ON STOCHASTIC STRUCTURAL TOPOLOGY OPTIMIZATION

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1 SUMMARY

We consider structural topology optimization problems including unilateral constraints arising from, for example, non-penetration conditions in contact mechanics or non-compression conditions for elastic ropes. To construct more realistic models and to hedge off possible failures or inefficient behaviour of optimal structures, we allow parameters (for example, loads) defining the problem to be stochastic. The resulting nonsmooth stochastic optimization problem is an instance of stochastic mathematical programs with equilibrium constraints (MPEC), or stochastic bilevel programs. The existence as well as the continuity of optimal solutions with respect to the lower bounds on the design variables are established. The question of continuity of optimal solutions with respect to small changes in probability measure is analyzed. For a subclass of the problems considered the answer is affirmative, thus showing the robustness of optimal solutions.

2 INTRODUCTION

Does the introduction of multiple load cases into a topology optimization problem always lead to robust optimal designs? The large number of publications aiming to achieve robust solutions by optimizing for several (in some cases the continuum) load cases suggests that the answer should be positive.

The answer of course depends on the exact definition of "robustness" and the type of optimization problem under consideration. The reason for considering several load cases is to incorporate the *uncertain* nature of the loads into the model, while the desired property of a robust design is to change continuously as a model of reality (loading conditions, material properties, etc.) changes. To thoroughly answer the posed question it is necessary to measure the closeness of two models of (uncertain) reality.

In this paper we consider two of the most natural and classic structural topology optimization problems: the finding of a maximally stiff truss under a volume constraint, and the finding of a truss of minimal weight under stress constraints. The uncertainty due to several factors (such as loads unknown in advance, varying material properties, manufacturing errors, etc.) is taken into account. Capturing the uncertainty in the model through the use of probability theory allows us to construct general models, and through the associated probability measure, it is possible to interpret the "continuous change in the model of reality" as a continuous change in a topological space of measures.

To include a wide range of applications we allow mechanical structures to be *unilaterally* constrained, i.e., some parts of the structure might come into unilateral frictionless contact with rigid obstacles, while some other parts might sustain only tensile forces. Practical applications of unilateral contact include such machine elements as joints, hinges, press-fits, and examples of structures with tensile-only members include suspension bridges and cranes.

Practice has shown that the idea of allowing truss topology to change may lead to exceedingly efficient designs. On the other hand, these designs may be very inefficient or can even fail when loading conditions slightly change. An attempt to maintain the efficiency of topology optimization while hedging off possible failures or inefficient behaviour has given rise to the field of *robust topology optimization*. The anticipated fact that the "real" probability model is never known, and the reported high sensitivity of solutions to stochastic structural optimization problems with respect to small changes in probability measure [1, pp. 20–22], caused the development in the area to concentrate on probability-free worst-case ("pessimistic") approaches to uncertainty. An efficient numerical approach to solve topology optimization problems with a simple convex uncertainty model is known [2].

An alternative approach to robust topology optimization based on treating uncertainty via probability theory is analyzed in [3], [4] and [5]; an interested reader can find all the proofs of the results listed here in these papers. In addition to extending "classic" structural topology optimization results (existence of optimal designs, convergence of ε-perturbations) to the general stochastic setting, we analyze the continuity of optimal solutions with respect to changes in the probability measure. The results of this analysis give us explicit information about when the introduction of uncertainty into the structural topology optimization models indeed leads to robust optimal designs. We also illustrate the convergence of various approximations for the stochastic weight minimization problem and show the qualitative behaviour of optimal solutions with numerical examples.

2.1 Mechanical equilibrium

Given positions of the nodes the *design* (and topology in particular) of a truss can be described by the following sets of *design* variables: $x_i \geq 0$, $i = 1, \dots, m$, representing the volume of material, allocated to the bar i in the structure; $X_j \geq 0$, $j = 1, \dots, r_2$, representing the volume of material, allocated to the cable j. We introduce two index sets of the present (or active) members in the structure: $\mathcal{I}(x) = \{ i = 1, \dots, m \mid x_i > 0 \}$ and $\mathcal{J}(X) = \{ j = 1, \dots, r_2 \mid X_j > 0 \}$.

Let $(\Omega, \mathfrak{S}, \mathrm{P})$ be a complete probability space. Given a particular design the status of the linear elastic mechanical system is governed by the principle of minimum complementary energy $(\mathcal{C})_{(x,X)}(\omega)$ (in our case it is an (x, X, ω)-parametric minimization problem) where the functions in the problem have the following meaning from a mechanical point of view: $E(\omega)$ and $E_c(\omega)$ are Young's moduli for the structure and cable materials respectively; $B_i(\omega)$ is the kinematic transformation matrix for the bar i; $\gamma_j(\omega)$ is the unit direction vector of the cable j; $(g_2(\omega))_j$ is the initial slack of the cable j; $L_j(\omega)$ is the length of the cable j; $C_1(\omega)$ is the quasi-orthogonal kinematic transformation matrix for rigid obstacles; $g_1(\omega) \geq 0$ is the vector of the initial gaps; and $f(\omega)$ is the vector of external forces.

For the problem to be tractable we assume that all functions listed above are $\mathfrak{S}$-measurable. We further assume that the matrix C_1 is quasi-orthogonal, that is, that $C_1 C_1^T = I$. That condition is fulfilled if at each node either there is at most one rigid support or multiple supports "act" in orthogonal directions to each other.

$$(\mathcal{C})_{(x,X)}(\omega) \begin{cases} \min_{(s,S,\lambda)} \mathcal{E}(x,X,s,S,\lambda,\omega) := \frac{1}{2} \sum_{i \in \mathcal{I}(x)} \frac{s_i^2}{E(\omega) x_i} + g_1^T(\omega)\lambda \\ \qquad\qquad + \sum_{j \in \mathcal{J}(X)} \Big(\frac{(L_j(\omega) S_j)^2}{2E_c(\omega) X_j} + (g_2(\omega))_j S_j \Big), \\ \text{s.t.} \begin{cases} C_1^T(\omega)\lambda + \sum_{i \in \mathcal{I}(x)} B_i^T(\omega) s_i + \sum_{j \in \mathcal{J}(X)} S_j \gamma_j(\omega) = f(\omega), \\ \lambda \geq 0, \quad S_{\mathcal{J}(X)} \geq 0, \quad s_{\mathcal{I}^c(x)} = 0, \quad S_{\mathcal{J}^c(X)} = 0. \end{cases} \end{cases}$$

The variables in the problem $(\mathcal{C})_{(x,X)}(\omega)$ have the following interpretation: s_i is the tensile force in the bar times its length; S_j is the tensile force in the cable; λ is the vector of contact forces. Using the quasi-orthogonality of C_1, contact forces λ are uniquely determined by (s, S) and depend continuously on them; this fact will be used without backward reference.

2.2 General stochastic minimum compliance problem

The general stochastic minimum compliance problem is:

$$(\mathcal{P}_1) \begin{cases} \min_{(x,X,s(\cdot),S(\cdot))} c^f(x,X,s(\cdot),S(\cdot),\lambda(\cdot)) := \int_\Omega \mathcal{E}(x,X,s(\omega),S(\omega),\lambda(\omega),\omega)\, \mathrm{P}(d\omega) \\ \text{s.t.} \begin{cases} \underline{x} \leq x \leq \overline{x}, \qquad 1_m^T x \leq v, \\ \underline{X} \leq X \leq \overline{X}, \quad 1_{r_2}^T X \leq V, \\ (s(\omega), S(\omega), \lambda(\omega)) \text{ solves } (\mathcal{C})_{(x,X)}(\omega), \text{ P-a.s.,} \end{cases} \end{cases}$$

where v and V are the limits on the amount of cable and structure material correspondingly. In this problem we minimize the *average* value of compliance for multiple load cases. In topology optimization we set lower bounds $\underline{x} = 0$ and $\underline{X} = 0$.

2.3 Stochastic stress constrained weight minimization problem

The formal problem formulation is as follows:

$$(\mathcal{P}_2) \begin{cases} \min_{(x,X,s(\cdot),S(\cdot))} w(x,X) := \rho_1 1_m^T x + \rho_2 1_{r_2}^T X \\ \text{s.t.} \begin{cases} \underline{x} \leq x \leq \overline{x}, \\ \underline{X} \leq X \leq \overline{X}, \\ |s_i(\omega)| \leq \overline{\sigma}_1 x_i, & i = 1,\dots,m, \quad \text{P-a.s.,} \\ L_j S_j(\omega) \leq \overline{\sigma}_2 X_j, & j = 1,\dots,m, \quad \text{P-a.s.,} \\ (s(\omega), S(\omega)) \text{ solves } (\mathcal{C})_{(x,X)}(\omega), & \qquad\qquad\quad \text{P-a.s.,} \end{cases} \end{cases}$$

where $\overline{\sigma}_1$ and $\overline{\sigma}_2$ are the maximal allowable effective stresses in, and ρ_1 and ρ_2 the densities of, the structure and the cable materials respectively. In this problem we require stress constraints to hold for *almost all* load cases, or we allow them to be violated *with probability zero*. In topology optimization we set lower bounds $\underline{x} = 0$ and $\underline{X} = 0$.

3 THEORETICAL RESULTS

3.1 Existence of solutions

Theorems 3.1 and 3.2 summarize conditions sufficient for the existence of optimal solutions to problems $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$.

Theorem 3.1 (Existence of solutions to $(\mathcal{P}_1)$). *Suppose that for some feasible point $(x_0, X_0, s(\cdot), S(\cdot))$ in the problem $(\mathcal{P}_1)$ we have $c^f(x, X, s(\cdot), S(\cdot), \lambda(\cdot)) < \infty$. Then, there exists at least one optimal solution to $(\mathcal{P}_1)$.*

Theorem 3.2 (Existence of solutions to $(\mathcal{P}_2)$). *Suppose that the following assumptions are satisfied: (i) the feasible set of the problem $(\mathcal{P}_2)$ is nonempty; (ii) $\mathrm{P}(E(\cdot) \geq c) = \mathrm{P}(E_c(\cdot) \geq c) = 1$ for some constant $c > 0$; (iii) the functions $L_j(\cdot)$, $g_1(\cdot)$, $g_2(\cdot)$, $C_1(\cdot)$, $B_i(\cdot)$ and $f(\cdot)$ are essentially bounded. Then there exists at least one optimal solution to the problem $(\mathcal{P}_2)$.*

3.2 Convergence of ε-perturbations

The replacement of the lower design bounds $(\underline{x}, \underline{X}) = 0$ with a small positive value $\varepsilon > 0$ tending to zero (whence the name — ε-perturbation), or the approximation with a sequence of *sizing* optimization problems, has become a classical solution approach to structural topology optimization problems. For compliance minimization problems such an approach is sufficient for approximating optimal solutions.

The situation with the stress constrained weight minimization is far more complicated. Sved and Ginos [6] observed that the problem may have singular solutions, which cannot be approximated by the simplistic approach outlined above. Cheng and Guo [7] proposed a more sophisticated relaxation procedure, where not only lower bounds but also stress constraints were perturbed. They showed the convergence of optimal values of perturbed problems to the optimal value of the original problem, while Petersson [8] showed the convergence of optimal solutions. Patriksson and Petersson [9] generalized the result for stochastic topology optimization problems with unilateral constraints and *discrete* probability measures.

Consider the following ε-perturbation of the problem $(\mathcal{P}_1)$:

$$(\mathcal{P}_1^{\varepsilon}) \begin{cases} \min\limits_{(x,X,s(\cdot),S(\cdot))} c^f(x, X, s(\cdot), S(\cdot), \lambda(\cdot)) \\ \text{s.t.} \begin{cases} \varepsilon 1_m \leq x \leq \overline{x}, & 1_m^T x \leq v, \\ \varepsilon 1_{r_2} \leq X \leq \overline{X}, & 1_{r_2}^T X \leq V, \\ (s(\omega), S(\omega), \lambda(\omega)) \text{ solves } (\mathcal{C})_{(x,X)}(\omega), \text{ P-a.s.} \end{cases} \end{cases}$$

Theorem 3.3 (Convergence of ε-perturbations for $(\mathcal{P}_1)$). *Suppose that for some $\varepsilon_0 > 0$ there is a solution $(x_0, X_0, s_0(\cdot), S_0(\cdot), \lambda_0(\cdot))$ that is feasible in $(\mathcal{P}_1)$ with $(x_0, X_0) \geq \varepsilon_0 1_{m+r_2}$ and $c^f(x_0, X_0, s_0(\cdot), S_0(\cdot), \lambda_0(\cdot)) < \infty$. For each $\varepsilon_0 \geq \varepsilon > 0$, let $(x_\varepsilon^*, X_\varepsilon^*, s_\varepsilon^*(\cdot), S_\varepsilon^*(\cdot), \lambda_\varepsilon^*(\cdot))$ denote an arbitrary optimal solution to $(\mathcal{P}_1^\varepsilon)$. Then any limit point of the sequence $\{(x_\varepsilon^*, X_\varepsilon^*, s_\varepsilon^*(\cdot), S_\varepsilon^*(\cdot), \lambda_\varepsilon^*(\cdot))\}$ (and there is at least one) is an optimal solution to $(\mathcal{P}_1^0) = (\mathcal{P}_1)$.*

For the stress constrained weight minimization problem we restrict ourselves to the very important special case of a truss without unilateral constraints, under stochastic loading.

Consider the following ε-perturbation of the problem $(\mathcal{P}_2)$:

$$(\mathcal{P}_2^\varepsilon)\begin{cases} \min\limits_{(x,s(\cdot))} w(x) \\ \text{s.t.} \begin{cases} o(\varepsilon)1_m \le x \le \overline{x} + o(\varepsilon)1_m, \\ |s_i(\omega)| \le \overline{\sigma}_1 x_i + \varepsilon, \qquad i = 1,\dots,m, \quad \text{P-a.s.}, \\ s(\omega) \text{ solves } (\mathcal{C})_x(\omega), \qquad \text{P-a.s.}, \end{cases} \end{cases}$$

where from the function $o : \mathbb{R}_{++} \to \mathbb{R}_{++}$ we only require the properties that $\{o(\varepsilon)/\varepsilon\}$ converges to zero while $\{o(\varepsilon)/\varepsilon^2\}$ is bounded away from zero (e.g., $o(\varepsilon) = \varepsilon^2$ satisfies these requirements).

Theorem 3.4 (Convergence of ε-perturbations for $(\mathcal{P}_2)$). *Suppose that the only source of uncertainty — loads $f(\cdot)$ are essentially bounded. Assume further that for some $\varepsilon_0 > 0$ there is a solution $(x_0, s_0(\cdot))$ that is feasible in $(\mathcal{P}_2)$ with $x_0 \ge o(\varepsilon_0)1_m$. For each $\varepsilon_0 \ge \varepsilon > 0$ let $(x_\varepsilon^*, s_\varepsilon^*(\cdot))$ denote an arbitrary optimal solution to $(\mathcal{P}_2^\varepsilon)$. Then any limit point of the sequence $\{(x_\varepsilon^*, s_\varepsilon^*(\cdot))\}$ (and there is at least one) is an optimal solution to $(\mathcal{P}_2^0) = (\mathcal{P}_2)$.*

Being an interesting theoretical result, the ε-perturbation based method has several drawbacks when it comes to algorithms. The presence of stress constraints in the problem does not allow us to use many numerical algorithms designed for MPECs without state constraints. Specifically, in the stochastic setting the presence of stress constraints does not allow us to construct discretizations of the problem: even though the stress constraints must hold with probability one they can be violated on some set of measure zero, which may happen to contain our discretization points. Therefore, we introduce an alternative convergent scheme, which in addition to adding the small lower bounds on the design variables moves the stress constraints into the objective function using a convex penalty function.

Let $G(x, s) := \sum_{i=1}^m [|s_i| - \overline{\sigma}_1 x_i]_+^2 / x_i$. Using the usual convention $0/0 = 0$ and $a/0 = \infty$ for any $a > 0$, the function G can be evaluated on any nonnegative design x. It is easy to check that G is l.s.c. on $\mathbb{R}_+^m \times \mathbb{R}^m$. Let $\mu : \mathbb{R}_{++} \to \mathbb{R}_{++}$ be an arbitrary function having a property $\lim_{\varepsilon\to+0} \mu(\varepsilon) = +\infty$ but $\lim_{\varepsilon\to+0} \varepsilon\mu(\varepsilon) = 0$. Consider the penalized problem:

$$(\bar{\mathcal{P}}_2^\varepsilon)\begin{cases} \min\limits_{(x,s(\cdot))} w^\varepsilon(x, s(\cdot)) := w(x) + \mu(\varepsilon) \int_\Omega G(x, s(\omega))\,\mathrm{P}(d\omega) \\ \text{s.t.} \begin{cases} \varepsilon 1_m \le x \le \overline{x} + \varepsilon 1_m, \\ s(\omega) \text{ solves } (\mathcal{C})_x(\omega), \quad \text{P-a.s.} \end{cases} \end{cases}$$

Theorem 3.5 (Penalty function approach to $(\mathcal{P}_2)$). *Suppose that the only source of uncertainty — loads $f(\cdot)$ are essentially bounded. Suppose further that for some $\varepsilon_0 > 0$ there is a solution $(x_0, s_0(\cdot))$, which is feasible in $(\mathcal{P}_2)$ with $x_0 \ge \varepsilon_0 1_m$. Then for any $0 < \varepsilon \le \varepsilon_0$ the problem $(\bar{\mathcal{P}}_2^\varepsilon)$ has an optimal solution $(x_\varepsilon, s_\varepsilon(\cdot))$. Any limit point of the sequence $\{(x_\varepsilon, s_\varepsilon(\cdot))\}$ is an optimal solution to $(\mathcal{P}_2)$.*

3.3 Distribution sensitivity

The analysis of stability of optimal solutions with respect to small changes in probability measure is of great importance. From the computational point of view it allows one to replace the original stochastic problem by a sequence of simpler problems, involving approximations (discretizations) of the probability measure. From the practical point of view, it asserts that solutions to the problem obtained using statistical estimations of the unknown stochastic distribution are "close" to exact solutions. From the theoretical point of view, it confirms the robustness of the probabilistic approach with respect to possible errors in the probability distribution.

Throughout this subsection we assume that Ω is a compact metric space, $\mathfrak{S} = \mathcal{B}(\Omega)$ and the only sources of uncertainty are the loads $f(\cdot)$, gaps $g_1(\cdot)$ and slacks $g_2(\cdot)$, which in addition are assumed to be continuous functions. We also assume that for all $(x, X) > 0$ the problem $(\mathcal{C})_{(x,X)}(\omega)$ is feasible for almost any $\omega \in \Omega$. We need to impose additional regularity properties on the sequence $\{\mathrm{P}_k\}$. Namely, we suppose that each measure P_k has a density $p_k(\cdot)$ with respect to a Lebesgue measure on Ω and that the sequence $\{p_k(\cdot)\}$ converges to a density $p(\cdot)$ of P Lebesgue-almost everywhere. The existence of densities is not a very restrictive assumption from the theoretical point of view, and it is usually assumed in engineering applications of probability theory.

Under these assumptions it is possible to show the continuity of the optimal solutions to the stochastic compliance minimization problem. We denote by $(\mathcal{P}_1)^k$ the problem $(\mathcal{P}_1)$ in which the measure P is substituted by P_k.

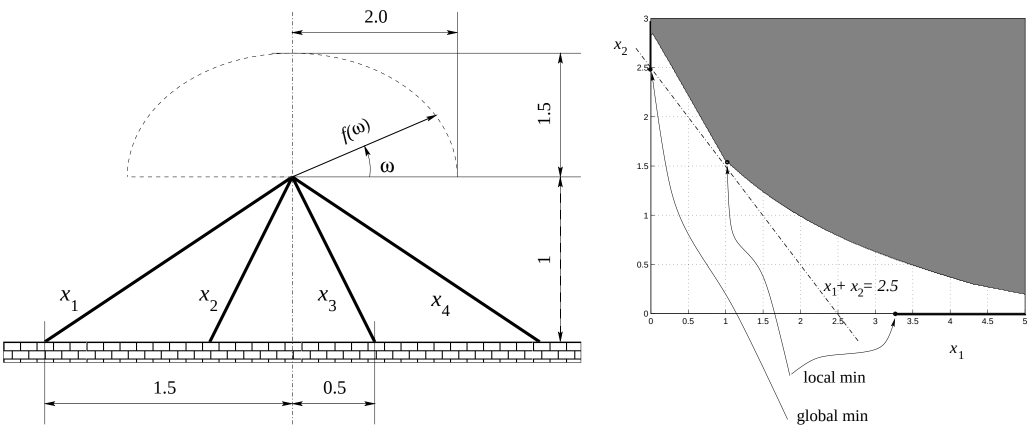
Theorem 3.6 (Robustness of solutions to $(\mathcal{P}_1)$). *Let $\{(x_k, X_k)\}$ be a sequence of optimal in $\{(\mathcal{P}_1)^k\}$ designs. Then any limit point (and there is at least one) of this sequence is a design, which is optimal in the limiting problem $(\mathcal{P}_1)$.*

Unfortunately, solutions to stress constrained weight minimization problems are not in general continuous with respect to small changes in probability measure. To ensure the continuity we need to make a very restrictive assumption that approximating measures and a limit measure have the same support, or else relax the model [5].

4 NUMERICAL EXAMPLES

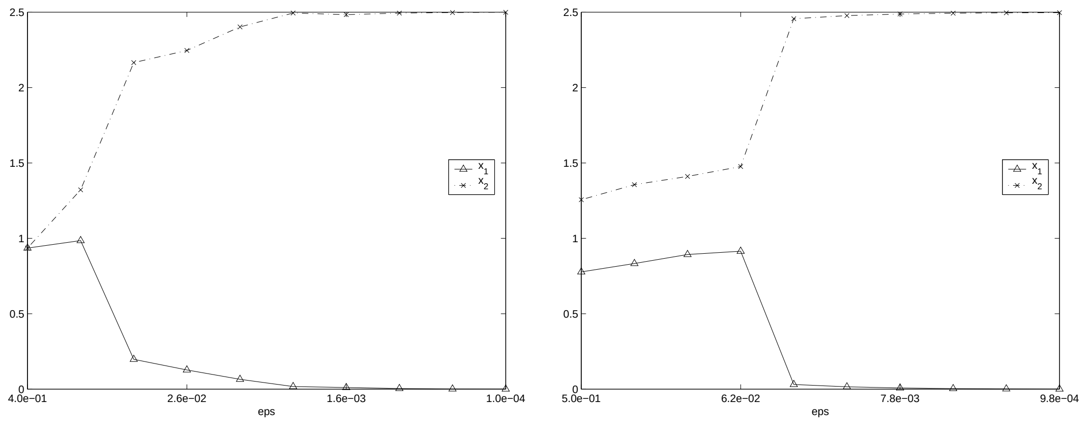
4.1 Convergence of ε-perturbations

Consider the problem of minimizing the weight of the 4-bar structure shown in Figure 1 (a). The stress limit for each bar is $\overline{\sigma} = 1$, and the Young's modulus is $E = 1$.



(a) (b)

Figure 1: The 4-bar truss problem (a) and the corresponding feasible design domain (b).



(a) (b)

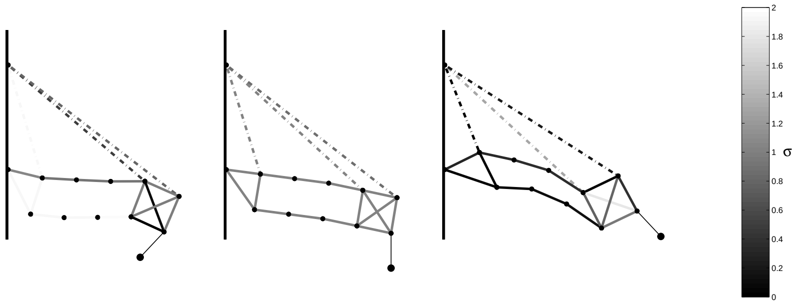
Figure 2: Convergence of the ε-perturbations (a) and the penalty function approach (b) for the 4-bar truss problem.

Assume that the upper design bounds are inactive, and that the force vector $f(\omega)$ equals $(2\cos(\omega), 1.5\sin(\omega))$, where $0 \leq \omega \leq \pi$. The probability measure is the uniform one on $[0, \pi]$. Since the initial structural topology as well as the loading conditions are symmetric, we can expect symmetric optimal solutions (i.e., $x_1^* = x_4^*$, $x_2^* = x_3^*$). Figure 1 (b) shows the projection of the set of feasible designs onto the linear subset $\{ x \in \mathbb{R}^4 \mid x_1 = x_4, x_2 = x_3 \}$. Note that the feasible set is not *a finite union of polyhedra*, because we work with an infinite number of load cases. Despite the large number of load cases, at the globally optimal solution, $x^* = (0, 2.5, 2.5, 0)$, the structural topology was modified (i.e., bars 1 and 4 were removed).

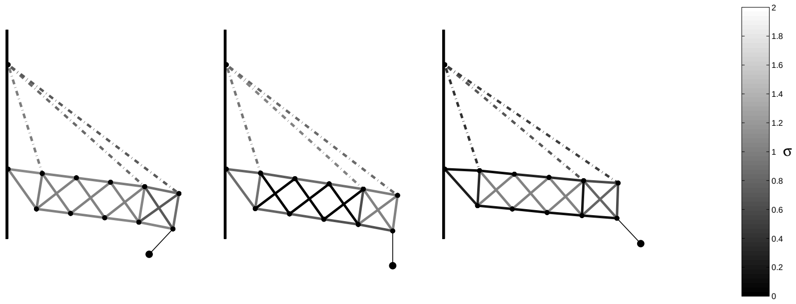
There are three local minima, two of which (including the globally optimal solution) are singular. The difference in the objective function value between the best non-singular local minimum and the globally optimal solution can be made much larger by a suitable choice of constants. The nonsingular non-global, local minimum of the original problem is the global minimum for the "naively" perturbed problem for all small values of ε. Therefore, we cannot approximate the globally optimal solution by the "naive" ε-perturbation. Both the "correct" ε-perturbation scheme and the penalty function approach (where we used $\mu(\varepsilon) = \varepsilon^{-0.8}$) allow us to recover the globally optimal solution, as shown in Figure 2.

4.2 Qualitative behaviour of optimal solutions

We consider a problem of finding a minimal weight of a cable suspended crane under stochastic loading. In this example the force is a unit vector with the direction uniformly distributed on $[-3\pi/4, -\pi/4]$. The number of bars is $m = 23$, and the number of cables is $r_2 = 5$. We set $\rho_1 = \rho_2 = 1.0$, the maximal cross-sectional area for both cables and bars equals to 1.0, the maximal stresses are $\overline{\sigma}_1 = 1.4$ and $\overline{\sigma}_2 = 0.8$, Young's moduli are $E = E_c = 1.0$, and initial slacks $g_1 = 0$. The behaviour of the single, average-load, optimal design under various loading conditions is shown in Figure 3 (a). The behaviour of a design optimized for a 625-point approximation of the probability measure is shown in Figure 3 (b).



(a)



(b)

Figure 3: Stresses and displacements for various random forces for optimal designs, corresponding to (a) single average-load and (b) 625 load cases. Note, for the sake of better visualization of stresses, line thicknesses are *not* proportional to cross-sectional areas, but instead choosen to have the *same* thickness. Lighter color means bigger stress.

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