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A column generation procedure for the side constrained traffic equilibrium problem

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Abstract

We present a column generation procedure for the side constrained traffic equilibrium problem. A dual stabilization scheme is introduced to improve the computational performance. Computational experiments for the case of linear side constraints are presented. The test problems are well known traffic equilibrium instances where side constraints of link flow capacity type and general linear side constraints are added. The computational results are promising especially for instances with a relatively small number of side constraints.

Keywords: Traffic assignment; Column generation; Stabilization techniques

1. Introduction

Our main motivation for studying the side constrained traffic equilibrium model is its usefulness in the context of strategic traffic management. The functionality of a traffic system can often be measured in terms of the traffic flows (or travel times) on the links in the network. By formulating management goals as restrictions on link flows and solving the resulting side constrained traffic equilibrium problem we obtain link flows that satisfy the formulated goals and which are equilibrium flows for a set of adjusted link travel times. The adjustments may be utilized in a strategic

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traffic management procedure for finding appropriate measures and placements thereof in the traffic network.

Another application of the side constrained traffic equilibrium problem is for the refinement of the basic traffic equilibrium model. A traffic equilibrium model where the travel delay is modelled with link separable travel delay functions does not capture complex relationships between flows and travel times well in all situations. This is often the case in operational planning where more detailed analyses are made. One way of refining the model is to extend it to include more complex travel delay functions. This type of extension often results in an asymmetric traffic equilibrium problem. A complementary way of refining the traffic equilibrium model is to introduce side constraints on the flows in the network. The use of the side constrained traffic equilibrium problem for refined traffic modeling is discussed in, for example, Hearn (1980); Ferrari (1995) and Larsson and Patriksson (1999).

We study the side constrained traffic equilibrium problem for the special case where the travel demand is fixed for each origin and destination in the network. Further, the travel delay functions are assumed to be strictly increasing, positive, continuous and separable in the link flows. With these assumptions, and given the link flow solution to the side constrained traffic equilibrium problem, one set of travel time adjustments is easily computed through the use of Lagrange multipliers (e.g., Larsson and Patriksson, 1999). The use of the adjustments of the travel times in the context of traffic management is further discussed in Larsson et al. (2000).

In this paper, a column generation procedure for the side constrained traffic equilibrium problem is presented. The procedure is based on a relaxation of the side constraints, resulting in a nonlinear column generation sub-problem which is a traffic equilibrium problem. The columns generated are weighted together in a *linear* master problem. The basic column generation scheme presented is an application of the scheme given for example in Dantzig (1963, Chapter 24). To increase the stability and to improve the performance of the procedure, a Boxstep restriction (Marsten et al., 1975) is introduced. In Section 2, the side constrained traffic equilibrium is stated. Related work on the side constrained traffic equilibrium problem is reviewed in Section 3. In Section 4, the column generation procedure for the problem is outlined along with a description of a stabilization technique. The section also summarizes the algorithm and establishes its convergence. A initialization heuristic is described in Section 5. In Section 6, implementation details are given together with computational results for three types of linear side constraints.

2. The side constrained traffic equilibrium problem

Let the traffic network be defined by the nodes $\mathscr{N}$ and the directed links $\mathscr{A}$. Assume that network is strongly connected and that the travel demand in the network is known and independent of the travel times and let d_{pq} denote the fixed travel demand from the origin p to the destination q. Let $\mathscr{C}$ be the set of origin–destination pairs (p, q). Let $\mathscr{R}_{pq}$ denote the non-empty set of simple routes in pair (p, q) and let h_r denote the flow on route $r \in \mathscr{R}_{pq}$. The compact set of feasible route flows can now be defined as the solutions to the system

$$\sum_{r\in\mathscr{R}_{pq}} h_r = d_{pq}, \quad (p,q) \in \mathscr{C}, \tag{1a}$$

$$h_r \geqslant 0, \quad r \in \mathscr{R}_{pq}, \ (p, q) \in \mathscr{C}. \tag{1b}$$

Let $\delta_{ra} = 1$ if route r includes link a, and 0 otherwise. Let f_a denote the flow on link a. The link flow on link a is calculated as a function of the route flows

$$f_a(h) = \sum_{(p,q)\in\mathscr{C}} \sum_{r\in\mathscr{R}_{pq}} \delta_{ra} h_r, \quad a \in \mathscr{A}. \tag{1c}$$

Let $\mathscr{F}$ denote the set of link flows that satisfy (1). We wish to describe a traffic equilibrium which follows the Wardrop conditions (e.g., Patriksson, 1994). These conditions state that every travler chooses his/her fastest route. Let c_r represent the travel time on route $r \in \mathscr{R}_{pq}$. The Wardrop conditions may be stated as

$$h_r > 0 \Rightarrow c_r = \pi_{pq}, \quad (p, q) \in \mathscr{C}, \ r \in \mathscr{R}_{pq}, \tag{2a}$$

$$h_r > 0 \Rightarrow c_r \geqslant \pi_{pq}, \quad (p, q) \in \mathscr{C}, \ r \in \mathscr{R}_{pq}, \tag{2b}$$

subject to (1), where π_{pq} can be interpreted as the minimum route travel time in pair (p, q). We presume that the travel time on link a, t_a, is a separable, continuous, positive and strictly increasing function. Assuming additive link travel times, the travel time of a route $r \in \mathscr{R}_{pq}$ is calculated as $c_r(h) = \sum_{a\in\mathscr{A}} \delta_{ra} t_a(f_a)$ for any pair (h, f) satisfying (1c). Given convex functions g_k on $f \in \mathscr{F}$ let

$$g_k(f) \leqslant b_k, \quad k \in \mathscr{K}, \tag{3}$$

define the side constraints.

The side constrained traffic equilibrium problem is

$$\min T(f) \triangleq \sum_{a\in\mathscr{A}} \int_0^{f_a} t_a(s)\,\mathrm{d}s, \tag{4a}$$

subject to

$$f \in \mathscr{F}, \tag{4b}$$

$$g_k(f) \leqslant b_k, \quad k \in \mathscr{K}. \tag{4c}$$

The side constrained traffic equilibrium problem is a nonlinear multicommodity network flow problem with side constraints. The problem (4) is convex. We make the assumption that a link flow solution $f \in \mathscr{F}$ exists for which $g_k(f) < b_k$ for all $k \in \mathscr{K}$, holds. In terms of generalized link travel times,

$$\tilde{t}_a(f_a) \triangleq t_a(f_a) - \sum_{k\in\mathscr{K}} u_k^* \frac{\partial g_k(f)}{\partial f_a}, \tag{5}$$

the optimal link flow solution, f^*, to the side constrained traffic equilibrium problem is a Wardrop equilibrium link flow solution where $u^* \leqslant 0$ is the vector of optimal Lagrangian multipliers to the side constraints (4c). One interpretation of this is that the effect of the side constraints is transferred to the trip makers perception of the link travel times. In addition, the value of the sum in (5) may be interpreted as the time equivalent of a toll that the travelers on the link are willing to pay to use the link (cf. the discussion in Jorgensen, 1963; Hearn, 1980; and Larsson and Patriksson, 1999). If a vector u^* of multipliers is known, then the side constrained traffic equilibrium problem

(4) could be solved as a standard traffic equilibrium problem. The link travel times $\tilde{t}_a(f_a)$ may be summed up into generalized route travel times as

$$c_r \triangleq \sum_{a \in \mathscr{A}} \delta_{ra} t_a(f_a) - \sum_{k \in \mathscr{K}} u_k^* \left(\sum_{a \in \mathscr{A}} \delta_{ra} \frac{\partial g_k(f)}{\partial f_a} \right),$$

which satisfy the Wardrop conditions (2). However, the dual multipliers u^* are generally not unique, which in turn make the generalized link, and route, travel times non-unique. This is proved and further discussed in Larsson and Patriksson (1999).

Our primary interest is in instances of problem (4) where the side constraints model flow restrictions on individual links or linear combinations thereof, and where the number of side constraints are small relative to the number of links in the network.

The side constrained traffic equilibrium with elastic demand is a generalization of the fixed demand mode. Special cases of the elastic demand model that can be modelled as optimization problems and the demand is bounded from above, can be transformed into a fixed demand model. This can be made by the procedure described in for example Gartner (1980). Using the this transformation, methods for the fixed demand case, can be applied to the elastic demand case.

3. Related work

The traffic equilibrium problem with side constraints is a side constrained nonlinear multicommodity network flow problem. Instances of this problem have been studied in several earlier papers and a number of solution algorithms have been proposed. Examples of applications of the nonlinear multicommodity network flow problem with side constraints can be found in Ahuja et al. (1993, Chapter 14); Marín (1995) and Pınar and Zenious (1993).

The capacitated nonlinear multicommodity network flow problem is a well studied special case of the side constrained nonlinear multicommodity network flow problem. In Hearn and Ribera (1981), restrictive assumptions on the travel delay functions are formulated such that the capacitated traffic equilibrium problem can be solved using standard methods for the traffic equilibrium problem. A dual ascent algorithm for the capacitated equilibrium problem is presented in Hearn and Lawphongpanich (1990). In their algorithm, the link/route coupling constraints (1c) are dualized. An master problem is constructed. The master problem is a linear problem where each constraint represents a tangent plane which overestimates the optimal dual value. This problem is solved alternating with two sub-problems. The two sub-problems solved in each iteration are, firstly, the problem of finding a shortest path from each origin to each destination in the network and secondly, a one-dimensional convex problem which is solved analytically. A new tangent plane is constructed from the solution to the sub-problem and is added to the master problem. Encouraging computational results for a small-scale network are given in the paper. In Larsson et al. (1997) a dual method for the traffic equilibrium problem where the link/route coupling constraints are dualized, is discussed (cf. Hearn and Lawphongpanich, 1990). The master problem for finding optimal dual values is solved with sub-gradient optimization. The extension to the case of link flow capacities is presented but no computational results are given. In Larsson and Patriksson (1995) an augmented Lagrangian approach is applied to the capacitated traffic

TRB 393
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equilibrium problem. The capacity constraints are dualized. A master problem for updating the dual multipliers and a penalty parameter is solved alternating with an sub-problem that is a traffic equilibrium problem. In the implementation, the disaggregate simplicial decomposition method is used for the sub-problem.

In Lawphongpanich (2000) a simplicial decomposition procedure is described for the nonlinear multicommodity network flow problem with linear side constraints. The simplicial decomposition procedure consists of one master and one sub-problem. The master problem consists of minimizing a convex nonlinear objective function over a restricted set of flow patterns, all of which is feasible with respect to the demand constraints and the side constraints. This master problem and an sub-problem which is a linear multicommodity network flow problem with side constraints are solved in alternation. The sub-problem is solved with a truncated Danzig–Wolfe decomposition algorithm where the side constraints are dualized. Computational results for an implementation in the GAMS software package (Brooke et al., 1998) are presented. The test problems are instances of capacitated traffic equilibrium problems.

4. A stabilized column generation procedure

In this section we present a column generation procedure for the convex side constrained traffic equilibrium problem. The basic column generation procedure to be presented in Section 4.1 is an application of the schemes given in Dantzig (1963, Chapter 24), Lasdon (1970, Chapter 4.4) and Magnanti et al. (1976). A Boxstep extension (Marsten et al., 1975) of the column generation scheme is presented in Section 4.2. The proposed column generation procedure is summarized and the convergence is proved in Section 4.3. An advanced start procedure is discussed in Section 4.4.

4.1. Basic scheme

Dualizing the side constraints in the problem (4) with multipliers $u \leqslant 0$ results in the Lagrangian dual problem

$$\max_{u \leqslant 0} h(u),$$

where

$$h(u) = \min_{f \in \mathscr{F}} \left\{ T(f) - \sum_{k \in \mathscr{K}} u_k (g_k(f) - b_k) \right\}.$$

This problem can be reformulated as

$$\max v, \tag{6a}$$

subject to

$$\sum_{k \in \mathscr{K}} u_k (g_k(f) - b_k) + v \leqslant T(f), \quad f \in \mathscr{F}, \tag{6b}$$

$$u_k \leqslant 0, \quad k \in \mathscr{K}, \tag{6c}$$

which has an infinite number of constraints. Denote problem (6) the dual *master problem.*

A *relaxed* dual master problem is constructed by considering only a subset of the points in $\mathscr{F}$. Let the set $\mathscr{I}$ hold the indices of a finite number of link flow vectors $f^i \in \mathscr{F}$. The relaxed dual master problem is

$$\max v, \tag{7a}$$

subject to

$$\sum_{k \in \mathscr{K}} u_k(g_k(f^i) - b_k) + v \leqslant T(f^i), \quad i \in \mathscr{I}, \tag{7b}$$

$$u_k \leqslant 0, \quad k \in \mathscr{K}. \tag{7c}$$

Dualizing the linear problem (7) gives the dual problem

$$\min \sum_{i \in \mathscr{I}} T(f^i)\lambda_i, \tag{8a}$$

subject to

$$\sum_{i \in \mathscr{I}} g_k(f^i)\lambda_i \leqslant b_k, \quad k \in \mathscr{K}, \tag{8b}$$

$$\sum_{i \in \mathscr{I}} \lambda_i = 1, \tag{8c}$$

$$\lambda_i \geqslant 0, \quad i \in \mathscr{I}, \tag{8d}$$

where λ is the vector of dual variables for the constraints (7b). For simplicity, we assume that the problem (8) has at least one feasible solution. A procedure for finding a feasible solution is discussed in Section 4.4. The problem (8) is called the restricted master problem. Let $\hat{\lambda}$ denote the solution to problem (8) and let $(\hat{u}, \hat{v})$ denote the corresponding dual solutions to (8b) and (8c), respectively. A link flow solution $f^* = \sum_{i \in \mathscr{I}} f^i \lambda_i$ is computed from the solution to the restricted master problem. The solution f^* is optimal to the side constrained traffic equilibrium problem (4) if the corresponding dual solution $(\hat{u}, \hat{v})$ satisfies

$$\sum_{k \in \mathscr{K}} \hat{u}_k(g_k(f) - b_k) + \hat{v} \leqslant T(f), \quad \forall f \in \mathscr{F}.$$

Solving a *sub-problem*,

$$\min_{f \in \mathscr{F}} \left\{ T(f) - \sum_{k \in \mathscr{K}} \hat{u}_k(g_k(f) - b_k) \right\}, \tag{9}$$

will check whether this is the case. Let $\hat{f}$ denote the optimal solution to the sub-problem. Note that since $\hat{v}$ is constant in the sub-problem it has been omitted there. If the *reduced cost* $\hat{r}$ defined as

$$\hat{r}(f) \triangleq T(\hat{f}) - \sum_{k \in \mathscr{K}} \hat{u}_k(g_k(\hat{f}) - b_k) - \hat{v}, \tag{10}$$

is negative, then the vector $(g(\hat{f})^{\mathrm{T}}, 1)^{\mathrm{T}}$ corresponds to a new column with potential to reduce the objective function value of problem (8) The index set $\mathscr{I}$ is augmented and the new column is added to the restricted master problem (8). If the reduced cost is non-negative, the column generation procedure is terminated and the optimal solution to (4) is given by the solution to the restricted master problem (cf. Lemma 1, Section 4.3). From well known properties of Lagrangian duality, the optimal value to problem (9) provides a lower bound to the optimal value to problem (4), since

$$\begin{aligned} T(f^*) = \min_{f\in\mathscr{F},g(f)\leqslant b} T(f) &\geqslant \min_{f\in\mathscr{F},g(f)\leqslant b}\left\{T(f) - \sum_{k\in\mathscr{K}} \hat{u}_k(g_k(f) - b_k)\right\} \\ &\geqslant \min_{f\in\mathscr{F}}\left\{T(f) - \sum_{k\in\mathscr{K}} \hat{u}_k(g_k(f) - b_k)\right\}. \end{aligned}$$

The sub-problem (9) is a traffic equilibrium problem. It is solved a number of times in the solution procedure, each time with changes only in the objective function, since u is updated in each iteration. Therefore a method that has good reoptimization facilities is most suitable. We have chosen the disaggregate simplicial decomposition (DSD) method of Larsson and Patriksson (1992).

The DSD method is based on a route-restricted equilibrium problem and a sub-problem for generating new routes to the restriction. Assume that a non-empty subset of the simple routes $\bar{\mathscr{R}}_{pq} \subset \mathscr{R}_{pq}$ for all $(p,q) \in \mathscr{C}$ is known and denote the flow on route $r \in \bar{\mathscr{R}}_{pq}$ by h_r. The restricted equilibrium problem in the DSD method is given by

$$\min_f \left\{T(f) - \sum_{K\in\mathscr{K}} \hat{u}_k(g_k(f) - b_k)\right\}, \tag{11a}$$

subject to

$$f_a = \sum_{(p,q)\in\mathscr{C}} \sum_{r\in\bar{\mathscr{R}}_{pq}} \delta_{ra} h_r, \quad a \in \mathscr{A}, \tag{11b}$$

$$\sum_{r\in\bar{\mathscr{R}}_{pq}} h_r = d_{pq}, \quad (p,q) \in \mathscr{C}, \tag{11c}$$

$$h_r \geqslant 0, \quad (p,q) \in \mathscr{C}, \; r \in \bar{\mathscr{R}}_p q. \tag{11d}$$

The problem (11) is a convex problem where the constraints have a Cartesian product structure with respect to the origin–destination pairs. The problem (11) is solved by a projected gradient method (see e.g., Bertsekas and Garni, 1983). The optimal solution to this problem is feasible in (9) and is therefore an upper bound on the optimal value to problem (9). Let the solution to the restricted equilibrium problem (11) be denoted $\hat{y}$. The DSD algorithm proceeds by linearizing the objective function of (11) with respect to the link flow variables at $\hat{y}$. The linearized problem is

$$\min_y \left\{T(\hat{y}) - \sum_{k\in\mathscr{K}} \hat{u}_k(g_k(\hat{y}) - b_k) + \sum_{a\in\mathscr{A}} \left(t_a(\hat{y}_a) - \sum_{k\in\mathscr{K}} \hat{u}_k \frac{\partial g_k(\hat{y})}{\partial y_a}\right)(y_a - \hat{y}_a)\right\}, \tag{12a}$$

subject to

$$y_a = \sum_{(p,q)\in\mathscr{C}} \sum_{r\in\mathscr{R}_{pq}} \delta_{ra} h_r, \quad a \in \mathscr{A}, \tag{12b}$$

$$\sum_{r\in\mathscr{R}_{pq}} h_r = d_{pq}, \quad (p,q) \in \mathscr{C}, \tag{12c}$$

$$h_r \geqslant 0, \quad r \in \mathscr{R}_{pq} \;\; (p,q) \in \mathscr{C}. \tag{12d}$$

This problem amounts to finding a fastest route in each of the origin–destination pairs. To efficiently solve the fastest route problem we need to disregard negative cycles. Let $\mathscr{K}_1$ and $\mathscr{K}_2$ denote the index set for linear and nonlinear side constraints respectively. We require that $\sum_{k\in\mathscr{K}_1} u_k \frac{\partial g_k(y)}{\partial y_a} \leqslant t_a(0)$ for $a \in \mathscr{A}$ and that $\frac{\partial g_k(y)}{\partial y_a} \geqslant 0$ for $a \in \mathscr{A}$ and $k \in \mathscr{K}_2$. These restrictions guarantee that the coefficients corresponding to y_a for all $a \in \mathscr{A}$ in the objective function (12a) are non-negative for all $\hat{y} \in \mathscr{F}$. Non-negative link travel times imply non-negative cycle times. We have not implemented any mechanism for avoiding such cycles, since they, in our application, have not appeared.

For each origin destination pair $(p,q) \in \mathscr{C}$ the set $\widehat{\mathscr{R}}_{pq}$ is augmented by the generated route if this is not already present in the set. Due to the convexity of problem (4),

$$\begin{aligned} T(f^*) &\geqslant \min_{f\in\mathscr{F}} \left\{ T(f) - \sum_{k\in\mathscr{K}} \hat{u}_k (g_k(f) - b_k) \right\} \\ &\geqslant \left\{ \min_{y\in\mathscr{F}} T(\hat{y}) - \sum_{k\in\mathscr{K}} \hat{u}_k (g_k(\hat{y}) - b_k) + \sum_{a\in\mathscr{A}} \left(t_a(\hat{y}_a) - \sum_{k\in\mathscr{K}} \hat{u}_k \frac{\partial g_k(\hat{y})}{\partial y_a} \right) (y_a - \hat{y}_a) \right\}, \end{aligned} \tag{13}$$

the optimal value to the fastest route problem provides a lower bound on the optimal value to problem (9) and therefore also a lower bound to the optimal value of problem (4). The DSD method alternates between solving the two problems Eqs. (11) and (12). The route generation procedure is terminated when the relative gap between the upper and the lower bounds is smaller than some prespecified value.

4.2. Boxstep modification

The cutting plane procedure, and the dual equivalent column generation procedure, suffers from inherent instability (see e.g., Hiriart-Urruty and Lemaréchal, 1993, Chapter 15). The instability may cause that consecutive iterates are very distant. To prevent instability in the column generation procedure, the procedure presented in Section 4.1 is modified to include a Boxstep restriction (Marsten et al., 1975). Through the Boxstep modification of the procedure, adaptive bounds on individual dual variables are introduced in the master problem to prevent large moves in the dual space. For linear problems a modification similar to the Boxstep restriction has been used for stabilizing column generation procedures with promising computational results in du Merle et al. (1999).

Box constraints are added on individual dual variables in the relaxed dual master problem (7). Given multiplier values $\hat{u} \leqslant 0$ the Boxstep modification of the relaxed dual master problem is

$$\max v, \tag{14a}$$

subject to

$$\sum_{k\in\mathscr{K}} u_k(g_k(f^i) - b_k) + v \leqslant T(f^i), \quad i \in \mathscr{I}, \tag{14b}$$

$$u_k^{lb} \leqslant u_k \leqslant U_k^{ub}, \quad k \in \mathscr{K}, \tag{14c}$$

$$u_k \leqslant 0, \quad k \in \mathscr{K}, \tag{14d}$$

where

$$u_k^{lb} := \hat{u}_k - \mu, \qquad u_k^{ub} := \hat{u}_k + \mu \quad \text{if} \quad \hat{u}_k = u_k^{lb}, \tag{15a}$$

$$u_k^{lb} := \hat{u}_k - \mu, \qquad u_k^{ub} := \hat{u}_k + \mu \quad \text{if} \ \hat{u}_k = u_k^{ub}, \tag{15b}$$

$$u_k^{lb} := u_k^{lb}, \qquad u_k^{ub} := u_k^{ub} \quad \text{if} \ u_k^{lb} < \hat{u}_k < u_k^{ub}, \tag{15c}$$

for $k \in \mathscr{K}$ and where μ is a box size parameter. The corresponding Boxstep modified restricted master problem is then

$$\min\left\{\sum_{i\in\mathscr{I}} T(f^i)\lambda_i + \sum_{k\in\mathscr{K}} (u_k^{ub}\alpha_k - u_k^{lb}\beta_k)\right\}, \tag{16a}$$

subject to

$$\sum_{i\in\mathscr{I}} g_k(f^i)\lambda_i + \alpha_k - \beta_k \leqslant b_k, \quad k \in \mathscr{K}, \tag{16b}$$

$$\sum_{i\in\mathscr{I}} \lambda_i = 1, \tag{16c}$$

$$\lambda_i \geqslant 0, \quad i \in \mathscr{I}, \tag{16d}$$

$$\alpha_k, \beta_k \geqslant 0, \quad k \in \mathscr{K}, \tag{16e}$$

where λ is the vector of the dual variables for the constraints (7b), and α and β are vectors of the dual variables for the upper and lower bounds on u, respectively. The values of multipliers $\hat{u}$ are taken as the values of the dual variables to the constraints (16b).

Applying the Boxstep procedure can result in an optimal value to problem (16) which is not necessarily an upper bound on the optimal value to problem (4) since the solution $\sum_{i\in\mathscr{I}} f^i\lambda_i$ is not necessarily feasible with respect to the side constraints. The computation of a lower bound given by (4.1) is still valid.

The Boxstep modification stabilizes the column generation scheme, and it also decreases the computational burden of the sub-problem, because it leads to a faster reoptimization.

4.3. Algorithm and convergence

The column generation procedure can be summarized in the following steps.

`Step 0.` Find an f^0 such that $g_k(f^0) < b_k$, $k \in \mathscr{K}$ and set the vector $u^1 = 0$. Choose some $\epsilon > 0$. Set $\mathscr{I} := \{0\}$ and $i := 1$.

`Step 1.` Solve the master problem (16) with Boxstep restrictions according to (15). If $g_k(\sum_{i\in\mathscr{I}} f^i\lambda_i) \leqslant b_k$, for all $k \in \mathscr{K}$, then update the upper bound.

`Step 2.` Solve the sub-problem (9). Update the lower bound of $T(f^*)$. If the corresponding reduced cost $\hat{r} \geqslant 0$ in (10) then the algorithm terminates. Otherwise, let f^i denote the solution to problem (9). Augment the index set $\mathscr{I} := \mathscr{I} \cup \{i\}$.

`Step 3.` Terminate if the relative gap between the upper and the lower bound is smaller than ϵ. Let $i := i + 1$ and go to Step 1.

A heuristic procedure for finding the strictly feasible link flow vector f^0 in Step 0 is discussed in Section 4.4. Below, we show how the algorithm has an asymptotic convergence. The proof is based on the technique used in Zangwill (1969, Chapter 14).

Lemma 1 (Convergence test). *If, for a fixed $\hat{u} \leqslant 0$ and $\hat{v} \in \Re$*

$$\min_{f\in\mathscr{F}} \hat{r}(f) \geqslant 0 \tag{17}$$

holds, then $\hat{u}$ and $\hat{v}$ are optimal in problem (6) *and the minimizer $\hat{f}$ to problem* (9) *is optimal in problem* (4).

Proof. See Dantzig (1963, Chapter 24, Theorem 2). □

Theorem 1. *If the sequence of iterates generated by the algorithm $\{f^i\}$ is finite, then the last iterate solves problem* (4). *If the sequence is infinite then for some $\bar{\mathscr{I}} \subseteq \mathscr{I}$, $\{f^i\} \stackrel{i\in\bar{\mathscr{I}}}{\rightarrow} f^\infty$, where f^∞ satisfies the convergence test* (17).

Proof. If the sequence of iterates is finite then the algorithm has terminated in the convergence test and we conclude that the last iterate f^i solves the sub-problem (9) and problem (4) according to Lemma 1.

Assume that the sequence of iterates is infinite. Denote the sequence of dual iterates by $\{(u^i, v^i)\}_1^\infty$. First we show that the primal and dual iterates are contained in compact sets. Construct the problem

$$\max v, \tag{18a}$$

$$\sum_{k\in\mathscr{K}} u_k(g_k(f) - b_k) + v \leqslant T(f), \quad f \in \mathscr{F}, \tag{18b}$$

$$u_k^{lb} \leqslant u_k \leqslant u_k^{ub}, \quad k \in \mathscr{K}, \tag{18c}$$

$$u_k \leqslant 0, \quad k \in \mathscr{K}. \tag{18d}$$

Define a point f' as

$$T(f') = \min_{f\in\mathscr{F}} T(f)$$

for which $v^i \geqslant T(f')$ holds. Let $u = 0$ and from (18b) and the strictly feasible solution f^0 we have that $v^i \leqslant T(f^0)$. Therefore, v^i is bounded. From the fact that $u \leqslant 0$ and from (18b) we get

$$u_k(g_k(f^0) - b_k) \leqslant \sum_{k\in\mathscr{K}} u_k(g_k(f^0) - b_k) \leqslant T(f^0) - v \leqslant T(f^0) - T(f').$$

Therefore

$$u_k \geqslant \frac{T(f^0) - T(f')}{g_k(f^0) - b_k},$$

that is, u^i is bounded. This boundedness ensures that there exists a set $\bar{\mathscr{I}} \subseteq \mathscr{I}$ for which

$$(u^i, v^i) \to (u^\infty, v^\infty), \quad i \in \bar{\mathscr{I}}, \tag{19}$$

$$f^i \to f^\infty, \quad i \in \bar{\mathscr{I}}. \tag{20}$$

Since in iteration i a constraint of the form

$$H(f^i) = \left\{ (u, v) \,\middle|\, \sum_{k\in\mathscr{K}} u_k(g_k(f^i) - b_k) + v \leqslant T(f^i) \right\},$$

is added, we have that $(u^l, v^l) \in H(f^i)$ for all $l \geqslant i + 1$, that is,

$$\sum_{k\in\mathscr{K}} u_k^l(g_k(f^i) - b_k) + v^l \leqslant T(f^i), \quad l \geqslant t + 1.$$

From (19),

$$\sum_{k\in\mathscr{K}} u_k^\infty(g_k(f^i) - b_k) + v^\infty \leqslant T(f^i).$$

Since $\mathscr{F}$ is a compact set and $T(f) - \sum_{k\in\mathscr{K}} u_k(g_k(f) - b_k)$ is a continuous function in u, the solution of the sub-problem (9) defines a closed point-to-point map $\Delta : R^{|\mathscr{K}|+1} \to R^{|\mathscr{A}|}$. Therefore, from (20)

$$\sum_{k\in\mathscr{K}} u_k^\infty(g_k(f^\infty) - b_k) + v^\infty \leqslant T(f^\infty). \tag{21}$$

Now, assume that (u^∞, v^∞) does not satisfy condition (17). Since the map Δ is closed, f^∞ is a solution to the sub-problem (9) for $(u, v) = (u^\infty, v^\infty)$. But for such (u^∞, v^∞) from Lemma 1,

$$\sum_{k\in\mathscr{K}} u_k^\infty(g_k(f^\infty) - b_k) + v^\infty > T(f^\infty). \tag{22}$$

The point $(u', v') = (0, T(f'))$ is feasible in $\cap_{i\in\mathscr{I}} H(f^i)$. Since the box constraints (18c) are always inactive there is at least one feasible point (u, v) in the set

$$\left\{ \bigcap_{i\in\mathscr{I}} H(f^i) \right\} \bigcap \{ u_k^{lb} \leqslant u_k \leqslant u_k^{ub} \ \forall k \in \mathscr{K} \}.$$

As (21) and (22) are contradictory, (u^∞, v^∞) must satisfy (17).

Let $y^i = \sum_{i\in\mathscr{I}} \lambda_i f^i$ for $\lambda \geqslant 0$ which, by the convexity of T, satisfies $T(y^i) \leqslant \sum_{i\in\mathscr{I}} \lambda_i T(f^i)$. Choosing λ as an optimal solution to problem (8) we have $T(y^i) \leqslant \sum_{k\in\mathscr{K}} b_k u_k^i + v^i$. Now

$$\sum_{k\in\mathscr{K}} b_k u_k^i + v^i \geqslant T(y^i) \geqslant T(f^*).$$

Consequently the limit y^∞ is an optimal solution to problem (4). □

4.4. Initialization

The column generation procedure proposed in Section 4 starts from a strictly feasible link flow vector f^0. However, an initial strictly feasible solution is not necessary for the convergence of the column generation scheme. Initial computational experiments showed that for some test problems, the nonlinear column generation procedure has problems finding feasible solutions and that the efficiency of the procedure is improved when it is warm-started from a feasible solution.

A heuristic phase-one procedure is developed to produce such a starting solution. The procedure is similar to the column generation procedure described for problem (4), but with an added penalty function to the objective (4a) together with a shift of the right-hand sides of the side constraints (4c). This heuristic procedure for finding feasible solution has earlier been used in, for example, Yang and Yager (1994). The right-hand side shift is made by scaling the right-hand side vector b of (4c). Set $\xi_k = 0.99$ for $k \in \mathscr{K}$ where $b_k > 0$ and $\xi_k = 1.01$ for $k \in \mathscr{K}$ where $b_k < 0$. Let

$$g_k(f) \leqslant \xi_k b_k, \quad k \in \mathscr{K}, \tag{23}$$

define a set of strengthend side constraints. Define the penalty function

$$P(f, c) = \frac{c}{2} \sum_{k \in \mathscr{K}} (\max\{0, g_k(f) - \xi_k b_k\})^2.$$

This penalty function is added to the objective function (4a). These modifications are applied to the solution procedure until a feasible solution is generated, either that the generated column is feasible with respect to the side constraints, or that the master problem generates a feasible solution. When a feasible solution is found the right-hand sides are shifted back to their initial values by letting $\xi = 1.0$, the penalty function P is removed from the objective, and the phase one procedure is terminated. The phase-one procedure is heuristic since it fails if the the set defined by (23) is an empty set. The value of the parameter c has a great effect on the computational performance. In the case of negative components in the $\partial g(f)/\partial f_a$-vector, a large value of c may introduce negative cycles in the fastest path sub-problem and some caution is required.

5. Numerical experiments

We present results from numerical tests for three types of side constraints. The first type is for general linear side constraints. The linear constraints are constructed to represent typical flow restrictions in the context of strategic traffic management. The second type of side constraints are restrictions on individual links. A subset of the links in the test network is selected and for these links the flow is fixed to the system optimal link flows. The third type of side constraint consists of link flow capacities. The capacities are chosen as multiples of the system optimal link flow solution. This type of capacity constraint is used in Lawphongpanich (2000). The test networks in Table 1 are used in the computational experiments.

In each iteration the linear restricted master problem is solved to optimality using the CPLEX 6.5 Simplex callable library. The sub-problem is solved using a modified implementation of the

Table 1
Test networks

Network	Nodes	Arcs	Commodities
Nine node	9	18	4
Sioux Falls	24	76	528
Hull	501	798	142
Linköping	335	882	12,372

disaggregate simplicial decomposition method in Larsson and Patriksson (1992) where the restricted equilibrium problem is solved using a gradient projection procedure (see e.g., Bertsekas and Garni, 1983). The solution procedure for the sub-problem is heavily truncated in all numerical experiments. The problem of finding new fastest routes in the network is solved every 20th iteration. The computations have been performed on a SUN Ultra 10 with 300 MHz CPU and 320 MB of internal memory. The chosen values of the penalty parameter c and the box size parameter μ is determined by experiments.

5.1. Three traffic management scenarios modeled using linear side constraints

In this section results from numerical experiments for three test scenarios with linear side constraints are presented. The scenarios are motivated by planning goals, and the constraints are chosen to represent typical traffic management goals. The test network is the city of Linköping. The equilibrium traffic flows for the centre of the Linköping are shown in Fig. 1. The link widths are proportional to the link flows. The objective value for the traffic equilibrium problem solved

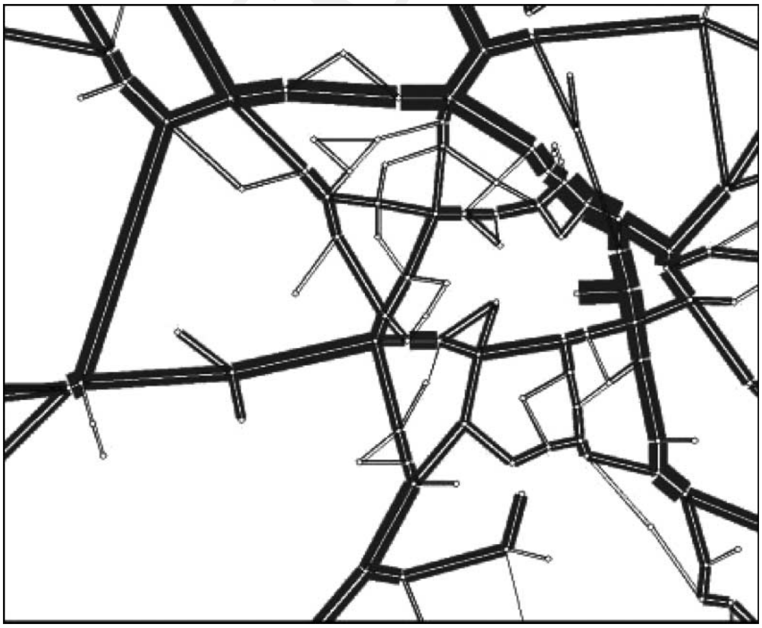

Fig. 1. Linköping equilibrium scenario.

Fig. 2. Linköping Scenario 1 30%.

for finding the equilibrium solution is 4.056018e+8 (with a lower bound on the optimal value of 4.056016e+8).

In the first scenario, the traffic inflow to the city center of Linköping is restricted. The restriction concerns the inflow into a zone of 28 nodes and is placed on 10 links and is defined by six linear side constraints. The links included in the six side constraints are shown in Fig. 2 where each black line represent one side constraint. The total flow on the 10 links in the Wardrop equilibrium flows is 40 265 vehicles during a period of 2 h and the travel demand from origins outside the defined city center to destinations in the city center is 21 608 vehicles during this time. Three sub-scenarios are constructed by requiring that the inflow is reduced by 30%, 20% and 10%, respectively, that is that 64%, 43% and 21% of the traffic whose destination is not within this zone is forced to reroute around the zone.

The penalty parameter c is set to 0.9 and a box size of $\mu = 100$ is chosen. The computational results are shown in Table 2 where the scenario is given in column 1, the number of iterations made in the column generation procedure in column 2, the number of route generation problems solved in column 3, the number of iterations in the column generation procedure until the solution

Table 2
Test results for the Linköping Scenario 1

Reduction (%)	#Iter.	#Route gen. prob.	#Iter. until feas.	Lower bound	Upper bound	CPU (s)
30	700	48	444	4.070729e+8	4.070762e+8	881
20	260	26	197	4.061485e+8	4.061505e+8	305
10	580	42	107	4.057255e+8	4.057282e+8	644

to the master problem is feasible in the side constraints, excluding the feasible solution found by the phase-one procedure, is given in column 4, the lower and upper bounds at termination in column 5 and 6, respectively, and the computational time in CPU seconds is given in column 7. The number of iterations, the number of fastest route problems solved and the CPU time include the computations made in the phase-one procedure. The solution procedure is terminated when a relative accuracy between the upper and the lower bound of 1e−3% is reached.

The six side constraints are all active in the optimal solutions to the three sub-scenarios. Fig. 3 shows the link flow difference between the unconstrained equilibrium solution and the case where the inflow is reduced by 30%. The link widths are proportional to the absolute difference, where the darker links indicate an increase and the lighter a decrease in flow. Fig. 4 shows link widths proportional to the relative flow difference between the Wardrop equilibrium flows and the flows in Scenario 1. In Fig. 5 the iteration history is shown for the objective values of the upper and lower bounds. The horizontal line in Fig. 6 represents a high quality lower bound on the optimal value, indicating that the quality of the upper bound is good compared to the quality of the lower bounds computed in the solution scheme.

In the second scenario the total flow, in both directions, on each of 13 road segments is restricted. The links with flow restrictions are all in the central part of Linköping. The locations of the restrictions are shown in Fig. 7. Three sub-scenarios are constructed. The flow restrictions in the three sub-scenarios are chosen such that the total road flow on individual road segments (i.e., the link flow in both directions) is reduced by 30%, 20% and 10% compared to the equilibrium flows. The penalty parameter c is set to 0.9 and a box size of $\mu = 1$ is chosen. The column generation scheme is terminated when a relative accuracy between the upper and the lower bound of 1e−3% is reached. The computational results are shown in Table 3. All but two of the side con-

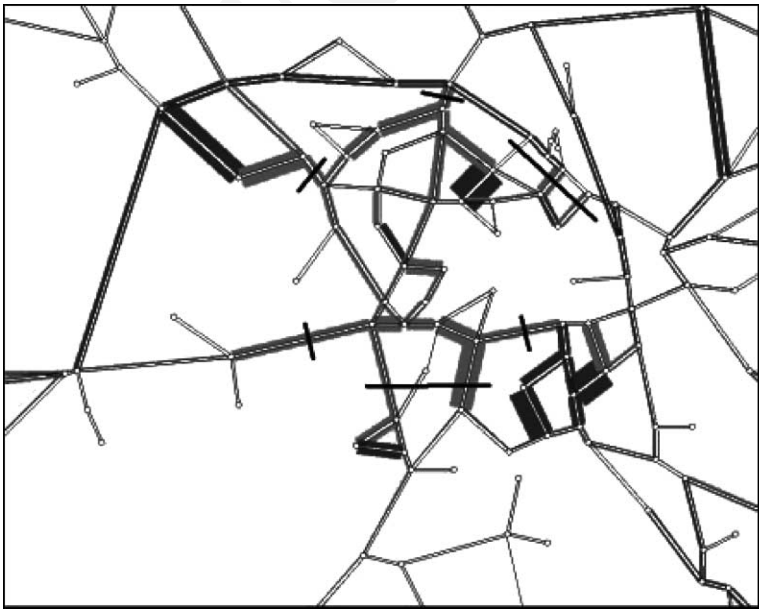

Fig. 3. Linköping Scenario 1 30% (relative difference).

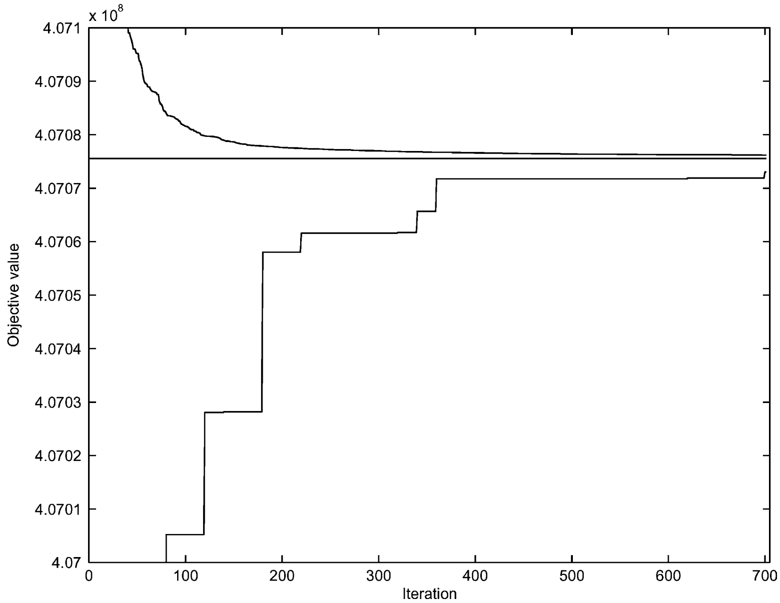

Fig. 4. Linköping Scenario 1 30% iteration history.

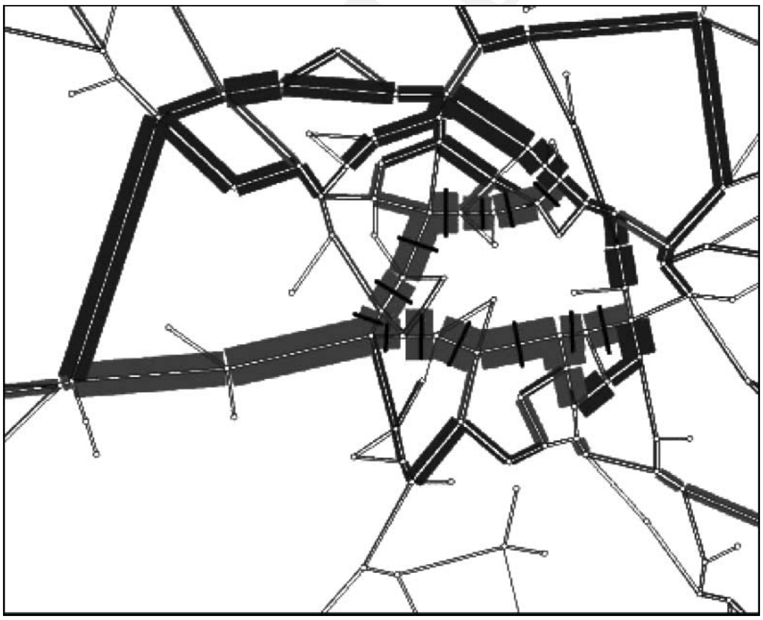

Fig. 5. Linköping Scenario 2 30%.

straints are active in the solution to the sub-scenario with a 30% reduction, and all but three in the 20% and 10% cases.

Fig. 6. Linköping Scenario 2 30% (relative difference).

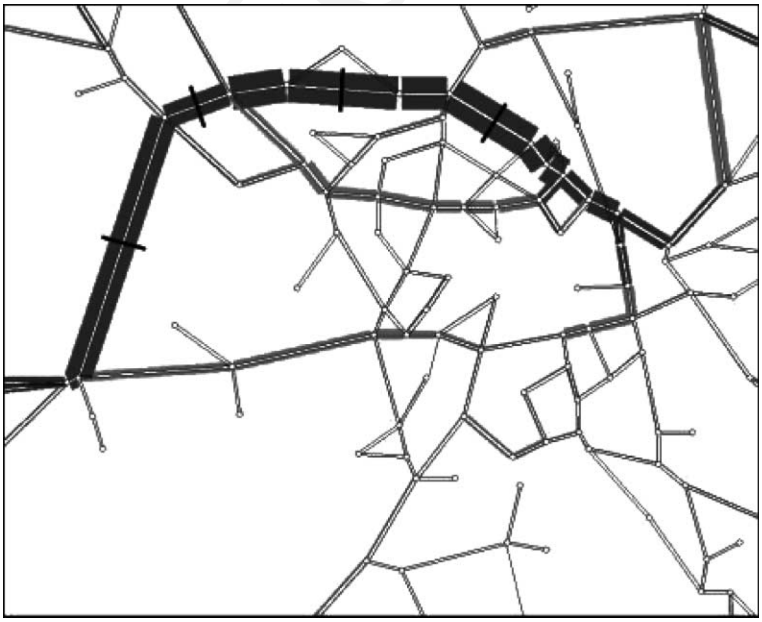

Fig. 7. Linköping Scenario 3.

Table 3
Test results for the Linköping Scenario 2

Reduction (%)	#Iter.	#Route gen. prob.	#Iter. until feas.	Lower bound	Upper bound	CPU (s)
30	780	52	154	4.062912e+8	4.062939e+8	1042
20	720	49	153	4.059475e+8	4.059512e+8	973
10	600	43	246	4.057128e+8	4.057149e+8	742

Fig. 8. Linköping Scenario 3 (relative difference).

In the third scenario, the total flow in both directions on each of four road segments, is restricted. The scenario is constructed such that some travelers are forced to use a bypass route instead of traveling through the central part of the city. The road segments where the total flow are restricted are shown in Fig. 8. In order to achieve this goal, the flow restrictions are chosen such that the link flow in both directions *increases* 20% compared to the equilibrium flow. That is, letting $\bar{f}$ denote the equilibrium link flow, the constraints added are of the form $f_{a1} + f_{a2} \geqslant 1.2(\bar{f}_{a1} + \bar{f}_{a2})$. The penalty parameter c is set to zero and a box size of $\mu = 1$ is chosen. The column generation procedure is terminated when a relative accuracy between upper and lower bounds on the objective value of 1e−3% is reached. The upper bound on the optimal objective value is at termination 4.058109e+8, and the lower bound is 4.058076e+8. All the side constraints are active in the solution at termination. The solution procedure requires 480 iterations and 37 fastest path problems are solved. The computational time is 526 s. Neither the choice of box size parameter nor the phase one procedure is crucial for the practical convergence speed of the algorithm for this type of test problem.

5.2. Partial system optimal flows

In this section, numerical results for test problems where some of the links in the network has a fixed flow are presented. The numerical experiments are performed using the Linköping network. Based on system optimal link flows $\tilde{f}$ for the Linköping network, two sub-scenarios are constructed. For each of the link in the network the total travel time $t_a(\tilde{f}_a)\tilde{f}_a$, where $\tilde{f}_a$ is the system optimal link flow for link $a \in \mathscr{A}$, is computed. For the two sub-scenarios, the 10% and 20% of the links with the highest value of $t_a(\tilde{f}_a)\tilde{f}_a$ are selected. From each of the selected links, a side constraint of the form $f_a = \tilde{f}_a$ is constructed. The selected links in the two sub-scenarios represent 85% and 92% of the total travel time respectively, and the sub-scenarios therefore ensure system optimal link flows on links with a noticeable effect on the total travel time. These test problems clearly violate our initial assumption of the existence of a strictly feasible solution to the side constraints. Further, the heuristic phase-one procedure proposed in Section 4.4 cannot be applied to the case of side contraints of equality type. Instead the column generation procedure is initialized with the (infeasible) Wardrop equilibrium links flows for the Linköping network.

The column generation procedure fails to produce a feasible solution in both of the sub-scenarios. However, near-feasible solutions are found. For the 10% sub-scenario, 89 links are selected resulting in 89 side constraints. (In practice, each of the equality side constraints is rewritten as two inequality side constraints.) The parameter value $\mu = 1.0$ is selected. The column generation procedure is terminated after 600 iterations. At termination, a link flow pattern with an objective value of 4.061930e+8 is obtained. A lower bound of 4.061786e+8 is recorded. The infeasibility of the best link flow solution found, computed as the residual $\sum_{k \in \mathscr{K}} \max\{0, g_k(f) - b_k\}$, equals 0.42, where $\sum_{k \in \mathscr{K}} b_k$ equals 5.2e + 5. The computational time is 934 CPU seconds. The route generation problem is solved 43 times.

The results for the 20%-scenario is the following. A set of 177 links is selected to comprise the links with the highest total travel time in the system optimal solution. The parameter value $\mu = 1.0$ is selected. After 600 iterations, an upper bound of 4.064737e+8 has been found and a lower bound of 4.064139e+8 is recorded. The residual for this solution is 5.03, where $\sum_{k \in \mathscr{K}} b_k$ equals 9.6e + 5. The computational time is 2109 CPU seconds and 43 route generation problems are solved.

To illustrate the importance of the Boxstep modification a numerical experiment with different box sizes is made. The data for the 20% scenario is used. The stabilized column generation procedure is run for a fixed CPU time of 800 s. In Fig. 9 the best lower bound is shown for different choices for the box size parameter μ. For small values of the box size parameter ($\mu = 0.01$) small dual steps are taken. For this choice, the reoptimization of the sub-problem is fast and a large number of iterations are made.

For large values of the box size parameter ($\mu = 5$) large dual steps are taken. In this case the procedure has an unstable behavior, the reoptimization of the sub-problem is not as efficient as for small changes in the dual variables, and a smaller number of iterations can be made within the given limit. (For even larger values of μ, the performance of the procedure becomes even worse.) The results from this experiment show how the quality of the lower bound can be improved by the introduction of the Boxstep procedure. The best computational performance is achieved with a compromise, where the reoptimization of the sub-problem is relatively fast, and where the dual steps are allowed to be sufficiently large (for example by the choice $\mu = 1$).

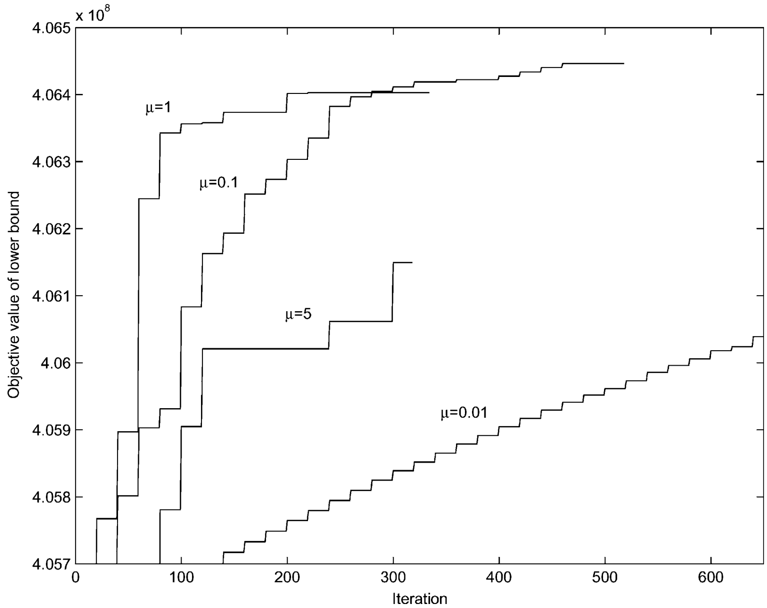

Fig. 9. Lower bounds for some box size parameters μ.

5.3. Capacity side constraints

In this section, results from numerical experiments are presented for the capacitated traffic equilibrium problem. All four networks in Table 1 have been used for the experiments.

Three scenarios are constructed for each of the test networks. In each of the scenarios the capacities are chosen as a percentage of the system optimal link flow solution. The percentages chosen are 105%, 110% and 120%. This type of test problems is also used in Lawphongpanich

Table 4
Test results for the capacitated traffic equilibrium problem

Network	Capacity (in %)	#Iter.	# Route gen. prob.	#Iter. until feas.	Lower bound	Upper bound	CPU (s)
Nine node	105	83	14	80	1912.50	1912.52	1
	110	80	10	74	1873.31	1873.32	1
	120	41	7	38	1829.56	1829.57	1
Sioux Falls	105	100	18	80	42.5326	42.5355	4
	110	60	16	55	42.3769	42.3796	2
	120	40	15	24	42.3169	42.3175	2
Hull	105	460	36	182	21799.4	21801.0	310
	110	380	32	177	21697.0	21698.6	191
	120	200	23	117	21603.4	21605.3	78
Linköping	105	340	30	–	4.061638e+8	4.063108e+8	1144
	110	180	22	–	4.058899e+8	4.060200e+8	476
	120	160	21	–	4.056554e+8	4.057481e+8	408

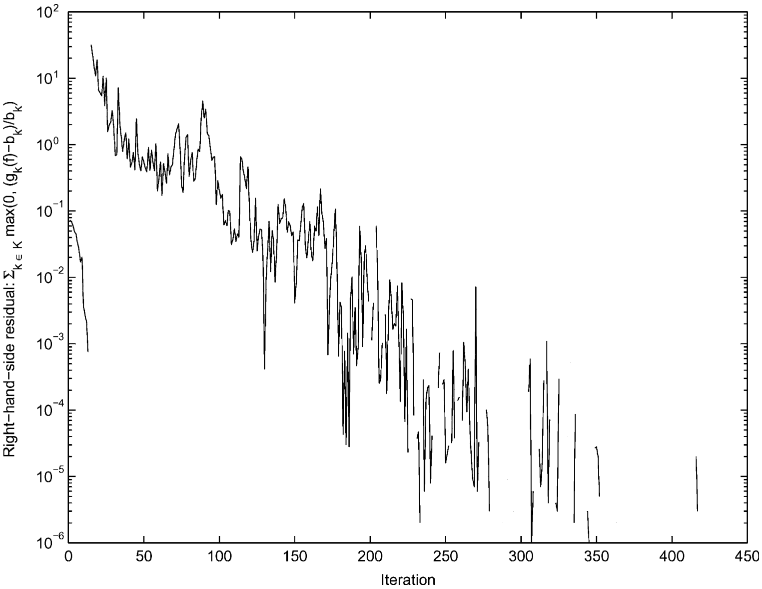

Fig. 10. Right-hand side residual in the Hull 105% case.

(2000). Links with zero flow in the system optimal solution are eliminated from the capacitated traffic equilibrium since the corresponding arc capacity will be zero. The box size is chosen as μ equal to 0.1, 0.001, 0.01, and 0.3, for the Nine node, Sioux Falls, Hull, and Linköping networks, respectively. The penalty parameter c is chosen to be 1.0, 0.1, 0.2, and 0.9, respectively. The solution procedure is terminated when a relative accuracy between the upper and the lower bound of 1e−3% for the Nine node network, 1e−2% for the Sioux Falls and Hull networks, and 5e−2% for the Linköping network, is reached. The results from the computational experiments are given in Table 4. For the Linköping network, only one feasible solution is generated, and it is generated by the phase-one procedure. In Fig. 10 the infeasibility with respect to the side constraints of the current solution to the master problem is given for the Hull test network and the 105% case. The first 19 iterations are made in the initialization procedure. Iterations where the value for the right-hand side residual is missing represent a feasible solution.

6. Conclusions

We have presented a column generation procedure for the side constrained traffic equilibrium problem. The column generation procedure is a modification of that given, for example, in Lasdon (1970, Chapter 4.4) where the stabilization technique in Marsten et al. (1975) is introduced to get a more computationally efficient algorithm. Computational results for three types of side constraints are presented with their focus on test problems which represent typical traffic management goals. The algorithm performs reasonably well and performs better for the cases where the number of side constraints is small. For the Nine node and the Sioux Falls network, the performance, measured in terms of the number of fastest route problems solved, is comparable to the result from the algorithm presented in Lawphongpanich (2000).

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