcal{P}_2)$.

The continuity of the objective function implies the convergence of the optimal values. The boundedness of energies at the optimal solution clarifies the usage of Proposition 1.1 (iii), hence the convergence of state variables also follows. □

We illustrate Theorem 3.4 with a small numerical example.

Example 3.5. (4-bar truss) Consider the problem of minimizing the weight of the 4-bar structure shown in Figure 1. The stress limit for each bar is $\overline{\sigma} = 1$, and the Young's modulus is $E = 1$. Assume that the upper

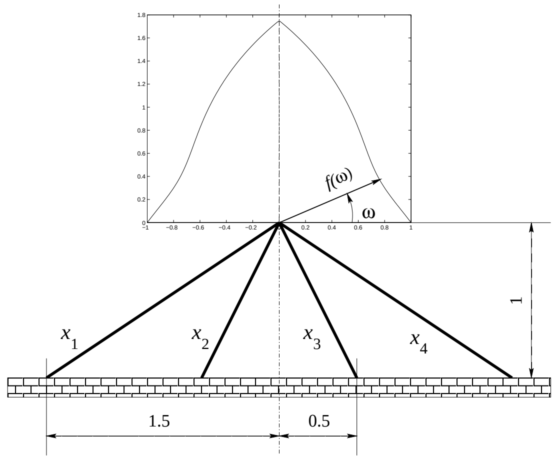

Figure 1: The 4-bar truss problem.

design bounds are inactive, and that the stochastic force is distributed as follows: the direction equals to ω (where $\omega \in [0, \pi]$); the magnitude is defined for $0 \leq \omega \leq \pi/2$ (and is defined by symmetry for $\pi/2 \leq \omega \leq \pi$) as a parabola passing through the points $(0, 1.0)$, $(\pi/6, 0.85)$, $(\pi/2, 1.75)$. The probability measure is the uniform one on $[0, \pi]$ (it is easy to see that only the support of the probability measure is necessary to define stress constrained weight minimization; see also remarks at the end of Section 4). Since the initial structural topology as well as the loading conditions are symmetric, we can expect symmetric optimal solutions (i.e., $x^*_1 = x^*_4$, $x^*_2 = x^*_3$). Figure 2 shows the projection of the subset of symmetric feasible designs (i.e., such that $x_1 = x_4$ and $x_2 = x_3$) onto the first two coordinates. Note that the feasible set is not *a finite union of polyhedra*, because we work with an infinite number of load cases (compare with the similar Problem 1 in [31]). Despite the large number of load cases, at the globally optimal solution, $x^* = (0.0, 1.294, 1.294, 0.0)$, the structural topology was modified (i.e., bars 1 and 4 were removed).

There are three local minima, two of which (including the globally optimal solution) are singular. The non-singular non-global, local minimum $x_l^* = (0.202, 1.173, 1.173, 0.202)$ of the original problem is the global minimum for the "naïvely" perturbed problem for all small but positive values of ε. Therefore, we cannot approximate the globally optimal solution by the "naïve" ε-perturbation.

The "correct" ε-perturbation scheme allows us to recover the global optimal solution. Figure 3 shows the convergence of the optimal solutions to the ε-perturbed problems to the solution of the original problem, as ε decreases to zero (variables x_3 and x_4 are not shown, owing to the symmetry of the optimal solutions). We have used the nested formulation (with eliminated state variables) and a finite difference approximation of the derivatives to solve the problem using an SQP algorithm.

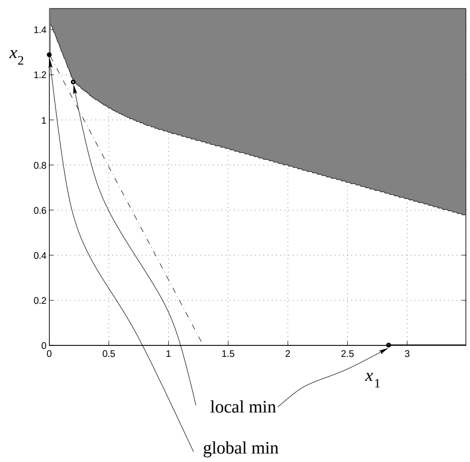

Figure 2: The feasible design domain of the 4-bar truss problem.

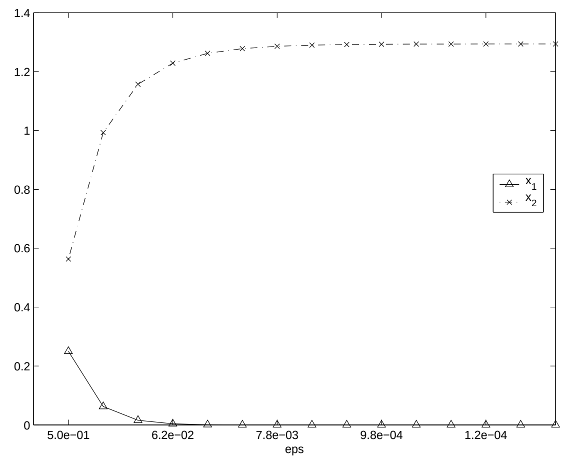

Figure 3: Convergence of the ε-perturbations for the 4-bar truss problem.

4 Distribution sensitivity

The analysis of stability of optimal solutions with respect to small changes in the probability measure is of great importance. From the computational point of view, it allows one to replace the original stochastic problem by a sequence of simpler problems, involving approximations (discretizations) of the probability measure. From the practical point of view, it asserts that solutions to the problem obtained using statistical estimations of the unknown stochastic distribution are "close" to exact solutions. From the theoretical point of view, it confirms the robustness of the probabilistic approach with respect to possible errors in the probability distribution.

Throughout this section we assume that Ω is a compact metric space, $\mathfrak{S} = \mathcal{B}(\Omega)$ and the only sources of uncertainty are the loads $f(\cdot)$, gaps $g_1(\cdot)$ and slacks $g_2(\cdot)$, which in addition are assumed to be continuous functions. Such a restriction is made in order to achieve the continuity of the energy with respect to uncertainty parameter ω, which is absolutely necessary, since we use energy integrals as the objective function in the compliance minimization problem, while we allow the underlying probability measure to change. The specific choice of the sources of uncertainty becomes clear if we recall that parameters (f, g_1, g_2) enter "linearly" into the state problem $(\mathcal{C})_{(x,X)}(\omega)$ (i.e., into the linear part of the objective function and into the right-hand sides of the constraints), so that the state variables as well as the energy depend continuously on them. On the other hand, our assumptions seem to be not very restrictive, since the main source of uncertainty in most problems are loads.

Even though we do not necessarily work with a complete probability space in such a setting, under additional assumptions about the feasibility of the lower-level problem it is possible to omit the adverb "almost" from the discussion.

Proposition 4.1. *Given a nonnegative design (x, X), suppose that the problem $(\mathcal{C})_{(x,X)}(\omega)$ is feasible for any ω. Then, the solution $(s(\omega), S(\omega), \lambda(\omega))$ exists, is unique and continuous, and the optimal value function $\mathcal{E}(x, X, s(\omega), S(\omega), \lambda(\omega), \omega)$ is continuous.*

Proof: Both existence and uniqueness were announced in Proposition 2.1. Continuity then follows from [2, Theorem 5.5.1]. □

In the remainder of the section we assume that the assumptions of Proposition 4.1 hold for any positive design (x, X). This assumption can easily be satisfied by choosing a "rich enough" ground structure, which can sustain loads for any ω.

Proposition 4.2. *Let the sequence $\{(x_k, X_k)\}$ of positive designs converge to a nonnegative design $(\bar{x}, \bar{X})$. Let $(s_k(\cdot), S_k(\cdot), \lambda_k(\cdot))$ be the continuous solution to $(\mathcal{C})_{(x_k,X_k)}(\cdot)$. Suppose that the sequence of energy estimates $\{\int_\Omega \mathcal{E}(x_k, X_k, s_k(\omega), S_k(\omega), \lambda_k(\omega), \omega)\,\mathrm{P}(d\omega)\}$ is bounded. Then the sequence $\{(s_k(\cdot), S_k(\cdot), \lambda_k(\cdot))\}$ converges on a set of measure one to a limit $(\bar{s}(\cdot), \bar{S}(\cdot), \bar{\lambda}(\cdot))$, which is the solution to the problem $(\mathcal{C})_{(\bar{x},\bar{X})}(\cdot)$. Furthermore, the sequence of optimal values $\{\mathcal{E}(x_k, X_k, s_k(\cdot), S_k(\cdot), \lambda_k(\cdot), \cdot)\}$ uniformly converges on a set of measure one to $\mathcal{E}(\bar{x}, \bar{X}, \bar{s}(\cdot), \bar{S}(\cdot), \bar{\lambda}(\cdot), \cdot)$.*

4.1 Stability of solutions to $(\mathcal{P}_1)$

Before proceeding with analysing the stability of solutions to the stochastic compliance minimization problem with respect to small changes in the probability measure, we note that in the settings of this section the problem $(\mathcal{P}_1)$ has a feasible solution $(x_0, X_0, s_0(\cdot), S_0(\cdot))$ such that $(x_0, X_0) > 0$, provided the bounds on the available material (v, V) are positive. We denote the optimal value of a problem $(\mathcal{P})$ by $\mathrm{val}(\mathcal{P})$.

Proposition 4.3. *The following equalities hold:*

$$\mathrm{val}(\mathcal{P}_1) = \inf_{\varepsilon>0} \mathrm{val}(\mathcal{P}_1^\varepsilon) = \lim_{\varepsilon\to 0} \mathrm{val}(\mathcal{P}_1^\varepsilon).$$

Proof: Follows immediately from Theorem 3.1 and Proposition 4.2. □

Consider a sequence of probability measures $\{\mathrm{P}_k\}$ defined on $\mathcal{B}(\Omega)$, together with a sequence of optimization problems:

$$(\mathcal{P}_1)^k \begin{cases} \min\limits_{(x,X,s(\cdot),S(\cdot))} c_k^f(x, X, s(\cdot), S(\cdot)) := \displaystyle\int_\Omega \mathcal{E}(x, X, s(\omega), S(\omega), \lambda(\omega), \omega)\,\mathrm{P}_k(d\omega) \\ \text{s.t.} \begin{cases} \underline{x} \le x \le \overline{x}, & 1_m^T x \le v, \\ \underline{X} \le X \le \overline{X}, & 1_{r_2}^T X \le V, \\ (s(\omega), S(\omega), \lambda(\omega)) \text{ solves } (\mathcal{C})_{(x,X)}(\omega), \ \mathrm{P}_k\text{-a.s.} \end{cases} \end{cases}$$

Lemma 4.4. *Suppose that the sequence of probability measures $\{\mathrm{P}_k\}$ weakly converges to* P. *Then* $\mathrm{val}(\mathcal{P}_1) \ge \limsup_{k\to\infty} \mathrm{val}(\mathcal{P}_1)^k$.

Proof: Consider an arbitrary, sufficiently small $\varepsilon > 0$ such that there is an optimal solution $(x_\varepsilon, X_\varepsilon, s_\varepsilon(\cdot), S_\varepsilon(\cdot))$ to the problem $(\mathcal{P}_1^\varepsilon)$. Owing to Proposition 4.1 this point is feasible for all problems $(\mathcal{P}_1)^k$, so we get:

$$\mathrm{val}(\mathcal{P}_1^\varepsilon) = c^f(x_\varepsilon, X_\varepsilon, s_\varepsilon(\cdot), S_\varepsilon(\cdot), \lambda_\varepsilon(\cdot)) = \lim_{k\to\infty} c_k^f(x_\varepsilon, X_\varepsilon, s_\varepsilon(\cdot), S_\varepsilon(\cdot), \lambda_\varepsilon(\cdot)) \ge \limsup_{k\to\infty} \mathrm{val}(\mathcal{P}_1)^k,$$

and, owing to Proposition 4.3:

$$\mathrm{val}(\mathcal{P}_1) = \inf_{\varepsilon>0} \mathrm{val}(\mathcal{P}_1^\varepsilon) \geq \limsup_{k\to\infty} \mathrm{val}(\mathcal{P}_1)^k. \quad \square$$

To prove the reverse inequality, we assume additional regularity properties on the sequence $\{\mathrm{P}_k\}$. Namely, we suppose that each measure P_k has a density $p_k(\cdot)$ with respect to a Lebesgue measure on Ω and that the sequence $\{p_k(\cdot)\}$ converges to a density $p(\cdot)$ of P Lebesgue-almost everywhere. The existence of densities is not a very restrictive assumption from the theoretical point of view, and it is usually assumed in engineering applications of probability theory (for just a few examples, see [17, 30, 33, 20]).

Theorem 4.5. *Let $\{(x_k, X_k, s_k(\cdot), S_k(\cdot))\}$ be a sequence of solutions to $\{(\mathcal{P}_1)^k\}$. Then any limit point (and there is at least one) of the sequence $\{(x_k, X_k, s_k(\cdot), S_k(\cdot))\}$ is a solution to the limiting problem $(\mathcal{P}_1)$.*

Proof: The sequence of design variables $\{(x_k, X_k)\}$ is bounded and has a limit point (x_0, X_0). Thus we may assume that the original sequence has converging design components.

Fatou's Lemma and Lemma 4.4 imply:

$$\begin{aligned}
&\int_\Omega \liminf_{k\to\infty} \mathcal{E}(x_k, X_k, s_k(\omega), S_k(\omega), \lambda_k(\omega), \omega)p(\omega)\, d\omega \\
\leq &\int_\Omega \liminf_{k\to\infty} [\mathcal{E}(x_k, X_k, s_k(\omega), S_k(\omega), \lambda_k(\omega), \omega)p_k(\omega)]\, d\omega \\
\leq &\liminf_{k\to\infty} \int_\Omega \mathcal{E}(x_k, X_k, s_k(\omega), S_k(\omega), \lambda_k(\omega), \omega)p_k(\omega)\, d\omega \leq \mathrm{val}(\mathcal{P}_1) < \infty.
\end{aligned}$$

Thus we see that the P-probability of the set $\Omega_f = \{\, \omega \in \Omega \mid \liminf_{k\to\infty} \mathcal{E}(x_k, X_k, s_k(\omega), S_k(\omega), \lambda_k(\omega), \omega) < \infty \,\}$ is one. Using Proposition 1.1 (*iii*) we can verify the existence of a limiting state $(s_0(\cdot), S_0(\cdot), \lambda_0(\cdot))$ corresponding to the design (x_0, X_0), and the P-a.s. convergence of $(s_k(\cdot), S_k(\cdot), \lambda_k(\cdot))$ to this state. Using the lower semicontinuity of $\mathcal{E}$, this implies:

$$\begin{aligned}
\mathrm{val}(\mathcal{P}_1) &\leq \int_\Omega \mathcal{E}(x_0, X_0, s_0(\omega), S_0(\omega), \lambda_0(\omega), \omega)p(\omega)\, d\omega \\
&\leq \int_\Omega \liminf_{k\to\infty} \mathcal{E}(x_k, X_k, s_k(\omega), S_k(\omega), \lambda_k(\omega), \omega)p(\omega)\, d\omega \\
&\leq \liminf_{k\to\infty} \int_\Omega \mathcal{E}(x_k, X_k, s_k(\omega), S_k(\omega), \lambda_k(\omega), \omega)p_k(\omega)\, d\omega = \liminf_{k\to\infty} \mathrm{val}(\mathcal{P}_1)^k.
\end{aligned}$$

Together with estimation of $\limsup_{k\to\infty} \mathrm{val}(\mathcal{P}_1)^k$ given by Lemma 4.4 this finishes the proof. $\square$

4.2 Example: Instability of solutions to $(\mathcal{P}_2)$

In the proof of continuity of solutions to the general stochastic compliance minimization problem with respect to small changes in the probability measure, we have used three properties of the problem:

- the boundedness of the objective function implies the convergence of the state variables;
- the optimal value of the problems $(\mathcal{P}_1^\varepsilon)$ monotonically decreases as ε goes to zero;
- any positive design defines a solution, which is feasible for all probability measures.

Unfortunately, the stress constrained weight minimization problem possesses neither of these properties. The following numerical example shows that optimal solutions to the stochastic weight constrained minimization problem are in general not continuous with respect to changes in the probability measure.

Example 4.6. (One bar with a cable) Figure 4 shows a simple one-dimensional structure, introduced and analysed in [23], that consists of a bar suspended with one cable. Suppose that $\Omega = [-1, 2]$, P is the uniform distribution on $[-1/2, 1]$, $f(\omega) = \omega$, $E = E_c = 1$, $\rho_1 = \rho_2 = 1$, $\overline{x} = 1$ and $\overline{X} = 2$. After eliminating the state

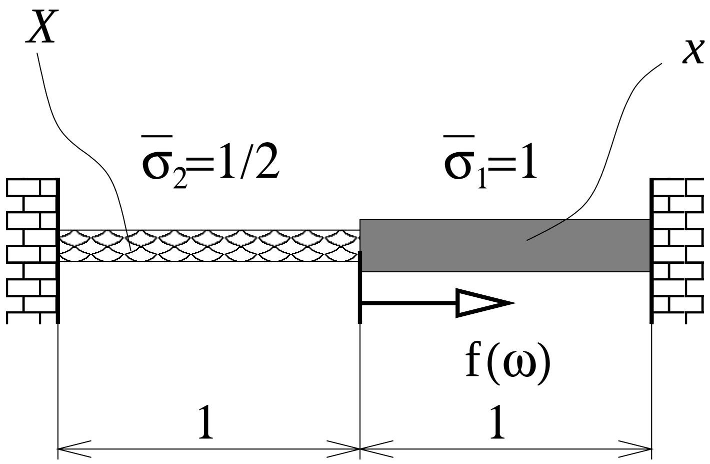

Figure 4: The cable suspended one-bar truss.

variables one obtains the following optimization problem:

$$\begin{cases} \min\limits_{(x,X)} x + X \\ \text{s.t.} \begin{cases} 0 \leq x \leq 1, & \\ 0 \leq X \leq 2, & \\ \dfrac{x\omega}{x+X} \leq x, & \text{P-a.s.,} \\ -\omega \leq x, & \text{P-a.s.,} \\ 0 \leq \dfrac{X\omega}{x+X} \leq X/2, & \text{P-a.s.} \end{cases} \end{cases}$$

Figure 5 shows the feasible domain for the design variables (x, X). Note that this domain consists of the union

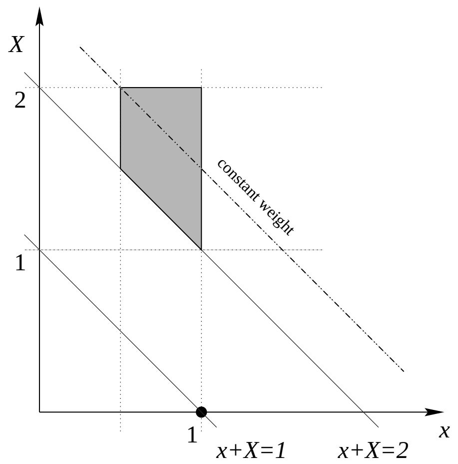

Figure 5: The admissible design domain. The optimal solution is at the black circle.

of a two-dimensional, convex domain and the isolated optimal point $(x^*, X^*) = (1, 0)$ with corresponding optimal weight $w^* = 1$.

Let P_k be a uniform distribution on $[-1/2, 1] \cup [2 - 1/k, 2]$. We note that the sequence $\{\mathrm{P}_k\}$ weakly converges to P and all the measures possess densities. On the other hand, for any vector (x_k, X_k) feasible to $(\mathcal{P})^k$ the inequality $X_k \geq 1$ holds, owing to the fact that $\mathrm{P}_k(f(\omega) > 1) > 0$. Thus the sequence of optimal solutions to $(\mathcal{P}_2)^k$ cannot converge to the optimal solution $(x^*, X^*) = (1, 0)$ of $(\mathcal{P}_2)$.

A few remarks are in order. The stress-constrained weight minimization problem depends only on the support of the probability measure, not on the measure itself. The assumption of fixed compact support of the approximating measures (the only condition implying the stability of the problem) is however extremely restrictive from both the theoretical and practical points of view. Since the state of the truss is uniquely determined by the design variables, we can eliminate the former variables from the problem (i.e., transform the problem into the *nested* form). By doing that we obtain a problem having only a finite number of variables and at the same time an infinite number of stress constraints, indexed by $\omega \in \Omega$. Therefore, the original stochastic problem can be considered as a semi-infinite (non-stochastic) problem, which solves all the problems with instabilities w.r.t. the underlying probability measure by simply eliminating the possibility of such a sensitivity analysis. However, the presence of a probabilistic structure could be exploited algorithmically. For example, when constructing penalty functions for the problem, one can impose larger penalties on more probable violations of the constraints [13].

5 Concluding remarks and further research

In Section 4 we have shown that the introduction of uncertainty into structural topology optimization problems does not necessarily lead to robust solutions, if one understands robustness as insensitivity to modelling errors. It is possible to prove the robustness of the optimal solutions to stochastic compliance minimization under regularity assumptions on the approximating probability measures. The main reason for the instability of the optimal solutions to stochastic stress constrained weight minimization is that stress constraints are imposed in a too restrictive way — we require them to hold with probability one. The alternative is to allow small *average* violations of the stress constraints (which, however, may result in large violations with very small probabilities); cf. [12].

Fredricson et al. [14] studied a problem, which is similar to the one studied in the present paper, namely the topology optimization of a discrete structure to achieve minimal compliance under multiple loading conditions with applications to the construction of a vehicle body. They consider much more general mechanical models — frames including flexible joints, but it seems possible to generalize our results for this class of structures. Therefore, the theoretical results obtained in this paper can be applied to the analysis and construction of algorithms for the optimization problems arising in the vehicle industry, where the decrease of a vehicle weight to reduce fuel consumption has a great importance, while uncertainty in the loading conditions enters the problem quite naturally (as in almost any practical problem).

One uncovered question in this paper is the possible construction of efficient numerical methods for stochastic topology optimization problems. The current research topics in this area include:

- the approximation of stochastic structural topology optimization problems with simpler finite-dimensional problems with discrete measures;
- alternative ε-perturbation approaches, removing the stress constraints from the original problem.

Some of the developments in this direction are discussed in [13]. This would allow us to apply existing algorithms for nonsmooth optimization (e.g. BT algorithm [22] or implicit programming algorithm [18]) to stochastic topology optimization problems.

Acknowledgement. The authors would like to thank the editor and an anonymous referee for their valuable comments and suggestions. This research is supported by the Swedish Research Council for Engineering Sciences (grant TFR 98-125).

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A Proofs of the auxiliary results

Proof of Proposition 2.1: Both existence and uniqueness follow from Proposition 1.1(i) applied for almost any ω.

The feasible set of the problem is defined by inequality and equality constraints involving only measurable mappings, whence a point-to-set mapping $\omega \to \{\,\text{feasible set of } (\mathcal{C})_{(x,X)}(\omega)\,\}$ is measurable for fixed (x, X), owing to Theorem 8.2.9 in [1]. Each function s^2/x is l.s.c. with respect to s by [26, p. 83], whence the objective function is measurable. Then we apply Lemma III.39 and its Application in [5] to verify the existence of a $\mathfrak{S}$-measurable solution. Since the solution is unique, we are done. □

Proof of Proposition 2.2: Let $\widetilde{\mathcal{E}}(\omega) := \liminf_{k\to\infty} \mathcal{E}(x_k, X_k, s_k(\omega), S_k(\omega), \lambda_k(\omega), \omega)$, which is non-negative and $\mathfrak{S}$-measurable being a lower limit of functions satisfying these properties (where measurability is owing to Proposition 2.1). Let $k(\omega)$ be a sequence of indices such that $\widetilde{\mathcal{E}}(\omega) = \lim_{k(\omega)\to\infty} \mathcal{E}(x_{k(\omega)}, X_{k(\omega)}, s_{k(\omega)}(\omega), S_{k(\omega)}(\omega), \lambda_{k(\omega)}(\omega), \omega)$.

Using the non-negativity of $\mathcal{E}$, the assumed feasibility of the problem, and Fatou's Lemma, we get:

$$0 \le \int_\Omega \widetilde{\mathcal{E}}(\omega)\,\mathrm{P}(d\omega) \le \liminf_{k\to\infty} \int_\Omega \mathcal{E}(x_k, X_k, s_k(\omega), S_k(\omega), \lambda_k(\omega), \omega)\,\mathrm{P}(d\omega) \le C < \infty.$$

Thus $\mathrm{P}(\widetilde{\mathcal{E}}(\omega) < \infty) = 1$ holds, and for almost all $\omega \in \Omega$ the sequence $\{\mathcal{E}(x_{k(\omega)}, X_{k(\omega)}, s_{k(\omega)}(\omega), S_{k(\omega)}(\omega), \lambda_{k(\omega)}(\omega), \omega)\}$ is bounded. Proposition 1.1(ii) for such ω implies that there exists $(\bar{s}(\omega), \bar{S}(\omega), \bar{\lambda}(\omega))$ solving the problem $(\mathcal{C})_{(\bar{x},\bar{X})}(\omega)$, and such that $(\bar{s}(\omega), \bar{S}(\omega), \bar{\lambda}(\omega)) = \lim_{k(\omega)\to\infty}(s_{k(\omega)}(\omega), S_{k(\omega)}(\omega), \lambda_{k(\omega)}(\omega))$. From Proposition 2.1 we know that $(\bar{s}(\cdot), \bar{S}(\cdot), \bar{\lambda}(\cdot))$ is measurable.

Now we can use the l.s.c. property of $\mathcal{E}$ for fixed ω (Lemma 3.2 in [23]):

$$\begin{aligned} 0 \le \mathcal{E}(\bar{x}, \bar{X}, \bar{s}(\omega), \bar{S}(\omega), \bar{\lambda}(\omega), \omega) &\le \lim_{k(\omega)\to\infty} \mathcal{E}(x_{k(\omega)}, X_{k(\omega)}, s_{k(\omega)}(\omega), \lambda_{k(\omega)}(\omega), S_{k(\omega)}(\omega), \omega) \\ &= \liminf_{k\to\infty} \mathcal{E}(x_k, X_k, s_k(\omega), S_k(\omega), \lambda_k(\omega), \omega), \end{aligned}$$

whence

$$\begin{aligned} 0 &\le \int_\Omega \mathcal{E}(\bar{x}, \bar{X}, \bar{s}(\omega), \bar{S}(\omega), \bar{\lambda}(\omega), \omega)\, \mathrm{P}(d\omega) \\ &\le \liminf_{k\to\infty} \int_\Omega \mathcal{E}(x_k, X_k, s_k(\omega), S_k(\omega), \lambda_k(\omega), \omega)\, \mathrm{P}(d\omega) \le C < \infty. \end{aligned}$$

The latter inequality and Proposition 1.1(iii) imply the almost sure convergence of the sequence $\{(s_k(\cdot), S_k(\cdot), \lambda_k(\cdot))\}$ to $(\bar{s}(\cdot), \bar{S}(\cdot), \bar{\lambda}(\cdot))$. □

Proof of Lemma 3.3: For each $\varepsilon > 0$ and almost each ω the problem $(\mathcal{Q})_{x_\varepsilon}(\omega)$ has a unique solution $(u_\varepsilon(\omega), s_\varepsilon(\omega))$. Owing to the Hoffman error bound [15], the inequality

$$\|s_\varepsilon(\omega) - s(\omega)\| \le \hat{\tau} \left\| \begin{pmatrix} 0 & B^T \\ D(x)B & -I \end{pmatrix} \begin{pmatrix} u_\varepsilon(\omega) \\ s_\varepsilon(\omega) \end{pmatrix} - \begin{pmatrix} f(\omega) \\ 0 \end{pmatrix} \right\| = \hat{\tau} \left\| \begin{pmatrix} 0 \\ \varepsilon D(\Psi) u_\varepsilon(\omega) \end{pmatrix} \right\|,$$

holds. Therefore, to finish the proof it is sufficient to show the uniform essential boundedness of $u_\varepsilon(\cdot)$ for $\varepsilon > 0$.

Owing to Theorem 3.2 in [24] (see also Theorem 3.2 in [23]) the inequality $\|u_\varepsilon(\omega)\|_\Psi \le \|u(\omega)\|_\Psi$ holds for almost each ω, where $u(\omega)$ is an arbitrary Lagrange multiplier for the problem $(\mathcal{C})_x(\omega)$, and $\|\cdot\|_\Psi$ is an elliptic norm associated with the positive definite matrix $\sum_{i=1}^m \Psi_i B_i^T E B_i$.

Let $\Omega_1 \in \mathfrak{S}$ be a set of probability one such that each problem $(\mathcal{C})_x(\omega)$ with $\omega \in \Omega_1$ has an optimal solution and the set $Q := \{\, f(\omega) \mid \omega \in \Omega_1 \,\}$ is bounded. The system $(\mathcal{Q})_x$, describing the KKT conditions for a quadratic problem $(\mathcal{C})_x$, is solvable for each vector f in the closure of the convex hull of the set Q, owing to Theorem 5.5.1 in [2]. Furthermore, holding x fixed, the problem $(\mathcal{C})_x(\cdot)$ satisfies the constant rank constraint qualification (CRCQ), which, in turn, implies the sequentially bounded constraint qualification (SBCQ) (cf. Proposition 1.3.8 in [18]). In particular, we can infer the existence of a constant C, independent of ω, bounding the minimum norm Lagrange multipliers $\bar{u}(\omega)$. This finishes the proof. □

Proof of Proposition 4.2: The pointwise convergence of optimal solutions on a set Ω_1 of measure one holds owing to Proposition 2.2. On the other hand, for each (x_k, X_k) the optimal value function to $(\mathcal{C})_{(x_k,X_k)}(\cdot)$ is a quadratic function of $(f(\cdot), g_1(\cdot), g_2(\cdot))$ owing to Theorem 5.5.2 in [2]. Using a characterization of the solubility set of a parametric quadratic programming problem with parameters in the linear part of the objective function and in the right-hand sides of the constraints (Theorem 5.5.1 in [2]), we can infer the convergence of solutions to $(\mathcal{C})_{(x_k,X_k)}$ towards the solution of $(\mathcal{C})_{(\bar{x},\bar{X})}$ for all (f, g_1, g_2) in some convex bounded polyhedral set, a.s. containing $\{\, (f(\omega), g_1(\omega), g_2(\omega)) \mid \omega \in \Omega_1 \,\}$. The convergence of quadratic functions on a bounded convex polyhedron is uniform, and this finishes the proof. □

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