TRAFFIC MANAGEMENT THROUGH LINK TOLLS—AN APPROACH UTILIZING SIDE CONSTRAINED TRAFFIC EQUILIBRIUM MODELS

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Abstract

We propose a systematic means for achieving a set of overall traffic management or planning goals with respect to the performance of the traffic network through the use of link tolls. The primary goals are defined by a set of link flow restrictions. The tolls that achieve these goals are obtained by solving a generalization of the classical user equilibrium model which includes a set of side constraints on the link flows. The set of toll prices obtained is not necessarily unique; this fact enables the traffic planner to choose a toll scheme which satisfies exogenous constraints and which may optimize a secondary goal, such as with respect to the toll itself. The overall model is derived as a special case of a mathematical program with equilibrium constraints (MPEC) describing a Stackelberg game involving the traffic manager and the users of the network. The model is shown to yield valuable information also in the case where the management goals and exogenous toll constraints are inconsistent with each other or with the underlying network. We give several examples of possible applications of the model, including the achievement of a system optimal flow and the derivation of actions for making public transport more attractive, and propose a conceptual algorithm for solving it. The paper is hoped to provoke continued research in both theoretical and applied directions.

Keywords: Stackelberg game; link tolls; mathematical programs with equilibrium constraints; side constrained traffic equilibrium; Lagrange multipliers; toll optimization; network design; system optimal flows; subsidies

1 Introduction

The criteria by which travellers choose their routes in the traffic network are, to some degree, in conflict with society's goal of utilizing the network efficiently: a traveller can be expected to most often choose a route which minimizes a combination of travel time and expenses while society's goal often is to have low average travel times and little damage to the environment. The traffic system may be viewed as a non-cooperative Stackelberg game, in which a traffic manager, represented as the leader, changes the infrastructure so as to achieve some overall management goal with respect to the distribution of the traffic in the network. The travellers are then modelled as the followers; they react to the infrastructure changes by modifying their behaviour, for example by adjusting their route choices or travel modes. If the manager's infrastructure changes are adequate, then the travellers' response is the desired one. Common means for achieving such a change in the traffic flows are to invest in traffic network capacity, to introduce traffic controls such as traffic lights and one-way traffic, to introduce tolls on some links, or some other network design measure, and to supply the travellers with information about alternative routes.

This work considers the use of a decentralized traffic control, namely the instrument of *link tolls*, to achieve a set of overall management or planning goals, and proposes a systematic means for deriving adequate toll prices. The primary goals considered can be quite general, and are defined by a set of desired restrictions on the link flows. Travellers are assumed to choose their routes in accordance with Wardrop's principle of user equilibrium; this means that the link tolls are to be set at a level such that the desired flow distribution is the result of an equilibration process where the travel costs equal the link tolls plus all other costs associated with the trips.

The means for deriving the proper link tolls are mathematical programming models which generalize the standard user equilibrium model by the introduction of *side constraints* describing the goals, and, in particular, the *Lagrange multipliers* associated with these constraints. The use of a mathematical model for the derivation of the link tolls is in contrast to heuristic toll schemes, based on trial and error, whose resulting traffic flows can never be anticipated. Further, the use of side constraints supports the process of identifying the proper goals to be formulated, and the definitions of the side constraints are quite flexible with respect to the goals that can be considered.

We analyze this model, in particular with regards to the properties of the set of tolls that will achieve the management goals, and the consequences of inconsistencies within the model. The latter is perhaps the most interesting part of the analysis, considering that in practice it is very likely that the management goals are unattainable, or that the tolls achieving them and those satisfying all exogenous toll constraints are incompatible. The work also supplies some example applications, and interesting topics for future research.

We limit ourselves to the study of a *static* model (an assumption which is essential[1]), with *fixed demands* and a *single class of users* having *additive link travel costs* (assumptions none of which are essential).

To introduce the setting of the problem studied and most of the notation used, we discuss the Wardrop equilibrium conditions in the next section. In Section 3, we introduce the general Stackelberg problem. Section 4 explains the basic properties of the special case of model studied in the context of toll networks. A conceptual algorithm, as well as the implications of inconsistencies in the model, is discussed in Section 5. Finally, Section 6 discusses some interesting applications.

2 Wardrop equilibrium

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A})$ be a strongly connected transportation network, where $\mathcal{N}$ and $\mathcal{A}$ are the sets of nodes and directed links (arcs), respectively. For certain ordered pairs of nodes, $(p, q) \in \mathcal{C}$, where node p is an origin, node q is a destination, and $\mathcal{C}$ is a subset of $\mathcal{N} \times \mathcal{N}$, there are positive travel demands d_{pq} (which for simplicity shall be assumed fixed) giving rise to a link traffic flow pattern.

Wardrop's user equilibrium principle states that for every origin–destination (OD) pair $(p, q) \in \mathcal{C}$, the travel costs of the routes utilized are equal and minimal. We denote by $\mathcal{R}_{pq}$ the set of simple (loop-free) routes in OD pair (p, q), by h_{pqr} the flow on route $r \in \mathcal{R}_{pq}$, and by $c_{pqr} = c_{pqr}(h)$ the travel cost on the route given the vector h of route flows; with this notation, an equilibrium flow is defined by the conditions

$$h_{pqr} > 0 \implies c_{pqr} = \pi_{pq}, \qquad \forall r \in \mathcal{R}_{pq}, \tag{2.1a}$$
$$h_{pqr} = 0 \implies c_{pqr} \geq \pi_{pq}, \qquad \forall r \in \mathcal{R}_{pq}, \tag{2.1b}$$

where the value of π_{pq} is the minimal (or, equilibrium) route cost in OD pair (p, q). An equilibrium state is reached precisely when no traveller can decrease his/her travel cost by shifting to another route.

The Wardrop conditions can be stated as a *variational inequality problem*: the conditions (2.1) are equivalent to h satisfying

[TAP-VIP-H]

$$-c(h) \in N_H(h),$$

[1]It does not, however, rule out the possibility of the existence of a natural dynamical extension of the model. Such a model could be of interest for the study of time-varying network conditions and corresponding management actions.

where $H = \{ h \in \Re_+^{|\mathcal{R}|} \mid \Gamma^{\mathrm{T}} h = d \}$ is the set of demand-feasible route flows (Γ^{T} is the route–OD pair incidence matrix), $c : \Re_+^{|\mathcal{R}|} \mapsto \Re_{++}^{|\mathcal{R}|}$ is the vector of route travel cost functions, $|\mathcal{R}|$ is the total number of simple routes in the network, and $N_H(h)$ denotes the *normal cone* to H at h, that is, the set

$$N_H(h) = \begin{cases} \{ z \in \Re^{|\mathcal{R}|} \mid z^{\mathrm{T}}(y - h) \leq 0, \quad \forall y \in H \}, & h \in H, \\ \emptyset, & h \notin H. \end{cases}$$

In the case where the travel cost of a route is the sum of the travel costs on the links defining it, or, in other words, the route costs are *additive*, then the Wardrop conditions can be described in terms of link flows. (This network description may be desirable, as the number of routes in a network can be extremely large.) Letting F denote the (bounded polyhedral) set of demand-feasible link flows, that is, the link flows f satisfying $f = \Delta h$ for some vector h in H where Δ^{T} is the link–route incidence matrix, the problem [TAP-VIP-H] can be equivalently written as

[TAP-VIP-F]

$$-t(f) \in N_F(f),$$

where $t : \Re_+^{|\mathcal{A}|} \mapsto \Re_{++}^{|\mathcal{A}|}$ is the vector of link travel cost functions. [The link and route costs are related by $c(h) = \Delta^{\mathrm{T}} t(f)$.] The flow polytope F can, equivalently by the positivity assumption on the link costs, be described by node conservation constraints.

For overviews of these models, see Nagurney (1993) and Patriksson (1994).

3 A general Stackelberg model

We start by formulating a general Stackelberg problem; later, we shall introduce reasonable assumptions on the nature of the actions taken by the traffic manager that will allow us to reformulate it into a much simpler problem.

Consider a situation in which the manager of a traffic system wishes to achieve certain goals with respect to the performance of the system. The goals are formulated as a set of flow restrictions of the form

$$g_k(f) \leq 0, \qquad k \in \mathcal{K}, \tag{3.1}$$

where $\mathcal{K}$ is a finite index set, and $g_k : \Re^{|\mathcal{A}|} \mapsto \Re$ are C^1 functions. The link flows that satisfy (3.1) form a closed set in $\Re^{|\mathcal{A}|}$, which we denote by G.

We let the vector $p \in \Re^{|\mathcal{A}|}$ represent the actions taken by the manager; the actions are assumed to result only in an influence upon the travel costs, according to some parameterization, $\bar{t}(f, p)$, of the link travel cost. There is a large freedom in interpretation of these actions, and several possible examples were given in the introduction.

These actions are further assumed to be restricted to some set P, which may be determined by political, practical, environmental and economical constraints, and possibly other considerations as well. Among the possible actions and flows in G, the manager optimizes a function, $\varphi : P \times F \mapsto \Re$ of the actions taken and resulting link flows. This function may include some further measures of network performance as well as measures of the cost and/or benefits associated with a given action. (We are deliberately imprecise in the description of this general model, since it will soon be reduced to a simpler model which we shall deal with in more detail.)

Example 3.1 (Capacity constraints) The most immediate example of a set of desired flow restrictions is that of upper bounds on some links' flows. In the framework of (3.1), such constraints are described by

$$g_a(f_a) := f_a - u_a \leq 0, \qquad a \in \underline{\mathcal{A}}, \tag{3.2}$$

where $\underline{\mathcal{A}} \subseteq \mathcal{A}$ and $u_a \geq 0$ is the upper bound on the flow on link a. □

Remark 3.2 (Types of restrictions) The restrictions are given in terms of *link flows*. Since link flows are the likely ones to be observed and those that may be affected through control measures, this is a natural assumption. We may, however, extend the situation described to cover also the cases where the side constraints are given in *route flow* variables, and the restrictions distinguish between *different classes of travellers and/or vehicles*.

The latter extension could serve as a modelling tool for taking into account different *values-of-time* among the travellers; further, it can be used to derive proper tolls on links leading into a given residential area for tourists but not for residents there, and tolls and/or additional queueing delays for private vehicles in order to favour public transport. We discuss these applications in more detail in Section 6, and note here only that the introduction of commodity-specific restrictions demands a further requirement of the underlying equilibrium model: if the restrictions are to be fulfilled through the use of (commodity-specific) tolls, then the equilibrium model must be such that the commodity flows are uniquely determined. □

This development results in a *mathematical program with equilibrium constraints*:

[MPEC]

$$\text{minimize } \varphi(p, f) \tag{3.3a}$$

subject to

$$\begin{aligned} p &\in P, && (3.3\text{b}) \\ f &\in G, && (3.3\text{c}) \end{aligned}$$

where

$$-\bar{t}(f,p) \in N_F(f). \qquad (3.3\text{d})$$

Remark 3.3 (Uniqueness of user equilibrium flows) Should $\bar{t}(\cdot,p)$ be a strictly monotone mapping for every feasible choice of p, the lower-level (follower) problem (3.3d) has a unique solution, and as such f can be described as $f(p)$. Then, the function φ is in fact a function of p only. □

Mathematical programs with equilibrium constraints generalize bilevel optimization problems, and are in general non-convex. [For an overview of bilevel problems in transportation research, see Migdalas (1995).]

Example 3.4 (Network design) A familiar form of the *equilibrium network design problem* is an instance of [MPEC]. Let p denote an investment in network capacity; the effect of an investment is that of a reduced travel time; $\bar{t}_a$ is often taken to be $\bar{t}_a(f,p) = t_a(f_a/p_a)$. An investment p_a is associated with an investment cost, $\psi_a(p_a)$. The goal is to minimize the total travel time (at a user equilibrium flow) plus the investment costs, that is, $\varphi(p,f) = \sum_{a\in\mathcal{A}}\{\bar{t}_a(f,p)f_a + \psi_a(p_a)\}$, while satisfying constraints on the investments made, $P = \{\, p \in \Re^{|\mathcal{A}|} \mid \ell \leq p \leq u;\ \sum_{a\in\mathcal{A}} p_a \leq U\,\}$. In the general formulation of [MPEC], network performance is measured also by constraints in the model. □

Example 3.5 (Signal control) A problem of a form similar to the equilibrium network design problem is the *signal setting problem.* The solution of this problem aims at finding a set of signal control parameter values which, under user equilibrium conditions, optimizes some measure of the performance of the network, such as the total queueing delay. In this case, then, the variables p are the control parameters, for example the *green times* allocated to the signal controls, and the parameterized travel cost mapping $\bar{t}(\cdot,p)$ measures the travel times and the delays at intersections. (See Cantarella and Sforza, 1987, and Smith and Van Vuren, 1993, and references therein for examples of traffic control policies and mathematical models.) □

Remark 3.6 (Generalized Nash games) A Stackelberg game in which the leader imposes constraints on the followers' strategies is discussed by Harker (1991) in the context of generalized Nash games. He also derives the incentives (taxes or subsidies) that must be given to the players in order to make them obey the constraints, in the form of their dual variables. The type of constraints considered is, however, rather special, and there is no apparent relationship between this model and the one developed in this paper. □

In the following, we shall introduce assumptions on what the actions p represent which reduce the problem [MPEC] to a sequence of two (potentially) well-behaved problems.

4 A toll-based model

We restrict our attention to considering an action of particular interest: on each link (at least potentially), a fixed toll is levied; thus, the influence on the travellers can be described by $\bar{t}(f,p) = t(f) + p$, where p is the toll fee.

Introducing some assumptions on this toll scheme, we first argue that it is natural to assume that link tolls are introduced only on those links where they are needed for achieving the goals; in other words, tolls should not be levied on links that do not contribute to the saturation of any restriction in (3.1). We also assume that the toll levied on a link can be decomposed into the contributions of all the different restrictions in (3.1), in a proportional manner according to their contribution to the saturation of the different restrictions; their contribution is measured in terms of derivatives of the constraint functions. (This decomposition principle makes it possible to investigate the economics of each flow restriction, and facilitate the economical evaluation of the toll scheme; the proportionality principle seems natural from the viewpoint of fairness.)

The implication of these conditions is that the link toll is distributed according to

$$p_a = \sum_{k \in \mathcal{K}(f)} \beta_k \frac{\partial g_k(f)}{\partial f_a}, \qquad a \in \mathcal{A}, \tag{4.1}$$

where $\mathcal{K}(f)$ is the subset of $\mathcal{K}$ for which $g_k(f) = 0$, and β_k, $k \in \mathcal{K}$, are nonnegative constants. (In vector notation, then, $p = \nabla g(f)\beta$, with $\beta^{\mathrm{T}} g(f) = 0$ and $\beta \geq 0$.)

Example 4.1 (Capacity constraints) In the case of simple upper bounds (see Example 3.1), the expression (4.1) reduces to $p_a = \beta_a$, $a \in \underline{\mathcal{A}}$. In applications to queueing networks (see Larsson and Patriksson, 1995a and 1995b), p_a is interpreted as the equilibrium queueing delay on link a. The above complementarity condition then states that there is a queueing delay ($p_a > 0$) only if the link is saturated ($f_a = u_a$). □

The effect of assumption (4.1) is more far-reaching than what might appear at first sight. The main effect is that of reducing the two-level problem into a sequence of two single-level ones; the reason is that what appears to be a mere definition is actually a complementarity condition, linking the two variable groups. Further, these single-level problems can, under some additional, but standard, assumptions, be formulated as convex programs.

To avoid having to deal with a lot of mathematical details without reaching a much deeper insight or generality, we will henceforth assume that G *is a convex set which further satisfies a constraint qualification* (e.g., Bazaraa *et al.*, 1993, Chapter 5) and, until further notice, that $F \cap G$ *is non-empty.* (The case of inconsistency is covered in Section 5.) (In the case where all functions g_k are affine, the former requirements are fulfilled automatically.) It follows immediately from the assumptions that $F \cap G$ is non-empty, compact and convex.

With these assumptions in place, we observe that the requirements (3.3c)–(3.3d) simplify to the primal–dual variational inequality problem of finding $(f, p) \in F \times \Re^{|\mathcal{A}|}$ satisfying[2]

$$
\begin{aligned}
-[t(f) + p] &\in N_F(f), \qquad (4.2a)\\
p &\in N_G(f). \qquad (4.2b)
\end{aligned}
$$

Remark 4.2 (Properties of (4.2)) (a) The assumptions made on the toll scheme have the effect of making it possible to place the constraints (3.3c) at the lower level of [MPEC]; this is instrumental in making the problem tractable.

(b) The formulation (4.2) is valid also when the management goals define non-convex constraints, through the proper re-definition of the normal cone mapping N_G.

(c) It is clear from the expression (4.2b) that the toll vector p is completely independent of the way in which the set G is described in terms of constraints. If, for example, the restrictions in (3.1) are given in different units or are scaled differently, a rescaling into the same units will not be necessary. Also, clearly, any presence of redundancy in the description of the set G will have no effect whatsoever on the resulting toll vector.[3] □

In order for the vector p to be naturally associated with a toll, that is, p is non-negative, $N_G(f) \subseteq \Re^{|\mathcal{A}|}_+$ must hold at every solution f to [TAP-VIP-

[2]The relation (4.2a) is equivalent to (3.3d), while (4.2b) is a restriction of (3.3c); the latter is easily verified by observing that, by convex analysis, the relation (4.2b) is equivalent to f being in the face of G exposed by the vector p, which is a subset of G.

[3]This fact is highlighted when describing the variational inequality in terms of normal cones; it was one of the main motivations for choosing this form of presentation. Another is that the traditional way of describing a variational inequality is not useful in the case of non-convex constraints.

SC-F]. A sufficient condition for this to hold is that $\frac{\partial g_k(f)}{\partial f_a} \geq 0$ for all $a \in \mathcal{A}$ and $k \in \mathcal{K}$ [cf. (4.1)]. If the condition is not fulfilled, then some link tolls can be negative, whence the "toll" should be regarded as a subsidy or a network improvement. (A natural condition is, however, that $t(f) + N_G(f) \subseteq \Re_+^{|\mathcal{A}|}$ holds.) (Applications of negative tolls are described in Section 6.4.)

Example 4.3 (Capacity constraints) The case of upper bounds on the link flows (cf. Examples 3.1 and 4.1) is one in which, clearly, tolls will be non-negative. □

We now observe that, by convex analysis, $N_{F\cap G}(f) = N_F(f) + N_G(f)$ holds everywhere, and so solving the system (3.3c)–(3.3d) reduces to finding a solution f to

[TAP-VIP-SC-F]

$$-t(f) \in N_{F\cap G}(f).$$

The above assumptions ensure the existence of a solution to [TAP-VIP-SC-F] (e.g., Hartmann and Stampacchia, 1966). This problem was studied in detail in Patriksson (1994) and Larsson and Patriksson (1995b and 1996) (in the optimization setting), and in Larsson and Patriksson (1994), in the context of descriptive models of traffic equilibria including *queueing effects*; the above development shows that the model is also intimately associated with *link tolls*.

Remark 4.4 (Queueing models) (a) In applications of side constrained traffic equilibria to queueing networks, we have in Larsson and Patriksson (1994 and 1996) derived natural conditions on the queueing dynamics that ensure the existence of an equilibrium in both link flows and queues. These conditions, and the results obtained, can be repeated verbatim (with the word 'queue' replaced by the word 'toll' throughout) for toll networks, thus obtaining an existence result for an equilibrium in both link flows and tolls, given natural conditions on the properties of the tolls during a non-stationary (disequilibrium) state. (Essentially, as time passes the toll contributed by a restriction in (3.1) increases to infinity if the flow violates the restriction, decreases to zero if the restriction is strictly fulfilled, and stays constant if the restriction is saturated exactly.)

(b) The descriptive and prescriptive modelling approaches can of course be combined into a single bilevel model, in which, then, the lower level describes a queue equilibrium model with toll parameters. □

One cannot assume in general that the set of multipliers β_k for the flow restrictions (3.1) is a singleton, and therefore, while the link flow solution is unique whenever the travel cost mapping t is strictly monotone, the toll vector p which yields the desired equilibrium solution is not. In the context of queueing networks, this can be considered a drawback, since the vector p is, in that case, interpreted as a queueing delay. So, in Larsson and Patriksson (1994 and 1996), sufficient conditions were therefore derived under which this equilibrium link queueing delay is unique. In the context of traffic management through link tolls, however, *the non-uniqueness of the multipliers β is a positive feature of the model, since it facilitates the fulfillment of the traffic management goals while also satisfying exogenous restrictions (practical, political, environmental and economical constraints) on the toll scheme. Furthermore, one could choose a toll vector which simultaneously optimizes the secondary performance function φ.* The non-uniqueness of the tolls is, seemingly, a property with many interesting consequences; we are, however, not aware of any previous observations of this fact, besides those for the system optimal problem (see Section 6.3), and of no model in which exogenous constraints on the tolls are considered. In the next section, we shall elaborate on this further, and study means for solving the toll optimization model.

5 Toll optimization

In this section, we study the upper-level problem of [MPEC] in terms of the toll vector p. We then introduce the further assumption that *t is strictly monotone on F*, and denote the unique link flow solution to [TAP-VIP-SC-F] by f^*. The set of toll vectors which achieves this solution while satisfying the requirement (4.1) is given by the set of solutions to (4.2) with $f = f^*$. This set, which we will denote by T, is clearly a non-empty polyhedral set, since F is described by linear constraints and G satisfies a constraint qualification, so the respective normal cones at f^* are polyhedral.

5.1 The toll optimization problem

The toll optimization problem is

[TOP]

$$\underset{p \in P \cap T}{\text{minimize}} \ \varphi(p).$$

This problem is convex, whenever *the set P of feasible toll schemes is convex, and the function $\varphi : \Re^{|\mathcal{A}|} \mapsto \Re$ is convex.* (It is further a linear program if P is polyhedral and φ is a linear function.) To be well-defined, a further requirement on the problem is that *$P \cap T$ is non-empty.* The convexity and non-emptiness conditions are assumed to hold for the present. We shall further assume throughout that P *is bounded*, and that *every vector* $p \in P$ *satisfies* $\sum_{a \in \mathcal{A}_C}[t_a(f^*) + p_a] \geq 0$ *for every compatible cycle* $\mathcal{A}_C$ *in the network*, that is, there is no feasible toll vector such that a traveller can benefit by travelling a full cycle. These natural conditions ensure the existence of an optimal solution to [TOP] (under the above conditions), and that the algorithm proposed for solving it is well-defined.

Example 5.1 (Secondary goals) Several optimization models of the form [TOP] are conceivable. We may wish to levy the least total toll in the network, whence we choose $\varphi(p) = \sum_{a \in \mathcal{A}} p_a f_a^*$, or precisely the opposite; the function φ may also include a measure of the cost of implementing toll stations on the different links, which we naturally would like to minimize. □

5.2 An algorithm for the toll optimization problem

5.2.1 Solving the problem [TAP-VIP-SC-F]

We consider solving [TAP-VIP-SC-F] using a nonlinear pricing scheme which simultaneously generates f^* and a vector β of multipliers for the side constraints. One such example is the *(proximal) method of multipliers* of Rockafellar (1978). Reasons for choosing such a scheme are, at least, threefold: (1) in the optimization setting, the corresponding *augmented Lagrangean* dualization schemes have been found to work very well in numerical experiments (Hearn and Ribera, 1980; Larsson and Patriksson, 1995a); (2) a vector β (and hence, through relation (4.1), a toll vector $p \in T$) is obtained through a convergent sequence of dual vectors; and (3) even in the case of inconsistency (that is, $P \cap T = \emptyset$), a toll vector with interesting properties is produced. (More on the last point below.) We refer the reader to Hearn and Ribera (1980), Patriksson (1994) and Larsson and Patriksson (1994 and 1995a) for further details on the properties of this class of schemes in the context of traffic problems.

Within this scheme a sequence of subproblems of the form [TAP-VIP-F] are constructed and solved; in these problems, the mapping t is augmented by primal–dual terms associated with the side constraints. When selecting an efficient algorithm for their solution, we note these important points: (1) the algorithm should have efficient primal reoptimization capabilities, to utilize that successive subproblems differ only in the travel cost mapping; this suggests using an efficient primal algorithm; and (2) when solving the toll problem [TOP], we must have an explicit description of the set T of optimal tolls, or a good approximation thereof; this suggests the use of column generation/cutting plane approaches.

The algorithm chosen is the *disaggregate simplicial decomposition* (DSD) method, which has been successfully applied in previous works on optimization versions of [TAP-VIP-F] (Larsson and Patriksson, 1992) and [TAP-VIP-SC-F] (Larsson and Patriksson, 1995a).[4]

We refer the reader to those references and Patriksson (1994, Sections 4.2.3, 4.3.5 and 5.3.5) for details on this method, but mention here that, at termination, the algorithm stores those routes in each OD pair that have a positive flow at some optimal route flow solution. In the application to [TAP-VIP-SC-F], then, the DSD algorithm will provide, for each OD pair (p,q), the routes in $\mathcal{R}_{pq}$ which are utilized in *one* route flow solution h to [TAP-VIP-SC-F]; these routes will also [cf. (4.2a)] all be least-cost routes given the fixed link travel cost vector $t(f^*)+p$, where $p = \nabla g(f^*)\beta$ and β is the vector of multipliers for (3.1) which is obtained in the limit of the multiplier scheme.

5.2.2 A regularity condition

Using linear programming duality on (4.2a), the set T can be described as the set of solutions in p of the following linear system in p, β and π:

$$\begin{aligned}
\Delta^{\mathrm{T}}[t(f^*)+p] - \Gamma\pi &\geq 0, && (5.1a)\\
[t(f^*)+p]^{\mathrm{T}} f^* - d^{\mathrm{T}}\pi &= 0, && (5.1b)\\
p - \nabla g(f^*)\beta &= 0, && (5.1c)\\
\beta^{\mathrm{T}} g(f^*) &= 0, && (5.1d)\\
\beta &\geq 0; && (5.1e)
\end{aligned}$$

(5.1a)–(5.1b) is a dual description of the normal cone inclusion of (4.2a), and (5.1c)–(5.1e) is an explicit description of the corresponding inclusion of (4.2b). The constraints (5.1a) render this system intractable in practice, since it contains as many constraints as there are routes in the network (cf. the dimensions of Δ^{T} and Γ); the constraints actually needed, however, are defined by the subset corresponding to the union of the routes being at minimum cost for *some* multiplier vector β, and should be comparatively very small.

Larsson and Patriksson (1994) introduce a regularity condition on the solution to [TAP-VIP-SC-F], so that the set of least-cost routes is invariant over the set of multipliers β for the side constraints, and hence over the set $p \in T$ (cf. ibid, Thm. 2.6). The condition says that, for each such vector, the Wardrop conditions are satisfied with *strict complementarity*, that is, that all routes with the minimal cost are actually used. In this case, using the DSD

[4] One complication with the DSD algorithm, as with any pricing scheme which includes shortest-route calculations, is that for some non-optimal multiplier values, negative cycles may appear. In such cases, one may either derive and incorporate a dual constraint in the scheme, or, preferably, continue the algorithm, ignoring the route generation stage.

algorithm for the traffic equilibrium subproblems in the multiplier scheme will yield all the information necessary to describe the (interesting subset of the) toll set T explicitly.

If this is not the case, then the least-cost routes are not the same for all values of the multiplier vector. Although, as remarked earlier, the description of the toll set T requires the union of the least-cost routes over *all* values of the multiplier vector β in the solution to [TAP-VIP-SC-F], the set of routes provided by the DSD algorithm will most probably be a very good approximation of the total number of least-cost routes needed (it is, for example, quite conceivable that regularity holds at least for some OD pairs), and that is the main motivation for using the DSD algorithm in this context.

5.2.3 The case of primal inconsistency

If the traffic manager has set unattainable goals, then the result will be $F \cap G = \emptyset$. Under this circumstance, the algorithm will still provide the manager with useful information. In fact, the algorithm will provide *the smallest possible adjustment of the manager's aspiration levels so that the goals become achievable*, and also provide *the toll vector* $\hat{p} = \nabla g(\hat{f})\hat{\beta}$ *that will achieve these slightly less ambitious aims.*

More formally, in the optimization setting, we may apply Theorem 4.1 of Golshtein and Tretyakov (1996, Sec. 3.4) to conclude that (a very small adjustment of) the algorithm stated above will yield a primal–dual pair $(\hat{f}, \hat{\beta})$ which solves a *perturbation* of [TAP-VIP-SC-F], in which the original restrictions (3.1) have been replaced by

$$g_k(f) \leq \varepsilon_k, \qquad k \in \mathcal{K},$$

the vector of ε_k being that of *smallest Euclidean norm* which makes the problem consistent.[5]

Example 5.2 (Goal programming approach) A traffic manager can utilize this scheme to evaluate several scenarios, possibly involving goals which are deliberately chosen to be unattainable. In contrast to the consistent case [cf. Remark 4.2.c], the resulting toll vector is sensitive to the introduction of scaling factors on the goals. Should the manager be uncertain if the goals are attainable, he/she can fulfill the most important goals while sacrificing others, through the use of different scaling factors on the goals according to their respective importance (cf. the use of aspiration levels and reference goals in multi-criteria optimization). □

[5]The perturbation ε found is actually the projection of the origin onto the effective domain of the perturbation function.

Having observed that the method detects primal inconsistency, and also in this case yields information that will be of use to the traffic manager, we move on to consider the problem [TOP] under the assumption that *the problem* [TAP-VIP-SC-F] *is consistent.*

5.2.4 An algorithm for [TOP]

The above discussion can be concluded with the remark that, at termination of the multiplier algorithm, the link flow solution f^* to [TAP-VIP-SC-F] and a toll vector $p \in T$ are given. The subset of routes generated within the DSD algorithm corresponds to a subset of the constraints in (5.1a). With this as the starting solution and information about the toll set T, a conceptual algorithm for [TOP] proceeds as follows.

We begin by noting that although the toll vector p is in T, it is not necessarily in the set P of feasible tolls. The fact that we do not know the set T explicitly further complicates the search for a toll vector which is simultaneously feasible (i.e., belongs to P) and optimal (i.e., belongs to T), should such a vector exist. The starting point in the solution of [TOP] is a *Phase-I* method, which attempts to find such a vector.

With the vector p as a starting point, we search for a point in the intersection of P and the set of tolls described by the subset of constraints in (5.1) currently known. (We shall use the notation $\overline{T}$ for this set.) This is an example of the problem of finding a point in the intersection of a finite number of convex sets, and a variety of methods exist for its solution (e.g., Bauschke and Borwein, 1996). Preferably, we should always stay in the set P, since it is to be considered to be described by *hard* constraints, while the set T is described by *soft* constraints.[6] We note that in the case where P is described by linear constraints, the Phase-I problem is an ordinary linear Phase-I problem, as solved within a simplex method.

Two outcomes are possible from this procedure. In the *first* case, the procedure reports that there is a positive distance between the sets P and $\overline{T}$. Since $T \subseteq \overline{T}$ holds, it must therefore be the case that the problem [TOP] is inconsistent, that is, $P \cap T = \emptyset$. (We refer to this property as *dual inconsistency.*)

The Phase-I method described above will, at termination, then have at hand a vector p which belongs to the set P (and is hence feasible), but which does not belong to the set T. This implies that *the link flow f^* solving* [TAP-VIP-SC-F] *is not possible to obtain through the use of a link toll scheme which is feasible with respect to the set P.* (The toll vector obtained through the solution of [TAP-VIP-SC-F], for example, then clearly is infeasible.) We can, however,

[6] We can envisage the set P to be described by much simpler constraints than those describing T, and hence such a procedure should be realistic.

expect that by using the toll vector obtained, at least a subset of the goals described by (3.1) will be achieved, and hence, *the toll vector obtained will achieve, by way of the Phase-I procedure, the highest aspiration levels possible within the limits set by the toll constraints.*

The second outcome possible is that the distance is reported to be zero, and then the output is a toll vector belonging to $P \cap \overline{T}$. The reader is, however, warned against interpreting this as a confirmation that the problem [TOP] is consistent.

This toll vector, $\overline{p}$, together with the vectors $\overline{\beta}$ and $\overline{\pi}$, obtained from the solution to the above Phase-I problem, is next used to investigate if $\overline{p} \in T$ actually holds, by checking whether any constraints in (5.1a) are violated. With the link costs $t(f^*) + \overline{p}$ at hand, the most violated, if any, constraint in (5.1a) for an OD pair (p, q) is obtained by finding the shortest route, $r \in \mathcal{R}_{pq}$ say; the constraints (5.1a) for the OD pair are satisfied if and only if $\sum_{a \in \mathcal{A}} \delta_{pqra}[t_a(f^*) + \overline{p}_a] \geq \overline{\pi}_{pq}$; otherwise, the most violated constraint has been identified, and the corresponding route is added to the subset of $\mathcal{R}_{pq}$ currently known. Performing this test for each OD pair will either result in termination of the Phase-I procedure, with a toll vector $\overline{p} \in T$ at hand, or to the generation of constraints (routes) in some OD pairs. In the latter case, a better outer approximation $\overline{T}$ of T has been found, and the Phase-I problem is resolved. This repeated procedure will terminate finitely, since the number of routes is finite. At termination, the procedure either reports that $P \cap T = \emptyset$, or provides a vector $\overline{p}$ in this set.

Having already discussed the case of dual inconsistency, we henceforth assume that *the Phase-I procedure was terminated with a toll vector in* $P \cap T$,[7] and hence *the problem* [TOP] *is consistent.* We next turn to the Phase-II part of the algorithm for [TOP].

In the second phase, we solve the modification of [TOP], where T is replaced by $\overline{T}$.[8] Let $(\overline{p}, \overline{\beta}, \overline{\pi})$ be a solution to this problem.[9] We next investigate if this solution solves [TOP], by checking whether any constraints in (5.1a) are violated. This is done in precisely the same manner as described above in the Phase-I procedure. Performing this test for each OD pair will either result in termination, with $(\overline{p}, \overline{\beta}, \overline{\pi})$ being a *proven optimal solution to* [TOP], or to the generation of constraints (routes) in some OD pairs. In the latter case, a better outer approximation of T has been found, and $(\overline{p}, \overline{\beta}, \overline{\pi})$ is used as an infeasible starting solution when solving the improved approximation of

[7]This solution also provides an upper bound on the optimal value of [TOP].

[8]We do not discuss particular methods for the solution of these restrictions in this paper; the adequate method will be a function of the properties of the side constraints and the toll restrictions.

[9]Since φ is optimized over the intersection of P and an outer approximation of T, $\varphi(f^*, \overline{p})$ is a lower bound on the optimal value of [TOP].

[TOP]. Obviously, since the number of routes is finite, this procedure is finite, and its output is a feasible and optimal toll vector, p^*, to [TOP].[10]

The algorithm stated here is by no means the only one possible; an alternative constraint generation approach would be the result of using an aggregate simplicial decomposition (SD) algorithm for the [TAP-VIP-F] type subproblems in place of the DSD algorithm. The route generation process is then replaced by the generation of all-or-nothing solutions. Convergence of the disaggregate algorithm given should, however, be much faster, since the latter algorithm corresponds to aggregating constraints generated in the former.

6 Applications and relations to previous work

6.1 Capacity constraints

6.1.1 Models

In the case of capacity constraints (see Example 3.1), the toll p_a equals the Lagrange multiplier β_a associated with the corresponding constraint in (3.2); see Example 4.1. Hearn (1980) uses these multipliers to define a perturbation of the travel time formulas so that an equivalent, un-capacitated assignment model can be given. Characterizations of solutions to capacitated traffic assignment models have been presented also in the case where the capacity side constraints are the result of the presence of traffic controls; then, the multipliers β are the equilibrium queueing delays at the signalized junctions. (See Larsson and Patriksson, 1995a, for an overview of capacitated models and their properties.) The use of the multipliers as tolls is implicit in the work of Beckmann and Golob (1974), albeit for a model governed by the principle of system optimality.

Ferrari (1995) considers an elastic demand model with *environmental* capacity side constraints, and establishes the existence of an equilibrium which satisfies the capacity constraints in the presence of proper additional costs on links that would otherwise produce unacceptable environmental damage.

It is to be noted that the above models define special cases of that developed here. Further, in none of these works is it observed that the tolls that should be levied to achieve the goals are not necessarily unique; consequently, the interesting applications that are brought forward by this fact are not recognized.

[10]Convergence of this algorithm can also be monitored by the bounds on the optimal value of [TOP] generated within the algorithm, and which are remarked upon in previous footnotes.

6.1.2 Methods

Algorithms for the generation of f^* solving this special case of [TAP-VIP-SC-F] appear in Hearn and Ribera (1980) and Larsson and Patriksson (1995a). Both of these are multiplier methods which simultaneously provide a multiplier vector β; the latter utilizes the DSD algorithm for the subproblems, and is therefore an instance of the algorithm for [TAP-VIP-SC-F] described above.

6.2 Obtaining a particular flow solution

An interesting special case of a goal described by a set of side constraints is that of achieving an *a priori* given flow solution f^*. We may consider several instances of such solutions; for example, f^* may be a calculated system optimal flow or be based on an observed flow. The model [MPEC] applies to such models, with the set of side constraints described by

$$g_a(f_a) := f_a - f_a^* = 0, \qquad a \in \mathcal{A}. \tag{6.1}$$

Tracing the above analysis, we find that the normal cone mapping N_G is non-empty exactly at f^*, where it equals $\Re^{|\mathcal{A}|}$. Hence, the requirement (4.2b) poses no further restrictions on p. This implies, in turn, that the solution of [TOP] is simplified substantially.

Example 6.1 (Extension of the toll set T) Let f^* be the solution to [TAP-VIP-SC-F]. Using model (6.1) will result in a larger freedom in choosing the tolls compared to that given by the toll set T; it will then be possible to levy a toll on a link which does not contribute to the saturation of any flow restrictions. □

If the given flow f^* does not belong to F, then it is not possible to achieve with any toll vector. In this case, the use of the multiplier scheme outlined in Section 5.2 will provide the toll which *achieves the demand-feasible flow which is closest in Euclidean norm to f^**. So, again we observe that the traffic management scheme given in this paper provides valuable information also in the case where the goals are unattainable.

Example 6.2 (Goal targets) In the model of Nagurney and Ramanujam (1996) the flow f^* is a *goal target*; deviations from it are penalized through the use of penalty functions which are used to derive taxes or subsidies on the links depending on whether a link flow is above or below the target value. Only in the case where f^* is consistent with F can the two models produce the same result; the output from our model in the case of inconsistency was discussed above; the output from their model depends heavily on the choice of penalty functions, guidelines to which are not provided. □

6.3 System optimal flows

Obtaining a system optimal flow by means of link tolls is a classic topic, with contributions tracing back to Pigou (1920). The traditional approach is *marginal cost pricing*, in which the toll levied equals the marginal cost minus user cost. Dafermos and Sparrow (1971) observe that *this toll is not the only one possible*, and that it may have certain drawbacks. (For example, a toll will be levied on every link.) They proceed to describe the set of toll vectors which will yield the desired link flow solution, and discuss possible secondary goals to direct the choice of toll vector.

Alternative tolls for achieving a system optimal flow have much later been considered also by Bergendorff and associates.

Bergendorff (1995) describes a particular case of capacitated model, wherein the upper bounds are the system optimal link flows. Based on this model, she develops a toll optimization problem similar to the one in this paper, but with no exogenous constraints on the toll vector, and presents a method, working in the spaces of commodity link flows and node prices, for solving linear instances of [TOP].

Bergendorff *et al.* (1996) present characterizations of the set of toll vectors additional to those of Dafermos, under various assumptions on the travel costs.

It is to be noted that the results reached in these references are, in essence, special cases of the results obtained in this paper, obtained by choosing side constraints of the form (6.1) with f^* being a system optimal solution.

Through the generality of the traffic management scheme described in this paper, we can consider more general goals than the achievement of system optimal flows; for example, we can introduce restrictions of the form (6.1) only on a subnetwork, thus achieving an approximate system optimal flow on a subset of the links, while the rest of the network is governed by the user optimality principle; the resulting model is reminiscent of the combined equilibrium models (e.g., Harker, 1988) used in the modelling and simulation of vehicle guidance systems.

6.4 Network improvement, subsidies and the redistribution of resources

Consider a situation in which the traffic manager would like to achieve a given link flow solution f^*. The absence of any conditions on the toll vector p other than (4.2a) makes it possible that some link tolls may be negative. An interesting *network design model* emanates from this fact. Indeed, while interpreting a positive value p_a as a toll levied on link a, a negative value of p_a may be interpreted as the *travel time improvement* (or, subsidy) that is necessary in order to make link a more attractive to some travellers; this information may be used, for example, in the calculation of necessary investments in network capacity. Note that the network design problem stated here has the reverse viewpoint to that of the traditional one.

Example 6.3 (Public transport) Side constrained traffic equilibrium models can be used to derive actions that favour public transport. Denoting the link flow for private (public) transport by f^{pr} (f^{pu}), side constraints describing a favourable situation for public transport could, for example, be constraints of the form

$$\gamma_a f_a^{\mathrm{pr}} - f_a^{\mathrm{pu}} \leq 0, \qquad a \in \mathcal{A},$$

where $\gamma_a \geq 0$ is the least ratio of public transport flows over that of private transport on link a; other possibilities include constraints describing maximal travel times for public transport vehicles.

The Lagrange multipliers associated with these constraints in the resulting side constrained *two-mode* traffic equilibrium model are then used to derive (1) an additional queueing delay for private vehicles, or (2) a subsidy to the public transportation sector. If, further, toll constraints of the form $P = \{\, p \in \Re^{|\mathcal{A}|} \mid \sum_{a\in\mathcal{A}}[p_a^{\mathrm{pr}} + p_a^{\mathrm{pu}}] = 0 \,\}$ are introduced in the model, then the subsidy takes the form of a reallocation of resources from the private to the public sector; the subsidy is realized, for example, through tolls levied on private vehicles only or through the design (or, improvement) of designated lanes for the public transport vehicles or lower ticket prices. □

6.5 Consistent toll schedules

Wardrop's user equilibrium principle states that costs are equal among the routes utilized in an OD pair, but it does not contain a mechanism by which their respective flows can be uniquely determined. (In technical terms, equilibrium commodity flows are not unique because the link cost mapping t is not strictly monotone in the commodity flow variables.) This inherent limitation of the Wardrop user equilibrium principle has two negative implications, which we shall shortly describe below; possible remedies to this limitation are then discussed.

6.5.1 The use of toll revenues

There are several reasons why one might wish to obtain disaggregated information about the traffic that passes a particular toll station, rather than just the total volume. Disaggregated traffic information could distinguish between (1) local, regional and long-distance traffic; (2) different vehicle modes such as cars, trucks and buses; (3) resident, commuter and commercial traffic; and (4) drivers with different values-of-time. A fundamental question raised in studies of toll schemes is: *how should the toll revenues be used?* As pointed out by Small (1992), "the people who benefit from congestion relief and revenue uses do not necessarily coincide with those who pay the fees or who suffer inconvenience in order to avoid them". Having the type of information mentioned above facilitates an analysis of the pricing policy, for example with the objective of determining which categories of travellers that are affected by the toll, and who among those that need to be compensated or who pays less than their contribution to congestion or environmental effects warrants. (See Rossi *et al.*, 1989, for a discussion on consistent impact-fee assessment in urban networks.) Based on such an analysis, the current revenue distribution could be re-evaluated. We then note that the Wardrop principle (in the case of additive cost structures) does not allow for these categories of traffic to be distinguished in the equilibrium solution, only their aggregates.

6.5.2 Commodity-specific management goals

Suppose that some management goals are described in terms of individual commodity flows. For example, we may consider restrictions of the form $g_{pq}(h_{pq}) \leq 0$, or $g_{pq}(f_{pq}) \leq 0$, where h_{pq} and f_{pq} is the vector of route flows and link flows, respectively, in commodity (p, q). (See Remark 3.2 for further examples of possible restrictions over individual flows.) If we are interested in obtaining flows satisfying these restrictions through the use of tolls, then the tolls would have to be imposed on the individual commodities, that is, it would be insufficient to introduce a toll based on the total link flow only. (This is quite evident from the equilibrium characterization of such a side constrained traffic equilibrium model.) Having solved the side constrained model, we obtain a unique link flow solution, and a consistent commodity flow which satisfies the side constraints. The commodity-specific toll which is automatically derived will, however, not necessarily influence the flow to satisfy the side constraints, even though the unique link flow solution will be obtained. The reason for this apparently counter-intuitive result is that while the link flow solution is unique, its decomposition into commodity flows is not, and among these various commodity flows there are some that do not satisfy all the side constraints.[11] (In technical terms, the tolled problem is non-strictly convex in the commodity flows, a property which in the context of Lagrangean duality bears the term *non-coordinability*.)

[11] The only way in which the situation can be dealt with (within the framework of additive cost structures) in the Wardrop equilibrium model is that *the toll schedule is not fixed*, but is a strictly monotone function of the commodity flows, that is, a *nonlinear price schedule* (Jennergren, 1972).

6.5.3 Remedies

There are basically two remedies to this problem.

The first, and most natural, remedy is the introduction of the principle of a *stochastic* user equilibrium, by which travellers are allowed to have different perceptions of the actual travel costs. (This has the additional advantage of being a more general model of route-choice.) In most stochastic models proposed (see, e.g., Patriksson, 1994, Sec. 2.8.1) the *stochastic equilibrium route flows are unique.* In the context of the discussion on the use of toll revenues, then, the implication of this property is that the different categories of traffic that passes the toll station could be distinguished; in the example concerning commodity-specific traffic management goals, this property has the immediate implication that tolls (or, taxes) that will achieve the traffic management goals could be determined. The results presented in this paper are easily transfered from the deterministic case considered to that of a framework of stochastic equilibrium models, with Wardrop's user equilibrium principle replaced by the principle of stochastic user equilibrium.

The second, and less general, remedy is to introduce assumptions in the original equilibrium model which makes the commodity flows unique; for example, one may introduce non-additive travel costs such that the route flows are strictly monotone.

6.6 Extensions and further research

The price-directive traffic management schemes described here may possibly be applied in other areas, where society through prices (taxes or subsidies) may influence markets. Examples of areas of application include protective duties in international trades, and environmentally derived taxes and subsidies.

This paper provides the basis for several interesting paths of research, both in directions of theoretical research and applications, several of which are mentioned.

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