Research paper

Struct Multidisc Optim 25, 189–198 (2003)
DOI 10.1007/s00158-003-0291-x

Stable relaxations of stochastic stress-constrained weight minimization problems

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Abstract The problem of finding a truss of minimal weight subject to stress constraints and stochastic loading conditions is considered. We demonstrate that this problem is ill-posed by showing that the optimal solutions change discontinuously as small changes in the modelling of uncertainty are introduced. We propose a relaxation of this problem that is stable with respect to such errors. We establish a classic ε-perturbation result for the relaxed problem, and propose a solution scheme based on discretizations of the probability measure. Using Chebyshev's inequality we give an a priori estimation of the probability of stress constraint violations in terms of the relaxation parameter. The convergence of the relaxed optimal designs towards the original (non-relaxed) optimal designs, as the relaxation parameter decreases to zero, is established.

Key words stochastic programming, robust optimization, ε-perturbation, stress constraints, discretization

1 Introduction

We consider the problem of finding a truss of minimal weight subject to stress constraints and stochastic loading conditions. The reason for introducing the stochasticity into the problem is that uncertainty due to loading conditions that are unknown in advance has to be taken into account to obtain robust optimal solutions. On the other hand, Evgrafov *et al.* (2003) showed that optimal solutions to stochastic stress-constrained weight minimization problems change discontinuously as small changes in the modelling of uncertainty are introduced. Therefore, the optimal solutions are not robust with respect to modelling errors, and their quality is very hard to estimate.

In this paper we impose the stress constraints in a relaxed manner, which makes the weight minimization problem stable with respect to changes in probability measure. By adjusting the relaxation parameter one can ensure that stress constrains are noticeably violated with an arbitrarily small probability, and that the relaxed optimal designs are close to the original (non-relaxed) optimal designs.

Given positions of the nodes the *design* (and topology in particular) of a truss can be described by *design* variables $x_i \geq 0$, $i = 1, \ldots, m$, representing the volume of material allocated to the bar i in the structure. We introduce an index set $\mathcal{I}(\mathbf{x}) = \{\, i = 1, \ldots, m \mid x_i > 0 \,\}$ of the present (or active) members in the structure.

Let $(\Omega, \mathcal{S}, \mathrm{P})$ be a complete probability space. The stochastic stress-constrained minimization problem can be formulated as follows:

$$(\mathcal{P}_2) \begin{cases} \min\limits_{(\mathbf{x},\mathbf{s}(\cdot))} w(\mathbf{x}) := \mathbf{1}_m^T \mathbf{x} \\ \text{s.t.} \begin{cases} \underline{\mathbf{x}} \leq \mathbf{x}\,, \\ |s_i(\omega)| \leq \overline{\sigma}_1 x_i, \qquad i = 1, \ldots, m, \quad \text{P-a.s.}\,, \\ \mathbf{s}(\omega) \text{ solves } (\mathcal{C})_{\mathbf{x}}(\omega), \qquad \text{P-a.s.}\,, \end{cases} \end{cases}$$

in which the minimization problem $(\mathcal{C})_{\mathbf{x}}(\omega)$ is the principle of minimum complementary energy:

$$(\mathcal{C})_{\mathbf{x}}(\omega) \begin{cases} \min\limits_{\mathbf{s}} \mathcal{E}(\mathbf{x}, \mathbf{s}) := \dfrac{1}{2} \sum\limits_{i=1}^{m} \dfrac{s_i^2}{E x_i}\,, \\ \text{s.t.} \left\{ \sum\limits_{i \in \mathcal{I}(\mathbf{x})} \mathsf{B}_i^T s_i = \mathbf{f}(\omega)\,. \right. \end{cases}$$

The data in the problem has the following meaning from a mechanical point of view:

- E is the Young modulus for the structure material;
- B_i is the kinematic transformation matrix for the bar i;
- $\mathbf{f}(\omega)$ is the vector of external forces.

Received: 24 April 2002
Revised manuscript received: 2 October 2002
Published online: 30 July 2003

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For the problem to be tractable we assume that the function $\mathbf{f}(\cdot)$ is $\mathcal{S}$-measurable. The variable s_i in the problem $(\mathcal{C})_{\mathbf{x}}(\omega)$ is the tensile force in the bar i times its length.

2
A (short) quest for a correct relaxation

One does not need a specially constructed example to demonstrate the discontinuity of optimal solutions to the stress-constrained weight minimization problem. The problem instance below is probably the simplest example one can imagine.

Example 1 (One-bar truss). Figure 1 shows a simple one-dimensional structure that consists of a single bar.

Suppose that $\Omega = [0,2]$, $\mathbf{f}(\omega) = \omega$, $\overline{\sigma}_1 = 1$, and $\underline{\mathbf{x}} = 0$. Let $\mathrm{P}^{(1)}$ be a uniform distribution on $[0,1]$, $\mathrm{P}^{(2)}$ be a uniform distribution on $[1,2]$, and $\mathrm{P}_k = (k-1)/k\mathrm{P}^{(1)} + 1/k\mathrm{P}^{(2)}$. The sequence $\{\mathrm{P}_k\}$ weakly converges to $\mathrm{P} = \mathrm{P}^{(1)}$, and each measure possesses a density.

The structure is statically determinable, thus the force $\mathbf{s}(\omega)$ is independent of $\mathbf{x}$ and equals $\mathbf{f}(\omega)$. The optimal solution $\mathbf{x}^*$ of $(\mathcal{P}_2)$ equals 1, while each optimal solution $\mathbf{x}_k^*$ of $(\mathcal{P}_2)^k$ equals 2. Therefore, the sequence $\{\mathbf{x}_k^*\}$ does not converge to $\mathbf{x}^*$ as k goes to infinity as one would want.

It is difficult to imagine the existence of any mild conditions under which the stochastic stress-constrained weight minimization problem is stable, when it is unstable even for the extremely simple structure of Example 1. Thus, it is reasonable to construct a relaxation of the problem, having the following properties:

(i) it is possible to recover a solution to the original problem as a limit point of the solutions to the relaxed problem as a relaxation parameter goes to 0;
(ii) the relaxed problem is stable with respect to changes in the probability measure;
(iii) it is possible to estimate the violation of the relaxed constraints; and
(iv) it is possible to numerically solve the relaxed problem.

One straightforward approach, which obviously satisfies the requirement (iii), is to choose a relaxation parameter $\delta > 0$ and to require that $\mathrm{P}(|s_i(\omega)| \le \overline{\sigma}_1 x_i + \delta) = 1$, $i = 1, \dots, m$. This approach is used when sizing approximations to the deterministic case of the problem $(\mathcal{P}_2)$ are considered (Cheng and Guo 1997; Petersson 2001). To show why such a relaxation of the problem is not enough, we consider the following example.

Example 2 (Two-bar truss). Figure 2 shows a simple structure that consists of two bars.

Suppose that $m = 2$, $\Omega = [0,2]$, $f_2(\omega) = \omega - 1$, $\overline{\sigma}_1 = 1$, $\underline{\mathbf{x}} = \mathbf{0}$, and

$$f_1(\omega) = \begin{cases} \omega, & \text{if } 0 \le \omega \le 0.5\,, \\ 1-\omega, & \text{if } 0.5 < \omega \le 1\,, \\ 0, & \text{otherwise}\,. \end{cases}$$

Let $\mathrm{P}_k^{(1)}$ be a uniform distribution on $[0,1/k]$, $\mathrm{P}^{(2)}$ be a uniform distribution on $[1,2]$, and $\mathrm{P}_k = 1/k^2\mathrm{P}^{(1)} + (k^2-1)/k^2\mathrm{P}^{(2)}$. The sequence $\{\mathrm{P}_k\}$ weakly converges to $\mathrm{P} = \mathrm{P}^{(2)}$ and each measure possesses a density.

As before, the force vector $\mathbf{s}(\omega)$ is independent of the design and equals $\mathbf{f}(\omega)$. The optimal solution to the non-relaxed problem $(\mathcal{P}_2)^k$ is $\mathbf{x}_k^* = (1/k, 1)^T$; thus the sequence of solutions $\{\mathbf{x}_k^*\}$ converges to the optimal solution $\mathbf{x}^* = (0,1)^T$ of the non-relaxed problem $(\mathcal{P}_2)$ as k goes to infinity for this example.

For any "small" $\delta > 0$ the optimal solution of the relaxed problem $(\mathcal{P}_2^\delta)$ exists and equals $\mathbf{x}^\delta = (0, 1-\delta)^T$. On the other hand, for $k > 1/\delta$ the feasible design space of the problem $(\mathcal{P}_2^\delta)^k$ is $(0,\infty) \times [1-\delta, \infty)$, and the objective function $w(\mathbf{x})$ does not attain its infimum on this set. Therefore, there is no optimal solution to the relaxed problem $(\mathcal{P}_2^\delta)^k$!

Example 2 clearly shows that the requirement (ii) is violated by the "straightforward" relaxation of stress constraints.

To introduce the "correct" relaxation scheme, for positive designs $\mathbf{x}$, we consider a convex, non-negative, and differentiable function that was used by Evgrafov and Patriksson (2003) to construct a penalty function for the stress-constrained weight minimization problem

$$G(\mathbf{x}, \mathbf{s}) := \sum_{i=1}^{m} \frac{[|s_i| - \overline{\sigma}_1 x_i]_+^2}{x_i}\,.$$

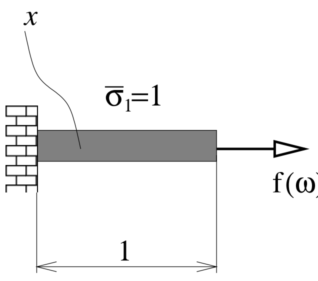

Fig. 1 The one-bar truss

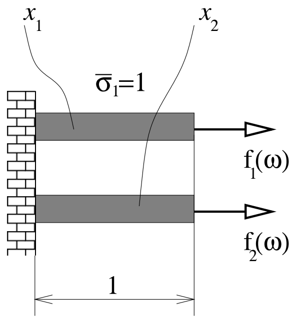

Fig. 2 The two-bar truss

Using the usual convention $0/0 = 0$ and $a/0 = \infty$ for any $a > 0$, the function G can be evaluated at any non-negative design $\mathbf{x}$, and, furthermore, it is l.s.c. on $\mathcal{R}^m_+ \times \mathcal{R}^m$.

Now, for a positive relaxation parameter $\delta > 0$ consider the following minimization problem:

$$(\mathcal{P}_2^\delta) \begin{cases} \min\limits_{(\mathbf{x},\mathbf{s}(\cdot))} \; w(\mathbf{x}) \\ \text{s.t.} \begin{cases} \underline{\mathbf{x}} \le \mathbf{x}\,, \\ \displaystyle\int_\Omega G(\mathbf{x}, \mathbf{s}(\omega))\, \mathrm{P}(\mathrm{d}\omega) \le \delta\,, \\ \mathbf{s}(\omega) \text{ solves } (\mathcal{C})_{\mathbf{x}}(\omega), \quad \text{P-a.s.} \end{cases} \end{cases}$$

Owing to the measurability of the solutions to $(\mathcal{C})_{\mathbf{x}}(\cdot)$ (see Evgrafov *et al.* 2003, Corollary 2.2), the problem $(\mathcal{P}_2^\delta)$ is indeed a relaxation of $(\mathcal{P}_2)$ (in the sense that the feasible set of the former problem contains that of the latter), and $(\mathcal{P}_2^0) = (\mathcal{P}_2)$.

Furthermore, owing to Chebyshev's inequality, for any $c > 0$ the following inequality holds:

$$\mathrm{P}\big(|s_i(\omega)| \ge \overline{\sigma}_1 x_i + c\big) \le \frac{\delta x_i}{c^2}\,, \tag{1}$$

i.e., by choosing a small δ the probability of violating any stress constraint can be made arbitrarily small. Therefore, the proposed relaxation satisfies the requirement (iii).

The rest of the paper is organized as follows. In Sect. 3 we investigate the properties of the feasible set of the problem $(\mathcal{P}_2^\delta)$, and show that it satisfies a Slater-type constraint qualification. Section 4 addresses the existence of solutions for the problem. In Sects. 5, 6, and 7 we show that the problem $(\mathcal{P}_2^\delta)$ possesses the properties we listed; in particular, Theorem 2 verifies the property (i), and Theorem 4 addresses the stability requirement (ii). Using Theorems 3 and 5 we can approximate the problem by a sequence of simple differentiable and finite-dimensional subproblems; this gives us the property (iv). Finally, we illustrate the theory with a numerical example in Sect. 8.

3 Auxiliary results

In this section we collect auxiliary results necessary for the following development.

The lemma below asserts the continuity of the mapping $\mathbf{x} \to \mathbf{s}(\cdot)$, where $\mathbf{s}(\cdot)$ solves $(\mathcal{C})_{\mathbf{x}}(\cdot)$, restricted to the feasible set of the problem $(\mathcal{P}_2^\delta)$. It is an important part of the proof of existence of solutions to $(\mathcal{P}_2^\delta)$, as it enables us to choose a feasible state corresponding to the limit of the design variables.

Lemma 1. *Suppose that the sequence* $\{(\mathbf{x}_k, \mathbf{s}_k(\cdot))\}$ *to* $(\mathcal{P}_2^\delta)$ *has design components converging to a limit* $\mathbf{x}_0$. *Then the sequence of state variables* P-a.s. *converges to a limit* $\mathbf{s}_0(\cdot)$ *solving* $(\mathcal{C})_{\mathbf{x}_0}(\cdot)$ *as k goes to infinity.*

Proof. The sequence of designs is bounded, so we can use Evgrafov and Patriksson (2003), Lemma 2.2, to conclude that the sequence of energy estimations $\left\{ \int_\Omega \mathcal{E}(\mathbf{x}_k, \mathbf{s}_k(\omega))\, \mathrm{P}(\mathrm{d}\omega) \right\}$ is bounded. Now the claim follows from Evgrafov *et al.* (2003), Proposition 2.3.

The following lemma is the crucial technical tool. It shows that a Slater-type constraint qualification holds for the relaxed stress constraints.

Lemma 2. *Suppose that* $(\mathbf{x}, \mathbf{s}(\cdot))$ *is a solution that is feasible in* $(\mathcal{P}_2^\delta)$ *and is such that* $\int_\Omega G(\mathbf{x}, \mathbf{s}(\cdot))\, \mathrm{P}(\mathrm{d}\omega) > 0$. *Fix an arbitrary* $\varepsilon > 0$. *Then it is possible to find a feasible point* $(\widetilde{\mathbf{x}}, \widetilde{\mathbf{s}}(\cdot))$ *such that* $\widetilde{\mathbf{x}} > \mathbf{x}$, $\|\widetilde{\mathbf{x}} - \mathbf{x}\| < \varepsilon$, *and* $\int_\Omega G(\widetilde{\mathbf{x}}, \widetilde{\mathbf{s}}(\omega))\, \mathrm{P}(\mathrm{d}\omega) < \int_\Omega G(\mathbf{x}, \mathbf{s}(\omega))\, \mathrm{P}(\mathrm{d}\omega)$.

Proof. Let $\bar{\mathbf{x}} = (1 + \varepsilon/3)\mathbf{x}$, and let $\bar{\mathbf{s}}(\cdot)$ be the solution to $(\mathcal{C})_{\bar{\mathbf{x}}}(\cdot)$. Then $\bar{\mathbf{s}}(\cdot) = \mathbf{s}(\cdot)$ and, since $\mathrm{P}\{G(\mathbf{x}, \mathbf{s}(\omega)) > 0\} > 0$, there is an index i such that $x_i > 0$ and $\mathrm{P}\{[|s_i(\cdot)| - \overline{\sigma}_1 x_i]_+^2 / x_i > 0\} > 0$. The continuity of $\int_\Omega [|s_i(\omega)| - \overline{\sigma}_1 x_i]_+^2 / x_i\, \mathrm{P}(\mathrm{d}\omega)$ w.r.t. x_i implies that $\int_\Omega G(\bar{\mathbf{x}}, \bar{\mathbf{s}}(\omega))\, \mathrm{P}(\mathrm{d}\omega) < \int_\Omega G(\mathbf{x}, \mathbf{s}(\omega))\, \mathrm{P}(\mathrm{d}\omega)$.

For some positive $p \ge 3$, to be determined later, set $\widetilde{\mathbf{x}} = \bar{\mathbf{x}} + \varepsilon/p \cdot \mathbf{1}_m$ and let $\widetilde{\mathbf{s}}(\cdot)$ be the solution of $(\mathcal{C})_{\widetilde{\mathbf{x}}}(\cdot)$. Using the directionally Lipschitz continuous dependence of solutions to $(\mathcal{C})_{\mathbf{x}}(\cdot)$ on $\mathbf{x}$ (see Evgrafov *et al.* 2003, Lemma 3.3), the continuity of $[|\widetilde{s}_i| - \overline{\sigma}_1 \widetilde{x}_i]_+^2 / \widetilde{x}_i$ for i such that $\bar{x}_i > 0$, and the inequality

$$\frac{[|\widetilde{s}_i(\cdot)| - \overline{\sigma}_1 \widetilde{x}_i]_+^2}{\widetilde{x}_i} \le \frac{p(\tau + \overline{\sigma}_1)^2 \|\widetilde{\mathbf{x}} - \bar{\mathbf{x}}\|^2}{\varepsilon} = \frac{\varepsilon(\tau + \overline{\sigma}_1)^2 \|\mathbf{1}_m\|^2}{p}\,,$$

for i such that $\bar{x}_i = 0$, we conclude that it is possible to choose a large enough p such that the inequality $\int_\Omega G(\widetilde{\mathbf{x}}, \widetilde{\mathbf{s}}(\omega))\, \mathrm{P}(\mathrm{d}\omega) < \int_\Omega G(\mathbf{x}, \mathbf{s}(\omega))\, \mathrm{P}(\mathrm{d}\omega)$ holds.

4 Existence of optimal solutions

From now on we make the following blanket assumptions:

(B1) for every positive design $\mathbf{x}$ the problem $(\mathcal{C})_{\mathbf{x}}(\omega)$ is feasible for almost any ω;

(B2) the problem $(\mathcal{C})_{\mathbf{0}_m}(\omega)$ is infeasible with a positive probability.

The first assumption is related to the "richness" of the ground structure and is easy to satisfy in practice. For example, one can start from a ground structure that is able to sustain *any* load. The second assumption eliminates the possibility of the empty structure being the optimal solution.

In view of Example 2 it is of prime importance to establish the existence of optimal solutions to the problem $\left(\mathcal{P}_2^\delta\right)$ for any $\delta > 0$.

Theorem 1. *For any $\delta > 0$ the problem $\left(\mathcal{P}_2^\delta\right)$ possesses at least one optimal solution $(\mathbf{x}^*, \mathbf{s}^*(\cdot))$.*

Proof. If there is at least one feasible solution $(\widetilde{\mathbf{x}}, \widetilde{\mathbf{s}}(\cdot))$ then we can bound the design space by introducing the additional constraint $w(\mathbf{x}) \le w(\widetilde{\mathbf{x}})$. Then from any minimizing sequence one can choose a subsequence with converging design components. Lemma 1 ensures that the corresponding subsequence of forces converges, and the limit is then feasible in $\left(\mathcal{P}_2^\delta\right)$ owing to the lower semicontinuity and non-negativity of G and Fatou's Lemma. Since the objective function is continuous in both design and state variables (it is independent of the forces), the limiting point is also an optimal solution.

Thus it remains to find a feasible solution. Following the proof of Lemma 2, we see that if $\widetilde{\mathbf{s}}(\cdot)$ solves $(\mathcal{C})_{\mathbf{1}_m}(\cdot)$, then it solves $(\mathcal{C})_{2^q\cdot\mathbf{1}_m}(\cdot)$ for any $q \ge 0$ as well. Thus we can make the value of $\int_\Omega G(2^q \cdot \mathbf{1}_m, \mathbf{s}(\omega))\,\mathrm{P}(\mathrm{d}\omega)$ arbitrarily small (but non-negative), if we choose a "large enough" q. Hence the point $(2^q \cdot \mathbf{1}_m, \widetilde{\mathbf{s}}(\omega))$ is feasible in $(\mathcal{P}_2^\delta)$ for some q.

Remark 1. For any optimal solution $(\mathbf{x}^*, \mathbf{s}^*(\cdot))$ to $\left(\mathcal{P}_2^\delta\right)$ the equality $\int_\Omega G(\mathbf{x}^*, \mathbf{s}^*(\omega))\,\mathrm{P}(\mathrm{d}\omega) = \delta$ holds.

Proof. If the strict inequality $\int_\Omega G(\mathbf{x}^*, \mathbf{s}^*(\omega))\mathrm{P}(\mathrm{d}\omega) < \delta$ held, then for some $0 < \mu < 1$ we would have $\int_\Omega G(\mu\mathbf{x}^*, \mathbf{s}^*(\omega)) \times \mathrm{P}(\mathrm{d}\omega) < \delta$ as well. Furthermore, $\mathbf{s}^*(\cdot)$ solves $(\mathcal{C})_{\mu\mathbf{x}^*}(\cdot)$, and $0 < w(\mu\mathbf{x}^*) < w(\mathbf{x}^*)$ (see assumption (B2)). The latter inequality contradicts the optimality of $(\mathbf{x}^*, \mathbf{s}^*(\cdot))$ in $\left(\mathcal{P}_2^\delta\right)$.

5 Continuity with respect to lower bounds and relaxation parameter

An additional motivation for considering the relaxed problems $(\mathcal{P}_2^\delta)$ is given by the following result, which ensures that by reducing the relaxation parameter to zero one recovers optimal solutions to the original problem $(\mathcal{P}_2)$.

We denote by $\mathrm{val}(\mathcal{P})$ the optimal value of any problem $(\mathcal{P})$.

Theorem 2. *Suppose that the problem $(\mathcal{P}_2)$ possesses an optimal solution, and let the sequence $\{\delta_k\}$ monotonically decrease to zero. Then any limit point of the sequence of optimal solutions $\left\{\mathbf{x}^*_{\delta_k}, \mathbf{s}^*_{\delta_k}(\cdot)\right\}$ (and there is at least one) is an optimal solution to $(\mathcal{P}_2)$.*

Proof. The inequality

$$\mathrm{val}(\mathcal{P}_2) \ge \limsup_{k\to\infty} \mathrm{val}\left(\mathcal{P}_2^{\delta_k}\right) \tag{2}$$

obviously holds.

On the other hand, the optimal solution to $(\mathcal{P}_2)$ is feasible in each problem $(\mathcal{P}_2^{\delta_k})$. In particular it means that the sequence of optimal designs $\{\mathbf{x}^*_{\delta_k}\}$ is bounded and has a limit point $\widetilde{\mathbf{x}}$. Lemma 1 implies that the corresponding sequence of forces $\{\mathbf{s}^*_{\delta_k}(\cdot)\}$ converges to a limit $\widetilde{\mathbf{s}}(\cdot)$ solving the problem $(\mathcal{C})_{\widetilde{\mathbf{x}}}(\cdot)$. The non-negativity and lower semicontinuity of G, and Fatou's Lemma, imply that $(\widetilde{\mathbf{x}}, \widetilde{\mathbf{s}}(\cdot))$ is feasible in $(\mathcal{P}_2)$, and thus we get

$$\mathrm{val}(\mathcal{P}_2) \le w(\widetilde{\mathbf{x}}) = \liminf_{k\to\infty} w(\mathbf{x}^*_{\delta_k}) = \liminf_{k\to\infty} \mathrm{val}\left(\mathcal{P}_2^{\delta_k}\right).$$

Together with (2), this proves the claim.

The function G, defining the constraints of our problem, is not upper-semicontinuous at the designs that are not strictly positive. Therefore, to apply numerical algorithms we would like to introduce a positive lower bound $\varepsilon\mathbf{1}_m$ on the design variables and eventually reduce ε to zero. This method, called ε-perturbation, is classic in topology optimization and is known to converge for compliance minimization problems (Achtziger 1998; Patriksson and Petersson 2002; Evgrafov *et al.* 2003). On the other hand, for stress-constrained weight minimization this simple procedure cannot approximate some optimal solutions, owing to the phenomena known as "stress singularities" and "singular topologies" (Sved and Ginos 1968; Kirsch 1990; Cheng and Jiang 1992; Rozvany and Birker 1994). More sophisticated numerical approaches are known to overcome this difficulty, for example the ε-perturbation by Cheng and Guo (1997) (see also Petersson 2001; Patriksson and Petersson 2002; Evgrafov *et al.* 2003) and a penalty function approach (Evgrafov and Patriksson 2003). It turns out that for our relaxation the simple approach outlined above is sufficient. To be more precise, for $\varepsilon > 0$ consider the following ε-perturbation of the problem $(\mathcal{P}_2^\delta)$:

$$\left(\mathcal{P}_2^{\delta,\varepsilon}\right)\begin{cases} \min\limits_{(\mathbf{x},\mathbf{s}(\cdot))} w(\mathbf{x}) \\ \text{s.t.} \begin{cases} \underline{\mathbf{x}} + \varepsilon\mathbf{1}_m \le \mathbf{x}\,, \\ \displaystyle\int_\Omega G(\mathbf{x}, \mathbf{s}(\omega))\,\mathrm{P}(\mathrm{d}\omega) \le \delta\,, \\ \mathbf{s}(\omega)\,\text{solves}\,(\mathcal{C})_{\mathbf{x}}(\omega)\,, \quad \text{P-a.s.} \end{cases} \end{cases}$$

Theorem 3. *Let $\{(\mathbf{x}^*_\varepsilon, \mathbf{s}^*_\varepsilon(\cdot))\}$ be a sequence of optimal solutions to the problems $\{(\mathcal{P}_2^{\delta,\varepsilon})\}$. Then any limit point of the sequence $\{(\mathbf{x}^*_\varepsilon, \mathbf{s}^*_\varepsilon(\cdot))\}$ as ε goes to zero (and there is at least one) is an optimal solution to the problem $(\mathcal{P}_2^\delta)$. Furthermore,*

$$\mathrm{val}\left(\mathcal{P}_2^\delta\right) = \inf_{\varepsilon>0} \mathrm{val}\left(\mathcal{P}_2^{\delta,\varepsilon}\right) = \lim_{\varepsilon\to 0} \mathrm{val}\left(\mathcal{P}_2^{\delta,\varepsilon}\right).$$

Proof. For any $\varepsilon_0 > 0$ there is a point $(\mathbf{x}_0, \mathbf{s}_0(\cdot))$, which for every $\varepsilon \in (0, \varepsilon_0)$ is feasible in each problem $(\mathcal{P}_2^{\delta,\varepsilon})$. In particular, it means that the sequence of optimal designs $\{\mathbf{x}_\varepsilon^*\}$ is bounded and has a limit point. We make another observation, namely that for $\varepsilon_1 < \varepsilon_2$ it holds that $\mathrm{val}(\mathcal{P}_2^{\delta,\varepsilon_1}) \le \mathrm{val}(\mathcal{P}_2^{\delta,\varepsilon_2})$.

Suppose that $\lim_{k\to\infty} \mathbf{x}_{\varepsilon_k}^* = \widetilde{\mathbf{x}}$ for some sequence ε_k converging to zero. Lemma 1 implies that the corresponding sequence of forces $\{\mathbf{s}_{\varepsilon_k}^*(\cdot)\}$ converges to a limit $\widetilde{\mathbf{s}}(\cdot)$ solving the problem $(\mathcal{C})_{\widetilde{\mathbf{x}}}(\cdot)$. The non-negativity and lower semi-continuity of G, and Fatou's Lemma, imply that $(\widetilde{\mathbf{x}}, \widetilde{\mathbf{s}}(\cdot))$ is feasible in $(\mathcal{P}_2^{\delta})$, and thus we get

$$\mathrm{val}\left(\mathcal{P}_2^{\delta}\right) \le w\left(\widetilde{\mathbf{x}}\right) = \lim_{k\to\infty} w\left(\mathbf{x}_{\varepsilon_k}^*\right) = \inf_{\varepsilon>0} \mathrm{val}\left(\mathcal{P}_2^{\delta,\varepsilon}\right). \quad (3)$$

On the other hand, Lemma 2 implies that any feasible solution $(\mathbf{x}, \mathbf{s}(\cdot))$ to $(\mathcal{P}_2^{\delta})$ such that $\int_\Omega G(\mathbf{x}, \mathbf{s}(\omega))\,\mathrm{P}(\mathrm{d}\omega) > 0$ holds can be arbitrarily closely approximated by feasible points of $(\mathcal{P}_2^{\delta,\varepsilon})$. In particular, any optimal solution to $(\mathcal{P}_2^{\delta})$ can be approximated in such a way as to give us the reverse inequality

$$\mathrm{val}\left(\mathcal{P}_2^{\delta}\right) = \lim_{k\to\infty} w\left(\mathbf{x}_{\varepsilon_k}\right) \ge \inf_{\varepsilon>0} \mathrm{val}\left(\mathcal{P}_2^{\delta,\varepsilon}\right).$$

Together with (3), this proves the claim.

The following proposition enables us to approximate the optimal value of $(\mathcal{P}_2^{\delta})$ from below in a different way.

Proposition 1. *Let the sequence $\{\delta_k\}$ monotonically increase to $\delta_\infty > 0$. Then $\mathrm{val}(\mathcal{P}_2^{\delta_\infty}) = \lim_{k\to\infty} \mathrm{val}(\mathcal{P}_2^{\delta_k})$.*

Proof. Obviously, the inequality

$$\mathrm{val}\left(\mathcal{P}_2^{\delta_\infty}\right) \le \liminf_{k\to\infty} \mathrm{val}\left(\mathcal{P}_2^{\delta_k}\right) \quad (4)$$

holds.

On the other hand, Lemma 2 implies that any solution $(\mathbf{x}, \mathbf{s}(\cdot))$ that is feasible in $(\mathcal{P}_2^{\delta_\infty})$ and is such that $\int_\Omega G(\mathbf{x}, \mathbf{s}(\omega))\,\mathrm{P}(\mathrm{d}\omega) > 0$ can be arbitrarily closely approximated by feasible points of $(\mathcal{P}_2^{\delta_k})$ for "large enough" k. In particular, any optimal solution to $(\mathcal{P}_2^{\delta_\infty})$ can be approximated in such a way, which gives us the reverse inequality

$$\mathrm{val}\left(\mathcal{P}_2^{\delta_\infty}\right) \ge \limsup_{k\to\infty} \mathrm{val}\left(\mathcal{P}_2^{\delta_k}\right).$$

Together with (4), this proves the claim.

6 Continuity with respect to changes in probability measure

In this section we prove the main result of the paper, showing that for fixed $\delta > 0$ the optimal solutions to the problem $(\mathcal{P}_2^{\delta})$ change continuously as the probability measure changes. Throughout the section we assume that Ω is a compact metric space, $\mathcal{S} = \mathcal{B}(\Omega)$, and the source of uncertainty $\mathbf{f}(\cdot)$ is assumed to be a continuous function.

Continuity allows us to omit the adverb "almost" when we talk about solutions of $(\mathcal{C})_{\mathbf{x}}(\cdot)$ for positive designs $\mathbf{x}$.

Proposition 2. *For positive design $\mathbf{x}$ and* each $\omega \in \Omega$ *the problem $(\mathcal{C})_{\mathbf{x}}(\omega)$ has a unique solution $\mathbf{s}(\omega)$, which is a continuous function of ω.*

Proof. We made an assumption that the problem $(\mathcal{C})_{\mathbf{x}}(\omega)$ is feasible for any ω for a positive design $\mathbf{x}$. The claim then follows from Evgrafov *et al.* (2003), Corollary 4.1.

Consider a sequence of probability measures $\{\mathrm{P}_k\}$ defined on $\mathcal{B}(\Omega)$, together with a sequence of optimization problems:

$$\left(\mathcal{P}_2^{\delta}\right)^k \begin{cases} \min\limits_{(\mathbf{x},\mathbf{s}(\cdot))} w(\mathbf{x}) \\ \text{s.t.} \begin{cases} \underline{\mathbf{x}} \le \mathbf{x}\,, \\ \displaystyle\int_\Omega G(\mathbf{x}, \mathbf{s}(\omega))\,\mathrm{P}_k(\mathrm{d}\omega) \le \delta\,, \\ \mathbf{s}(\omega) \text{ solves } (\mathcal{C})_{\mathbf{x}}(\omega), \quad \mathrm{P}_k\text{-a.s.} \end{cases} \end{cases}$$

Without any further regularity assumptions on the probability measure we can prove the following inequality.

Lemma 3. *Suppose that the sequence of probability measures $\{\mathrm{P}_k\}$ weakly converges to* P. *Then $\mathrm{val}(\mathcal{P}_2^{\delta}) \ge \limsup_{k\to\infty} \mathrm{val}(\mathcal{P}_2^{\delta})^k$.*

Proof. Fix arbitrary positive numbers $\varsigma < \delta$ and $\varepsilon > 0$. Consider an optimal solution $(\mathbf{x}^*, \mathbf{s}^*(\cdot))$ to $(\mathcal{P}_2^{\varsigma,\varepsilon})$. Owing to Proposition 2, $\mathbf{s}^*(\cdot)$ is a continuous function. Furthermore, since the energy $\mathcal{E}(\mathbf{x}^*, \mathbf{s}^*(\cdot))$ is continuous, we can deduce that $G(\mathbf{x}^*, \mathbf{s}^*(\cdot))$ is continuous as well. Since $\{\mathrm{P}_k\}$ weakly converges to P we conclude that $(\mathbf{x}^*, \mathbf{s}^*(\cdot))$ is feasible in $(\mathcal{P}_2^{\delta})^k$ for large enough k, and

$$\mathrm{val}\left(\mathcal{P}_2^{\varsigma,\varepsilon}\right) \ge \limsup_{k\to\infty} \mathrm{val}\left(\mathcal{P}_2^{\delta}\right)^k$$

holds.

Owing to Theorem 3 and Proposition 1, the following inequality holds:

$$\mathrm{val}\left(\mathcal{P}_2^{\delta}\right) = \inf_{\varepsilon>0} \inf_{\varsigma<\delta} \mathrm{val}\left(\mathcal{P}_2^{\varsigma,\varepsilon}\right) \ge \limsup_{k\to\infty} \mathrm{val}\left(\mathcal{P}_2^{\delta}\right)^k,$$

which is the desired result.

To prove the reverse inequality we assume additional regularity properties on the sequence $\{\mathrm{P}_k\}$. Namely, we suppose that each measure P_k has a density $p_k(\cdot)$ with respect to a Lebesgue measure on Ω, and that the sequence $\{p_k(\cdot)\}$ converges to a density $p(\cdot)$ of P Lebesgue-almost everywhere. This assumption is not very restrictive from a theoretical point of view, and it is usually assumed in engineering applications of probability theory.

Theorem 4. *Let $\{(\mathbf{x}_k, \mathbf{s}_k(\cdot))\}$ be a sequence of solutions to $\{(\mathcal{P}_2^\delta)^k\}$. Then any limit point (and there is at least one) of the sequence $\{(\mathbf{x}_k, \mathbf{s}_k(\cdot))\}$ is a solution to the limiting problem $(\mathcal{P}_2^\delta)$.*

Proof. As in the proof of Theorem 1, for large enough q we can find a point $(2^q \cdot \mathbf{1}_m, \widetilde{\mathbf{s}}(\cdot))$ that is feasible in $(\mathcal{P}_2^{\delta/2})$. Since $\{\mathrm{P}_k\}$ weakly converges to P and $\widetilde{\mathbf{s}}(\cdot)$ is continuous, for large enough k this point is feasible to $(\mathcal{P}_2^\delta)^k$. In particular, it means that the sequence $\{\mathbf{x}_k\}$ is bounded and has a limit point $\mathbf{x}_0$. Therefore, we may assume that the original sequence has converging design components.

The lower semi-continuity and non-negativity of G, and Fatou's Lemma, imply that

$$\int_\Omega \liminf_{k\to\infty} G(\mathbf{x}_k, \mathbf{s}_k(\omega))p(\omega)\, \mathrm{d}\omega \le$$
$$\int_\Omega \liminf_{k\to\infty} [G(\mathbf{x}_k, \mathbf{s}_k(\omega))p_k(\omega)]\, \mathrm{d}\omega \le$$
$$\liminf_{k\to\infty} \int_\Omega G(\mathbf{x}_k, \mathbf{s}_k(\omega))p_k(\omega)\, \mathrm{d}\omega \le \delta\,.$$

Thus we see that the P-probability of the set $\Omega_f = \{\, \omega \in \Omega \mid \liminf_{k\to\infty} G(\mathbf{x}_k, \mathbf{s}_k(\omega)) < \infty \,\}$ is one. Using Lemma 1 we can verify the existence of a limiting state $\mathbf{s}_0(\cdot)$ corresponding to the design $\mathbf{x}_0$, and the P-a.s. convergence of $\mathbf{s}_k(\cdot)$ to this state. Using the lower semi-continuity of G, this implies that

$$\int_\Omega G(\mathbf{x}_0, \mathbf{s}_0(\omega))p(\omega)\, \mathrm{d}\omega \le$$
$$\int_\Omega \liminf_{k\to\infty} G(\mathbf{x}_k, \mathbf{s}_k(\omega))p(\omega)\, \mathrm{d}\omega \le$$
$$\liminf_{k\to\infty} \int_\Omega G(\mathbf{x}_k, \mathbf{s}_k(\omega))p_k(\omega)\, \mathrm{d}\omega \le \delta\,.$$

The latter inequality shows that $(\mathbf{x}_0, \mathbf{s}_0(\cdot))$ is feasible in $(\mathcal{P}_2^\delta)$, and thus

$$\mathrm{val}\left(\mathcal{P}_2^\delta\right) \le w(\mathbf{x}_0) \le \liminf_{k\to\infty} \mathrm{val}\left(\mathcal{P}_2^\delta\right)^k.$$

Together with the estimation of $\limsup_{k\to\infty} \mathrm{val}(\mathcal{P}_2^\delta)^k$ given by Lemma 3 this finishes the proof.

To show the qualitative difference between the problems $(\mathcal{P}_2)$ and $(\mathcal{P}_2^\delta)$ we reconsider Example 1.

Example 3 (Example 1 revisited). Figure 3 shows the convergence of solutions to $(\mathcal{P}_2^\delta)^k$ to the solution of $(\mathcal{P}_2^\delta)$

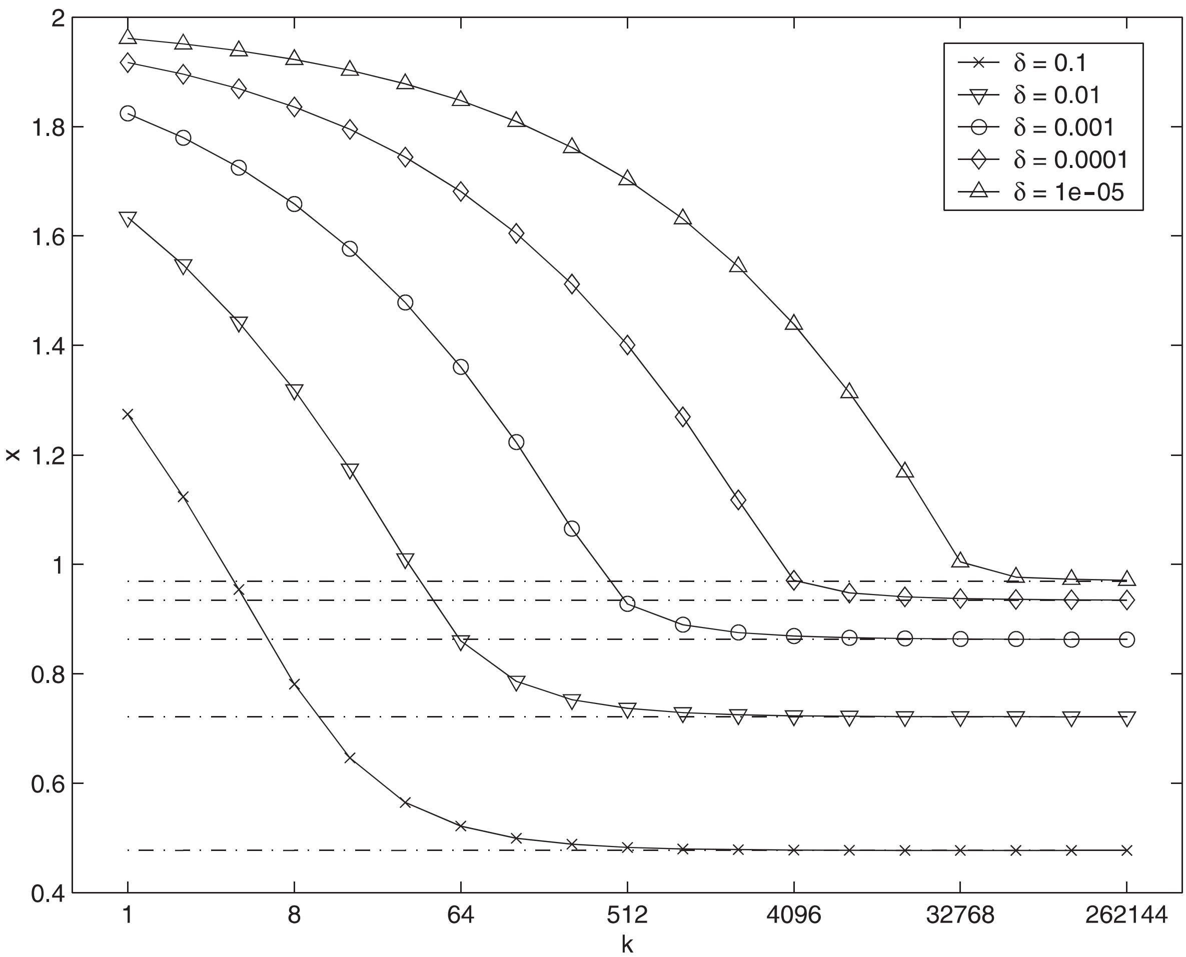

Fig. 3 Convergence of solutions to $(\mathcal{P}_2^\delta)^k$ to the solution of $(\mathcal{P}_2^\delta)$ for various values of δ

as k increases to infinity for various values of δ, as predicted by Theorem 4.

On the other hand, for a fixed k, the solutions to $(\mathcal{P}_2^\delta)^k$ converge to the optimal solution $x_k^* = 2$ of $(\mathcal{P}_2)^k$ as δ decreases to zero, in accordance with Theorem 2. Similarly, optimal solutions to $(\mathcal{P}_2^\delta)$ converge to the optimal solution $x^* = 1$ of $(\mathcal{P}_2)$.

This example shows that one cannot in general expect convergence as δ goes to zero and k goes to infinity *simultaneously.*

7
Discretization

The most popular method for solving a stochastic programming problem involving a non-discrete probability measure is to approximate it by a sequence of finite-dimensional problems with discrete measures. Unfortunately, we cannot apply Theorem 4 to our situation, because the approximating discrete measures do not possess densities. Without this assumption, the implementation of such a strategy seems to be impossible, owing to the discontinuity of the function G defining the constraints of our problem. Therefore, we discretize the sizing approximations $(\mathcal{P}_2^{\delta,\varepsilon})$ of $\left(\mathcal{P}_2^\delta\right)$; Theorem 3 shows the viability of such an approach.

In this section we sketch one possible discretization approach, which does not require us to assume the continuity of the load vector $\mathbf{f}(\cdot)$ with respect to ω. Evgrafov and Patriksson (2003) used this approach to discretize sizing approximations to the stochastic compliance minimization problem and to the original (non-relaxed) stress-constrained weight minimization problem. The interested reader is referred to the cited paper and references therein for the detailed development of the discretization theory.

Suppose that Ω is a compact metric space with a metric denoted by $\rho(\cdot,\cdot)$. Let $\mathcal{S} \supset \mathcal{B}(\Omega)$, $\mathrm{P}(\{\,\omega \mid \rho(\omega,\omega_0) < r\,\}) = \mathrm{P}(\{\,\omega \mid \rho(\omega,\omega_0) \le r\,\}) > 0$ for any $\omega_0 \in \Omega$, $r > 0$, and P be a regular measure.

Consider a sequence of *partitions* of Ω, $\mathcal{A}^k = \{\,A_1^k, \dots, A_k^k\,\}$, satisfying the following properties for each k and $1 \le l \le k$:

(M1) $\mathrm{P}\left(A_l^k\right) > 0$,

(M2) $\cup_{l=1}^k A_l^k = \Omega$,

(M3) $A_i^k \cap A_j^k = \emptyset$, $i \ne j$,

(M4) $\lim_{k\to\infty}\ \mathrm{diam}\left(A_l^k\right) = 0$,

(M5) $\mathrm{P}\left(\partial A_l^k\right) = 0$.

Note that the collection of sets $\{\mathcal{A}^k\}$, satisfying the properties (M1)–(M5), generates an algebra $\mathcal{S}_0 \subset \mathcal{S}$.

Define a sequence of discrete measures P_k with support $\mathrm{suppP}_k = \{\,\omega_1^k, \dots, \omega_k^k\,\}$, satisfying the following properties for each k and $1 \le l \le k$:

(M6) $\omega_l^k \in A_l^k$,

(M7) $\lim_{k\to\infty} \max_{1\le l\le k} \mathrm{P}_k(\omega_l^k)/\mathrm{P}(A_l^k) = 1$.

We further assume that

(D1) the function $\mathbf{f}(\cdot)$ is $\mathcal{S}_0$-measurable and bounded.

We denote by $(\mathcal{C})_{\mathbf{x}}^k(\omega_l^k)$ the following equilibrium principle:

$$\begin{cases} \min\limits_{\mathbf{s}} \mathcal{E}(\mathbf{x},\mathbf{s}) \\ \text{s.t.} \begin{cases} \displaystyle\sum_{i\in\mathcal{I}(\mathbf{x})} \mathsf{B}_i^T s_i = \mathbf{f}(\omega_l^k)\ . \end{cases} \end{cases}$$

In the following theorem we establish the convergence of discretizations for the problem $(\mathcal{P}_2^{\delta,\varepsilon})$. We note that from the weak* discrete convergence of the sequence $\{(\mathbf{x}_k^*, \mathbf{s}_k^*(\cdot))\}$ follows the (usual) convergence of the optimal designs.

Theorem 5. *Consider the following sequence* $\{(\mathcal{P}_2^{\delta,\varepsilon})^k\}$ *of discretizations of the problem* $(\mathcal{P}_2^{\delta,\varepsilon})$*:*

$$\left(\mathcal{P}_2^{\delta,\varepsilon}\right)^k \begin{cases} \min\limits_{(\mathbf{x},\mathbf{s}(\cdot))} w(\mathbf{x}) \\ \text{s.t.} \begin{cases} \underline{\mathbf{x}} + \varepsilon \mathbf{1}_m \le \mathbf{x}\,, \\ \displaystyle\int_\Omega G(\mathbf{x},\mathbf{s}(\omega))\,\mathrm{P}_k(\,\mathrm{d}\omega) \le \delta\,, \\ \mathbf{s}(\omega_l^k)\,\text{solves}\,(\mathcal{C})_{\mathbf{x}}^k(\omega_l^k), \quad l = 1,\dots,k\,. \end{cases} \end{cases}$$

Suppose that the assumptions (M1)–(M7) *and* (D1) *hold. Suppose further that there exists an optimal solution* $(\mathbf{x}^*, \mathbf{s}^*(\cdot))$ *to the problem* $(\mathcal{P}_2^{\delta,\varepsilon})$ *such that the energy functional* $\mathcal{E}(\mathbf{x}^*, \mathbf{s}^*(\cdot))$ *is essentially bounded.*

Owing to the positivity of $\mathbf{x}^*$ *and assumption* (B1) *the problems* $(\mathcal{C})_{(\mathbf{x}^*)}^k(\omega_l^k)$ *are feasible for any* k*,* $1 \le l \le k$*. Thus, there exists a sequence of optimal solutions to* $\{(\mathcal{P}_2^{\delta,\varepsilon})^k\}$*; we denote it by* $\{(\mathbf{x}_k^*, \mathbf{s}_k^*(\cdot))\}$ *Then any weak* discrete limit point of this sequence solves the limiting problem* $(\mathcal{P}_2^{\delta,\varepsilon})$*.*

Proof. We assume that the original sequence is weakly* convergent. The following two inequalities follow respectively from Propositions A.7 and A.8 in Evgrafov and Patriksson (2003):

$$\mathrm{val}\left(\mathcal{P}_2^{\delta,\varepsilon}\right) \le \liminf_{k\to\infty} \mathrm{val}\left(\mathcal{P}_2^{\delta,\varepsilon}\right)^k, \tag{5}$$

$$\mathrm{val}\left(\mathcal{P}_2^{\varsigma,\varepsilon}\right) \ge \limsup_{k\to\infty} \mathrm{val}\left(\mathcal{P}_2^{\delta,\varepsilon}\right)^k, \tag{6}$$

for any $0 < \varsigma < \delta$. Then, the claim follows from Proposition 1.

8 Numerical example

We consider the problem of finding a minimal weight of the beam-like structure shown in Fig. 4. In this example, the forces of magnitude one act independently of each other, with the directions uniformly distributed in the intervals schematically shown in the figure. The number of bars in the ground structure is $m = 49$. We set $E = 1.0$ and $\overline{\sigma}_1 = 1.0$, start with $\varepsilon = 0.05$, and successively multiply it by the factor 0.6 until it becomes as small as 5.0×10^{-4}.

We have solved the nested formulation of the problem (i.e., we have eliminated the state variables and treated them as functions of design) using an SQP-type algorithm. The starting point was the equally distributed material.

In Table 1 we report the optimal weights and statistics describing the violations of the stress constraints for various values of the number of discretization points k. The definitions of the statistics used are given below:

$$\max_\sigma := \max_{1\le \ell \le \hat{k}} \max_{1\le i\le m} [|\sigma|_{i\ell} - \overline{\sigma}_1]_+ \,,$$

$$\mathrm{avg}_\sigma := \sum_{\ell=1}^{\hat{k}} \left\{ \max_{1\le i\le m} [|\sigma|_{i\ell} - \overline{\sigma}_1]_+ \right\} P_{\hat{k}}\left(\omega_\ell^{\hat{k}}\right),$$

where $\sigma_{i\ell}$ is a tensile stress in the bar i under the loading condition ℓ, and $\hat{k} = 625$. The number $\max_\sigma$ characterizes the maximal stress violation in the structure for all load cases, whereas avg_σ is the average (for all load cases) maximal (among the structure members) stress violation. The way we formulate stress constraints only guarantees that avg_σ is small when δ is small. Nevertheless, $\max_\sigma$ turns out to be not very big and seems to decrease with δ for this problem.

The reduction of the relaxation parameter δ to the value 1×10^{-5} while keeping $k = 625$ gives us only a 3.6% increase in the optimal weight, whereas the corresponding numbers $\max_\sigma$ and avg_σ decrease drastically to 2.54% and 4×10^{-2}%, respectively (compare with the last row in Table 1).

Further increases of k do not lead to significant changes in the optimal design. Therefore, we assume that $k = 625$ is a reasonably good approximation to the problem's probability measure, and, in particular, use this approximation when calculating the statistics $\max_\sigma$ and avg_σ.

Two optimal designs corresponding to $k = 1$ and $k = 625$ are shown in Fig. 5. It is interesting to note that the multiple-load optimal design has fewer bars than the corresponding average-load design.

Their behaviour under various loading conditions is shown in Fig. 6.

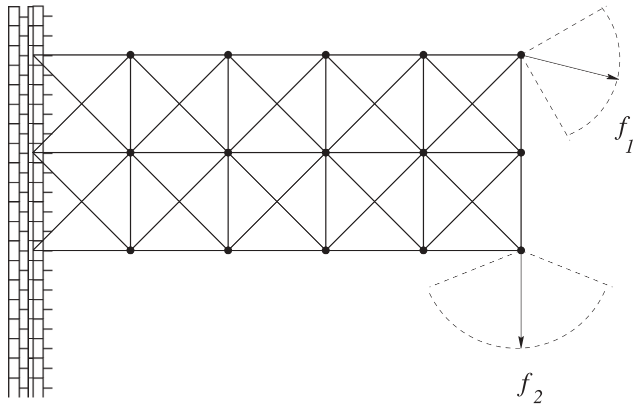

Fig. 4 The ground structure for the weight minimization problem

Table 1 Results of numerical calculation

k	w^*	$\max_\sigma$	avg_σ
1	33.599	745.5%	285.0%
25	45.447	18.95%	1.02%
625	45.967	13.27%	0.61%

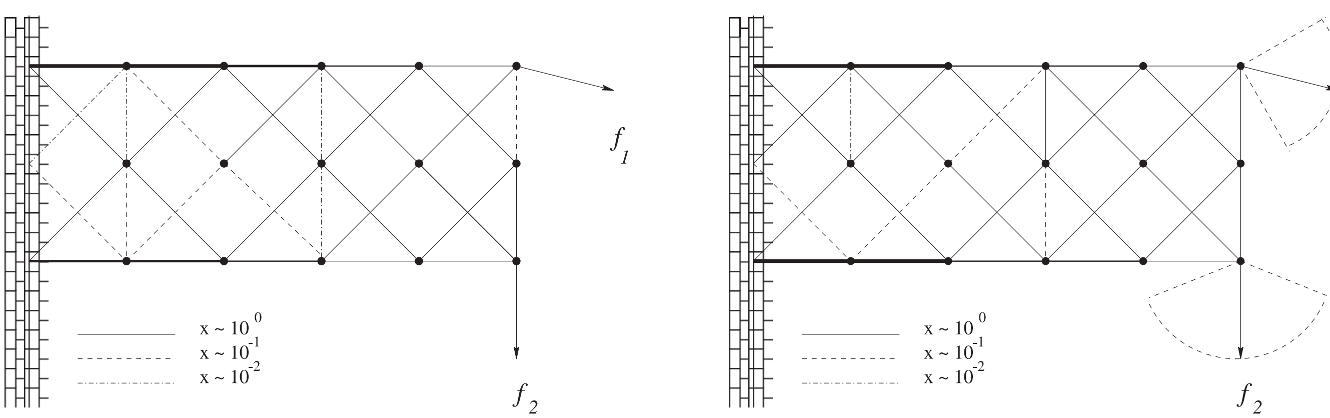

Fig. 5 The optimal designs for the weight minimization problem corresponding to (a) $k = 1$ and (b) $k = 625$. Line thicknesses are proportional to cross-sectional areas

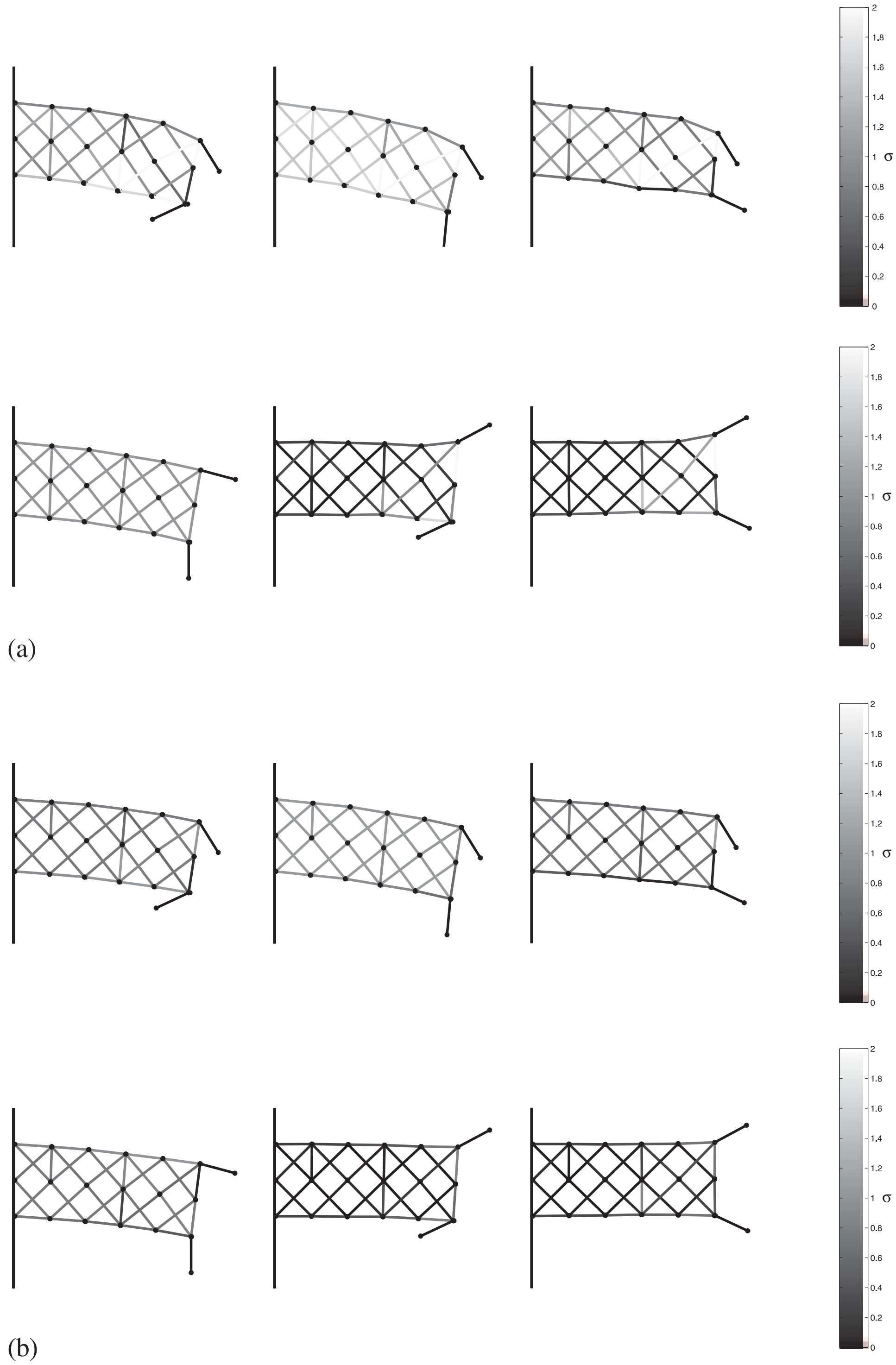

Fig. 6 Stresses and displacements for various random forces for optimal designs, corresponding to (a) $k = 1$ and (b) $k = 625$. Note: for the sake of a better visualization of stresses, line thicknesses are *not* proportional to cross-sectional areas. Lighter color means bigger stress

9 Conclusions

The relaxation of the stress-constrained weight minimization problem proposed in this paper offers a good trade-off between the strict satisfaction of the stress constraints and the robustness of the optimal solutions obtained with respect to changes in the modelling of uncertainty. The bound (1) on the constraint violations also allows one to choose a satisfactory value of δ before starting the optimization. For example, one can choose the boundary value c of the maximal acceptable violation of stress constraints, and then choose δ to be so small that the estimation $\delta\overline{x}/c^2$ of the probability of exceeding this boundary is negligible, where $\overline{x}$ is an upper bound for the design variables $\mathbf{x}$.

The ongoing research is concentrated on the development of efficient numerical methods for the problem $(\mathcal{P}_2^\delta)$ as well as on possible extensions of the results to more general mechanical models (e.g., trusses with unilateral constraints, frames, possibly with flexible joints).

Acknowledgements This research is supported by the Swedish Research Council for Engineering Sciences (grant TFR 98-125).

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