Institute for Advanced Studies for Basic Sciences

# Discontinuous Galerkin Method for Burgers Equation

*M.Sc. Thesis*

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**Abstract**

In present thesis we will consider the discontinuous Galerkin (DG) finite element method for 2D viscous Burgers equation. We will prove the stability of the DG method for this equation. Also, we will show the optimal rate of convergence for this equation by the DG method, and will show that the a priori error estimate is of order $\mathcal{O}(h^{k+1/2})$ when we use from the polynomials of degree $k$ and when the exact solution is smooth enough; here $h$ denote the mesh size for space discretization.

Also we implement the DG method for some examples in last chapter.

# Contents

# Chapter 1

# Preface

## 1.1 Introduction

Problems of practical interest in which *convection* plays an important role arise in applications as diverse as meteorology, weather-forecasting, oceanography, gas dynamics, aeroacoustics, turbomachinery, turbulent flows, granular flows, oil recovery simulation, modeling of shallow water, transport of contaminant in porous media, viscoelastic flows, semiconductor device simulation, magneto-hydrodynamics, and electro-magnetism, among many others. This is why devising robust, accurate and efficient methods for numerically solving these problems is of considerable importance and, as expected, has attracted the interest of many researchers and practitioners.

This endeavor, however, is far from trivial because of two main reasons. The first is that the exact solution of nonlinear purely convective problems develops discontinuities in finite time; the second is that these solutions might display a very rich and complicated structure near such discontinuities. Thus, when constructing numerical methods for these problems, it must be guaranteed that the discontinuities of approximate solution are the physically relevant ones. Also, it must be ensured that the appearance of a discontinuity in the approximate solution does not induce spurious oscillations that spoil the quality of the approximation; on the other hand, while ensuring this, the method must remain sufficiently accurate near that discontinuity in order to capture the possibly rich structure of the exact solution.

These difficulties were successfully addressed during the remarkable development of the *high-resolution finite difference* and *finite volume* schemes for nonlinear hyperbolic systems by means of suitably defined *numerical fluxes* and *slope limiters*. Since discontinuous Galerkin (DG) methods assume discontinuous approximate solutions, they can be considered as generalizations of finite volume methods. As a consequence, the DG methods incorporate the ideas of numerical fluxes and slope limiters into the *finite element* framework in a very natural way; they are able to capture the physically relevant discontinuities without producing spurious oscillations near them.

Owing to their finite element nature, the DG methods have the following main advantages over classical finite volume and finite difference methods:

- The actual order of accuracy of DG methods solely depends on the exact solution; DG methods of arbitrarily high formal order of accuracy can be obtained by suitably choosing the degree of the approximating polynomials.

- DG methods are highly parallelizable. Since the elements are discontinuous, the mass matrix is block diagonal and since the size of the blocks is equal to the number of degrees of freedom inside the corresponding elements, the blocks can be inverted by hand (or by using a symbolic manipulator) once and for all.

- DG methods are very well suited to handling complicated geometries and require an extremely simple treatment of the boundary conditions in order to achieve uniformly high-order accuracy.

- DG methods can easily handle adaptivity strategies since refinement or unrefinement of the grid can be achieved without taking into account the continuity restrictions typical of conforming finite element methods. Moreover, the degree of the approximating polynomial can be easily changed from one element to the other. Adaptivity is of particular importance in hyperbolic problems given the complexity of the structure of the discontinuities.

Although the original DG method has been known since 1973 by Reed and Hill, in the framework of neutron transport. It was only recently that DG methods have evolved in manner that made them suitable for use in computational fluid dynamics and aforementioned applications. The original DG finite element method was introduced in 1973 by Reed and Hill [46] for solving the neutron transport equation

$$\sigma u + \nabla \cdot (\mathbf{a} u) = f, \qquad \in \Omega,$$

where $\sigma$ is a real number and $\mathbf{a}$ a constant vector. The relevance of the method was recognized by LeSaint and Raviart who in 1974 [35] published its first mathematical analysis.

## 1.2 Analysis of the original DG method

**A priori error estimates.** In 1974 LeSaint and Raviart [35] made the first analysis of the DG method and proved a rate of convergence of $(\Delta x)^k$ in the $L_2(\Omega)$-norm for general triangulations and of $(\Delta x)^{k+1}$ for tensor products of polynomials of degree $k$ in one variable defined on Cartesian grids. In 1986, Johnson and Pitkaränta [33] proved a rate of convergence of $(\Delta x)^{k+1/2}$ for general triangulations. In a chain of papers [2, 3, 4, 5] the work by LeSaint and Raviart as well as Pitkaränta and Johnson were extended to the neutron transport equations in, e.g., cylindrical domains with polygonal cross-sections; where in particular the last two papers: [4, 5] employ interpolation and imbedding techniques between Besov ans Sobolev spaces to obtain optimal superconvergence rates. In 1991, Peterson [42] numerically confirmed this rate to be optimal. In 1988, Richter [44] obtained the optimal rate of convergence of $(\Delta x)^{k+1}$ for some structured two-dimensional non-Cartesian grids. The issue of the loss of order of convergence was addressed again in 1991 by Lin and Zhou [39] who proved that the standard Galerkin method using bilinear approximations defined on almost uniform Cartesian is of order 2; the order of this method for arbitrary meshes is only one. In 1994, Zhou and Lin [52] extended this result to piecewise-linear approximations in almost uniform triangulation. Then, in 1996 Lin, Yan, and Zhou [38] showed first order convergence for the DG method using piecewise-constant approximations. Their result holds for almost uniform grids of rectangles and for almost uniform grids of triangles; their technique is based on a key approximation result.

All the above mentioned papers assume that the exact solution is smooth. In 1993, Lin and Zhou [40] proved convergence to the weak solution assuming only that the exact solution belongs to $H^{1/2}(\Omega)$. More recently, Houston Schwab and Süli [30] proved spectral convergence of the DG method assuming that the exact solution is piecewise analytic. Cockburn, Luskin, Shu and Süli [22] showed that if the exact solution is in $L_2$ but is locally smoother, error estimates can be obtained between the exact solution and a suitably post-processed approximate solution.

**A posteriori error analysis.** In 1990, Stroubolis and Oden [47] studied a posteriori error estimates for the DG method. Later, Bey and Oden [13] obtained the first *hp*- a posteriori error estimates for the DG method; parallelization strategies based on these estimates were developed in 1995 by Bey, Patra, and Oden [15] and in 1996 by Bey, Oden and Patra [14]. A posteriori error analysis of finite element methods for hyperbolic problems, including a slight modification of the original DG method, have been studied in 1996 by Süli [48] and in 1997 by Süli and Houston [50]; see also the 1999 lectures notes on this subject by Süli [49].

## 1.3 Time discretization of parabolic equations

Also in 1978, Jmaet [31] used the DG method to discretize in time parabolic equations and showed that the method was of order $k$. Since then, several authors have studied this method. Thus, in 1985, K. Eriksson, C. Johnson and V. Thomée [28] proved that the method was of order $2k+1$ at nodes and later Eriksson and Johnson studied the issue of error control in a series of papers [23]-[27] starting in 1987 and ending in 1995. In 1997, Makridakis and Babuska [41] studied the effect on adaptive mechanisms on the stability of the method. Schötzau and Schwab have studied how to actually solve the system of equations defined by the DG methods; they show that it is possible to decouple the system into several scalar equations of the same type. Yet a more involved *hp* strategy for the Vlasov-Poisson-Fokker-Planck equation is developed in [8].

## 1.4 A DG method for convection-diffusion problems

In 1992, Richter [45] proposed a direct extension of the original DG method to linear convection-diffusion equations. Richter proved that if the convection is dominant, that is, if the viscosity coefficients were of the order of the meshsize, the optimal order of convergence is $k + 1/2$ when polynomials of degree $k$ are used.

# Chapter 2

# Preliminaries

## 2.1 Basic Definitions

**Definition 2.1 (The Burgers equation).** The scalar parabolic equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \varepsilon\frac{\partial^2 u}{\partial x^2} = 0, \tag{2.1.1}$$

was introduced in particular by Burgers as the simplest differential model for a fluid flow and is therefore often called the (viscous) Burgers equation.

Though very simple, this equation can be regarded as a model for decaying free turbulence. Burgers studied the limit equation when $\varepsilon$ tends to zero, which we write in conservation form

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0, \tag{2.1.2}$$

is the inviscid Burgers equation (or Burgers equation without viscosity), which for brevity, we shall simply call from now on Burgers'equation. It occurs in particular in wave theory to depict the distortion of waveform in simple waves (see [37], Sec. 2.9, [51], Sec. 2.8).

We can see that the Burgers'equation possesses all the features of a scalar convex equation given by

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0, \tag{2.1.3}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a convex smooth function, $f''(u) > 0$ for all $u$ (or, equally well, $f$ is concave with $f''(u) < 0$ for all $u$). In particular, the Cauchy problem for Burgers'equation may have discontinuous weak solutions even for a smooth initial data $u_0$, and solution of the Riemann problem is either a shock propagating or a rarefaction wave. The conservation law together with piecewise constant data having a single discontinuity is known as the Riemann problem. As an example, inviscid Burgers' equation with piecewise constant initial data

$$u(x,0) = \begin{cases} u_l & x < 0 \\ u_r & x > 0. \end{cases} \tag{2.1.4}$$

The form of the solution depends on the relation between $u_l$ and $u_r$.

Finally it is worth mentioning that the Cauchy problem for (2.1.1) has an explicit solution, obtained using the Cole-Hopf transform

$$u = -2\varepsilon \frac{\varphi_x}{\varphi}.$$

It eliminates the nonlinear term and transforms (2.1.1) into the heat equation

$$\frac{\partial \varphi}{\partial t} = \varepsilon \frac{\partial^2 \varphi}{\partial x^2},$$

for which explicit expression of the solution are known.

Let $\Omega$ be an open subset of $\mathbb{R}$, and let $f$, be a smooth function from $\Omega$ into $\mathbb{R}$; the general form of a scalar conservation law in one dimension can be written as

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0, \qquad x \in \mathbb{R}, \qquad t > 0, \tag{2.1.5}$$

where $u = u(x,t)$ is a scalar function from $\mathbb{R} \times [0,+\infty[$ into $\Omega$. The set $\Omega$ is called the set of states and the function $f$ is called the flux-function. One says that equation (2.1.5) is written in *conservation form.*

For such equations, initial value problem (IVP) is: Find a function $u : (x,t) \in \mathbb{R} \times [0,+\infty[\to u(x,t) \in \Omega$ which is a solution of (2.1.5) satisfying the initial condition

$$u(x,0) = u_0(x),$$

where $u_0 : \mathbb{R} \to \Omega$ is a given function.

Now, consider the problem (2.1.5) and assume $u_0$ be the given initial boundary data, we want to state precisely in which sense (2.1.5) is to be taken. Let $C_0^1(\mathbb{R} \times [0,+\infty[)$ denote the space of $C^1$ functions $\varphi$ with compact support in $\mathbb{R} \times [0,+\infty[$. We begin by noticing that if $u$ is $C^1$ and $\varphi \in C_0^1(\mathbb{R} \times [0,+\infty[)$, we obtain using integration by parts

$$\begin{aligned} &-\int_0^\infty \int_{\mathbb{R}} \left[\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u)\right] \varphi \, dx\, dt \\ &= \int_0^\infty \int_{\mathbb{R}} \left[\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial x} f(u)\right] dx\, dt + \int_{\mathbb{R}} u_0(x)\varphi(x,0)\, dx = 0. \end{aligned} \tag{2.1.6}$$

Now we are ready to define the weak solution of a differential equation.

**Definition 2.2 (Weak Solution).** The function $u(x,t)$ is called a weak solution of the conservation law (2.1.5) if (2.1.6) holds for all functions $\varphi \in C_0^1(\mathbb{R} \times [0,+\infty[)$.

The Sobolev spaces which will play an important role in the theory of finite element method are built on the function space $L_2(\Omega)$. $L_2(\Omega)$ consists of all functions $u$ which are square integrable over $\Omega$ in the Lebesgue sense. $L_2(\Omega)$ becomes a Hilbert space with the scalar product

$$(u,v)_0 := (u,v)_{L_2} = \int_\Omega u(x)v(x)\, dx$$

and the corresponding norm

$$|| \, u \, ||_0 = \sqrt{(u,u)_0}.$$

**Definition 2.3 (Weak Derivative).** $u \in L_2(\Omega)$ possesses the (weak) derivative of order $\alpha$: $v = \partial^\alpha u$ in $L_2(\Omega)$, provided that $v \in L_2(\Omega)$ and

$$(\phi, \mathcal{D}_w^\alpha u)_0 = (\phi, v)_0 = (-1)^{|\alpha|}(\partial^\alpha \phi, u)_0 \qquad \forall \phi \in C_0^\infty. \tag{2.1.7}$$

If such a $v$ exists, we define $\mathcal{D}_w^\alpha u = v$.

Here $C^\infty(\Omega)$ denotes the space of infinitely differentiable functions, and $C_0^\infty(\Omega)$ denotes the subspace of such functions which are nonzero only on a compact subset of $\Omega$.

If a function is differentiable in the classical sense, then its weak derivative also exists, and two derivatives coincide. In this case (2.1.7) becomes Green's formula or integration by parts.

**Example.** Take $\Omega = [-1, 1]$, and $f(x) = 1 - |x|$. We claim that $\mathcal{D}_w^1 f$ exists and is given by

$$g(x) := \begin{cases} 1 & x < 0 \\ -1 & x > 0. \end{cases}$$

To see this, we break the interval $[-1, 1]$ into the two parts in which $f$ is smooth, and we integrate by parts. Let $\phi \in C_0^\infty(\Omega)$. Then

$$\begin{aligned} \int_{-1}^1 f(x)\phi'(x)\,dx &= \int_{-1}^0 f(x)\phi'(x)\,dx + \int_0^1 f(x)\phi'(x)\,dx \\ &= -\int_{-1}^0 (+1)\phi(x)\,dx + f\phi|_{-1}^0 - \int_0^1 (-1)\phi(x)\,dx + f\phi|_0^1 \\ &= -\int_{-1}^1 g(x)\phi(x)\,dx + (f\phi)(0-) - (f\phi)(0+) \\ &= -\int_{-1}^1 g(x)\phi(x)\,dx \end{aligned}$$

because $f$ is continuous at 0.

**Example.** Let $\Omega = [-1, 1]$

$$f(x) = \begin{cases} -1 & x < 0 \\ 1 & x \geq 0. \end{cases}$$

Then $f \in L_2[-1, 1]$ and weak derivative of $f$ is $\mathcal{D}^1_w f = \delta$ (Dirac Delta) is not given by a locally integrable function and hence not by an $L_2$ function.

**Definition 2.4 (Sobolev Spaces).** Given an integer $m \geq 0$, let $H^m(\Omega)$ be the set of all functions $u$ in $L_2(\Omega)$ which possess weak derivatives $\partial^\alpha u$ for all $|\alpha| \leq m$. We can define a scalar product on $H^m(\Omega)$ by

$$(u, v)_m := \sum_{|\alpha| \leq m} (\partial^\alpha u, \partial^\alpha v)_0$$

with associated norm

$$||\, u \,||_m := \sqrt{(u, u)_m} = \sqrt{\sum_{|\alpha| \leq m} ||\, \partial^\alpha u \,||^2_{L_2(\Omega)}}.$$

The corresponding semi-norm

$$|\, u \,|_m := \sqrt{\sum_{|\alpha| = m} ||\, \partial^\alpha u \,||^2_{L_2(\Omega)}}$$

is also of interest.

## 2.2 Fundamental Definitions and Theorems

**Definition 2.5.** Let $V$ be a Hilbert space. A bilinear form $a : V \times V \to \mathbb{R}$ is called continuous provided that there exists a constant $C > 0$ such that

$$|a(u, v)| \leq C ||\, u \,|| ||\, v \,|| \qquad \forall u, v \in V.$$

A symmetric continuous bilinear form $a$ is called $V$-elliptic, or coercive, provided for some $\alpha > 0$,

$$a(v,v) \geq \alpha \|\, v \,\|^2 \qquad \forall v \in V.$$

**Theorem 2.6 (Lax-Milgram lemma).** *Let $V$ be a Hilbert space, let $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ be a continuous $V$-elliptic bilinear form, and let $f : V \to \mathbb{R}$ be a continuous linear form.*
*Then the abstract variational problem: that*

$$a(u,v) = f(v), \quad \forall v \in V \tag{2.2.1}$$

*has a unique solution $u \in V$.*

**Proof.** Let $M$ be a constant such that

$$\forall\, u, v \in V, \quad |a(u,v)| \leq M \|\, u \,\| \|\, v \,\|.$$

For each $u \in V$, the linear form $V \ni v \longmapsto a(u,v)$ is continuous and thus there exists a unique element $Au \in V'$ ($V'$ is the dual space of $V$) such that

$$\forall v \in V, \quad a(u,v) = Au(v).$$

Denoting by $\| \cdot \|^*$ the norm in space $V'$, then there exists a constant $M > 0$ such that

$$\|Au\|^* = \sup_{v \in V} \frac{|Au(v)|}{\|v\|} \leq M \|\, u \,\|.$$

Consequently, the linear mapping $A : V \to V'$, being bounded, is continuous, with

$$\|\, A \,\|_{\mathcal{L}(V;V')} \leq M.$$

Let $\tau : V' \to V$ denote the Riesz mapping defined by

$$\forall\, f \in V', \quad \forall\, v \in V, \quad f(v) = (\tau f, v),$$

where $(\cdot,\cdot)$ denotes the inner product in the space $V$. Then solving the variational problem (2.2.1) is equivalent to solving the equation $\tau Au = \tau f$. We will show that this equation has a unique solution by verifying that, for appropriate values of a parameter $\rho > 0$, the affine mapping

$$v \in V \to v - \rho(\tau Av - \tau f) \in V \tag{2.2.2}$$

is a contraction. To see this, we observe that

$$\begin{aligned} ||v - \rho\tau Av||^2 &= ||\, v\,||^2 - 2\rho(\tau Av, v) + \rho^2||\tau Av||^2 \\ &\le (1 - 2\rho\alpha + \rho^2 M^2)||\, v\,||^2, \end{aligned}$$

by coercivity of $a$ and since $A$ is a bounded operator it follows that

$$(\tau Av, v) = Av(v) = a(v, v) \ge \alpha||\, v\,||^2,$$

$$||\tau Av|| = ||Av||^* \le ||\, A\,||\,||\, v\,|| \le M||\, v\,||.$$

Therefore the mapping defined in (2.2.2) is a contraction whenever the number $\rho$ belongs to the interval $]0, 2\alpha/M^2[$ and the proof is complete. □

**Definition 2.7 (Galerkin Method).** Consider the linear abstract variational problem: Find $u \in V$ such that

$$\forall\, v \in V, \quad a(u, v) = f(v),$$

where the space $V$, the bilinear form $a(\cdot,\cdot)$, and the linear form $f$ are assumed to satisfy the assumptions of the Lax-Milgram lemma. Then the Galerkin method for approximating the solution of such a problem consists of defining similar problems in finite dimensional subspace of the space $V$.

More specifically, with any finite dimensional subspace $V_h$ of $V$, we associate the discrete problem: Find $u^h \in V_h$ such that

$$\forall\, v^h \in V_h, \quad a(u^h, v^h) = f(v^h).$$

Applying the Lax-Milgram lemma we infer that such a problem has one and only one solution $u^h$, which we shall call a discrete solution.

In order to apply Galerkin method, we face by definition, the problem of constructing finite dimensional subspaces $V_h$ of space an infinite dimensional space $V$; such as $L_2(\Omega), H_0^1(\Omega)$.

**Definition 2.8 (Finite Element Method).** The finite element method, in its simplest form, is a specific process of constructing subspaces $V_h$, which shall be called finite element spaces.

This construction is characterized by three basic aspects, in other words, a finite element is a triple $(\mathcal{T}_h, \Pi_h, \Sigma)$ with following properties:

(FEM1) The first aspect, and certainly the most characteristic, is that a triangulation $\mathcal{T}_h$ is established over the set $\overline{\Omega}$, i.e., the set $\overline{\Omega}$ is subdivided into a finite number of subsets $K$, called elements, in such a way that following properties are satisfied:

($\mathcal{T}_h$1) $\overline{\Omega} = \bigcup_{K\in\mathcal{T}_h} K$.

($\mathcal{T}_h$2) For each $K \in \mathcal{T}_h$, the set $K$ is closed and its interior $\overset{\circ}{K}$ is nonempty.

($\mathcal{T}_h$3) For each distinct pair $K_1, K_2 \in \mathcal{T}_h$, one has $\overset{\circ}{K}_1 \cap \overset{\circ}{K}_2 = \emptyset$.

($\mathcal{T}_h$4) For each $K \in \mathcal{T}_h$, the boundary $\partial K$ is Lipschitz continuous[1].

(FEM2) The second basic aspect of finite element method is that the spaces

[1]A function $f : \mathbb{R}^n \supset D \to \mathbb{R}^m$ is called *Lipschitz continuous* provided that for some number $c$, $||f(x) - f(y)|| \leq c||x - y||$ for all $x, y \in D$. A hypersurface in $\mathbb{R}^n$ is a graph whenever it can be represented in the form $x_k = f(x_1, ..., x_{k-1}, x_{k+1}, ..., x_n)$, with $1 \leq k \leq n$

$\Pi_h$, contain polynomials, or, at least contain functions which are "close to" polynomials. $\Pi_h$ is a subspace of $C(T)$ with finite dimension $s$. (Functions in $\Pi_h$ are called shape functions if they form a basis for $\Pi_h$).

(FEM3) The third basic aspect of the finite element method is that there exists at least one "canonical" basis in the space $\Pi_h$, i.e., $\Sigma$ is a set of linearly independent functions on $\Pi_h$. Every $P \in \Pi_h$ is uniquely defined by the values of the $s$ (the dimension of $\Pi_h$) functions in $\Sigma$.

In a two dimensional setting with triangular elements $K \in \mathcal{T}_h$ we define

$$h_K = \text{the diameter of } K\text{=the longest side of } K,$$

$$\varrho_K = \text{the diameter of the circle inscribed in } K,$$

$$h = \max_{K \in \mathcal{T}_h} h_K.$$

We shall assume that there is a positive constant $\beta$ independent of the triangulation $\mathcal{T}_h \in \{\mathcal{T}_h\}$, i.e., independent of $h$, such that

$$\frac{\varrho_K}{h_K} \geq \beta \qquad \forall\, K \in \mathcal{T}_h.$$

This condition means that the triangles $K \in \mathcal{T}_h$ are not allowed to be arbitrary thin, or equivalently, the angles of the triangle $K$ are not allowed to be arbitrary small; the constant $\beta$ is a measure of the smallest angle in any $K \in \mathcal{T}_h$ for any $\mathcal{T}_h \in \{\mathcal{T}_h\}$.

Also we shall assume that the family $\{\mathcal{T}_h\}$ of triangulations $\mathcal{T}_h = \{K\}$ satisfies the following conditions: There are positive constants $\beta_1$ and $\beta_2$ independent of $h = \max_{K \in \mathcal{T}_h} h_K$ such that for all $K \in \mathcal{T}_h$, $\mathcal{T}_h \in \{\mathcal{T}_h\}$,

$$h_K \geq \beta_1 h, \tag{2.2.3}$$

and some suitable domain in $\mathbb{R}^{n-1}$. A domain $\Omega \subset \mathbb{R}^n$ is called a *Lipschitz domain* provided that for every $x \in \partial\Omega$, there exists a neighborhood of $\partial\Omega$ which can be represented as the graph of a Lipschitz continuous function.

$$\frac{\varrho_K}{h_K} \geq \beta_2, \tag{2.2.4}$$

where $h_K$ and $\varrho_K$ are defined above. The condition (2.2.3) states that all elements $K$ of $\mathcal{T}_h$ are roughly of the same size. Such triangulations are said to be *quasi-uniform.*

**Definition 2.9 (Green's Formula).** Given two functions $u, v \in H^1(\Omega)$, the following fundamental Green's formula

$$\int_\Omega u\, \partial_i v\, dx = -\int_\Omega \partial_i u\, v + \int_\Gamma uv\, n_i\, d\sigma,$$

holds.

From this formula, other Green's formulas may be easily deduced. For example, replacing $u$ by $\partial u$, we get

$$\int_\Omega \nabla u \cdot \nabla v\, dx = -\int_\Omega \Delta u\, v\, dx + \int_\Gamma \frac{\partial u}{\partial n} v\, d\sigma \qquad \forall u \in H^2(\Omega),\, \forall v \in H^1(\Omega).$$

One form of Green's formula that we will use is

$$\int_\Omega (\nabla u \cdot \beta) v\, dx = \int_\Gamma uv\, (n \cdot \beta)\, d\sigma - \int_\Omega u\, (\nabla v \cdot \beta)\, dx - \int_\Omega uv\, div\, \beta\, dx,$$

where $\beta$ is a vector that depends on $x$.

**Theorem 2.10 (Grönwall Inequality).** *Suppose that $u(t) \geq 0$ and $\varphi(t) \geq 0$ are continuous, real valued functions defined on the interval $0 \leq t \leq T$ and $u_0 \geq 0$ is a constant. If $u$ satisfies the inequality*

$$u(t) \leq u_0 + \int_0^t \varphi(s) u(s)\, ds \qquad \forall\, t \in [0, T]$$

*then*

$$u(t) \leq u_0\, exp\left(\int_0^t \varphi(s)\, ds\right) \qquad \forall\, t \in [0, T].$$

*In particular if $u_0 = 0$ then $u(t) \equiv 0$.*

**Proof.** Suppose first that $u_0 > 0$. Let

$$U(t) = u_0 + \int_0^t \varphi(s)u(s)\,ds. \tag{2.2.5}$$

Then, since $u(t) \leq U(t)$, we have that

$$\dot{U} = \varphi u \leq \varphi U, \qquad U(0) = u_0.$$

Since $U(t) > 0$, it follows that

$$\frac{d}{dt}\log U = \frac{\dot{U}}{U} \leq \varphi.$$

Hence

$$\log U(t) \leq \log u_0 + \int_0^t \varphi(s)\,ds,$$

so

$$u(t) \leq U(t) \leq u_0\,exp\left(\int_0^t \varphi(s)\,ds\right). \tag{2.2.6}$$

If the inequality (2.2.6) holds for $u_0 = 0$, then it also holds for all $u_0 > 0$. Tacking the limit of (2.2.6) as $u_0 \to 0^+$, we conclude that $u(t) \equiv 0$, which proves the result when $u_0 = 0$. □

**Theorem 2.11 (Cauchy-Schwarz Inequality).** *If $f, g \in L_2(\Omega)$ then $fg \in L_1(\Omega)$ and*

$$\int_\Omega |f(x)g(x)|\,dx \leq ||f||\,||g||.$$

**Proof.** This is simply a special case of the Hölder inequality with $p = q = 2$. So we prove the more general form

$$\int_\Omega |f(x)g(x)|\,dx \leq \left\{\int_\Omega |f|^p\,d\mu\right\}^{\frac{1}{p}} \left\{\int_\Omega |g|^q\,d\mu\right\}^{\frac{1}{q}}. \tag{2.2.7}$$

Let $A$ and $B$ be the two factors on the right of (2.2.7). If $A = 0$, then $f = 0$ a.e.; hence $fg = 0$ a.e., so (2.2.7) holds. If $A > 0$ and $B = \infty$, (2.2.7) is again trivial. So we need consider only the case $0 < A < \infty$, $0 < B < \infty$. Put

$$F = \frac{|f|}{A}, \qquad G = \frac{|g|}{B}. \tag{2.2.8}$$

This gives

$$\int_{\Omega} F^p \, d\mu = \int_{\Omega} G^q \, d\mu = 1. \tag{2.2.9}$$

If $x \in \Omega$ is such that $0 < F(x) < \infty$ and $0 < G(x) < \infty$, there are real numbers $s$ and $t$ such that $F(x) = e^{\frac{s}{p}}$, $G(x) = e^{\frac{t}{q}}$. Since $\frac{1}{p} + \frac{1}{q} = 1$, the convexity of the exponential function implies that

$$e^{s/p+t/q} \leq p^{-1}e^s + q^{-1}e^t. \tag{2.2.10}$$

It follows that

$$F(x)G(x) \leq p^{-1}F(x)^p + q^{-1}G(x)^q, \tag{2.2.11}$$

for every $x \in \Omega$. Integrating of (2.2.11) yields

$$\int_{\Omega} FG \, d\mu \leq p^{-1} + q^{-1} = 1, \tag{2.2.12}$$

by (2.2.9); Inserting (2.2.8) into (2.2.12), we obtain (2.2.7). □

**Lemma 2.12.** *For $a, b \in \mathbb{R}$ and $\epsilon > 0$ we have the following inequality*

$$ab \leq \frac{\epsilon a^2}{C} + \frac{Cb^2}{\epsilon}.$$

The proof is straightforward.

## 2.3 Saddle Point Problem

**Definition 2.13 (Inf-Sup Condition).** Let $U$ and $V$ be Hilbert spaces. Then we say that the form $a : U \times V \to \mathbb{R}$ satisfies the *Inf-Sup condition* if there exists $\alpha > 0$ such that

$$\sup_{v \in V} \frac{a(u,v)}{||v||_V} \geq \alpha ||u||_U \qquad \forall \, u \in U.$$

The name for this condition comes from the equivalent formulation

$$\inf_{u\in U}\sup_{v\in V}\frac{a(u,v)}{||v||_V||u||_U}\geq\alpha>0.$$

We now turn to variational problems with constraints. Let $X$ and $M$ be two Hilbert spaces, and suppose

$$a : X\times X\to\mathbb{R},\quad b : X\times M\to\mathbb{R}$$

are continuous bilinear forms. We denote both the dual pairing of $X$ with $X'$ and that of $M$ with $M'$, associated by the scalar product $\langle\,\cdot,\cdot\,\rangle$. We consider the following minimization problem.

**Problem (M).** Let $f\in X'$ and $g\in M'$. Find the minimum over $X$ of

$$J(u)=\frac{1}{2}a(u,u)-\langle\, f,u\,\rangle$$

subject to the constraint

$$b(u,\mu)=\langle\, g,\mu\,\rangle\qquad\forall\,\mu\in M.$$

Our starting point is the same as in classical theory of Lagrange extremal problems. If $\lambda\in M$, then $J$ and the *Lagrange function*

$$\mathcal{L}(u,\lambda):=J(u)+[b(u,\lambda)-\langle\, g,\lambda\,\rangle]$$

have the same values on the set of all points which satisfy the constraints (see [19], Chapter II). Instead of finding the minimum of $J$, we can seek a minimum of $\mathcal{L}(\cdot,\lambda)$ with fixed $\lambda$. This raises the question of whether $\lambda\in M$ can be selected so that the minimum of $\mathcal{L}(\cdot,\lambda)$ over the space $X$ is assumed by an element which satisfies the given constraints. Since $\mathcal{L}(u,\lambda)$ is a quadratic expression in $u$ and $\lambda$, we are led to the following *saddle point problem*:

**Problem (S).** Find $(u,\lambda) \in X \times M$ with

$$\begin{cases} a(u,v) + b(v,\lambda) & = \langle\, f, v\,\rangle \qquad \forall\, v \in X, \\ b(u,\mu) & = \langle\, g, \mu\,\rangle \qquad \forall\, \mu \in M. \end{cases} \tag{2.3.1}$$

We want to find conditions implying existence and possibly uniqueness of solutions to this problem. If the bilinear form $a(\cdot,\cdot)$ is symmetric, equations (2.3.1) are optimality conditions of the saddle point problem. It is easy to see that every solution $(u,\lambda)$ of Problem (S) must satisfy the *saddle point property*

$$\mathcal{L}(u,\mu) \leq \mathcal{L}(u,\lambda) \leq \mathcal{L}(v,\lambda) \qquad \forall\,(v,\mu) \in X \times M.$$

Equation (2.3.1) defines a linear mapping

$$\begin{aligned} L : X \times M &\to X' \times M' \\ (u,\lambda) &\mapsto (f,g). \end{aligned} \tag{2.3.2}$$

To show that $L$ is an isomorphism we need the *Inf-Sup* condition. We introduce special notation for the affine space of admissible elements and for the corresponding linear spaces:

$$\begin{aligned} V(g) &:= \{v \in X; b(v,\mu) = \langle g,\mu\rangle \qquad \forall \mu \in M\}, \\ V &:= \{v \in X; b(v,\mu) = 0 \qquad \forall \mu \in M\}. \end{aligned} \tag{2.3.3}$$

Since $b$ is continuous, $V$ is a closed subspace of $X$.

Now we are ready for the main theorem for saddle point problems.

**Theorem 2.14.** *For the saddle point problem* (2.3.1)*, the mapping* (2.3.2) *defines an isomorphism $L : X \times M \to X' \times M'$ if and only if the following conditions are satisfied:*

*(i) The bilinear form $a$ is $V$-elliptic, i.e.,*

$$a(v,v) \geq \alpha\|\, v\,\|^2 \qquad \forall v \in V,$$

*where* $\alpha > 0$, *and* $V$ *is as in* (2.3.3).

*(ii) The bilinear form* $b$ *satisfies the Inf-Sup condition.*

For the proof see [16] page 127.

# Chapter 3

# Stability of the DG Method

## 3.1 The Continuous Problem

We consider the following initial-boundary value problem for the viscous Burgers equation: Find the scalar function $u \equiv u(\mathbf{x}, t) := u(x, y, t)$ such that

$$\begin{cases} u_t + uu_x + uu_y - \varepsilon \Delta u &= 0 \qquad (\mathbf{x}, t) \in \Omega \times I \\ u(\mathbf{x}, 0) &= u_0 \qquad \mathbf{x} \in \Omega, \\ u &= 0 \qquad (\mathbf{x}, t) \in \Gamma \times I, \end{cases} \tag{3.1.1}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^2$ with the boundary $\Gamma := \partial\Omega$, $u_t = \partial u/\partial t$, and we shall denote $uu_x + uu_y = (u, u) \cdot \nabla u$ by $\tilde{u} \cdot \nabla u$, with $\tilde{u} := (u, u)$ and $\nabla$ is the gradient operator with respect to $(x, y) \in \mathbb{R}^2$. Further $\varepsilon > 0$ is a small parameter and $\mathbf{n} = \mathbf{n}(\mathbf{x}) = (n_1(\mathbf{x}), n_2(\mathbf{x}))$ is the outward unit normal to the boundary $\Gamma$ at the point $\mathbf{x} \in \Gamma$. By $\tilde{u} \cdot \mathbf{n}$ we mean the usual scalar product $(u, u) \cdot (n_1, n_2) = (n_1 + n_2)u$. Finally, $u_0$ is the initial data, and $I = (0, T)$ is a given time interval.

## 3.2 The Discontinuous Galerkin Method

The discontinuous Galerkin (DG) method for (3.1.1) is based on using finite elements over the space-time domain $Q = \Omega \times I$, with interelement discontinuities, at the interelement boundaries, both in space and time variables. To define this method, let $\mathcal{T}_h = \{\tau\}$ be a finite element subdivision of $\Omega$ into triangular elements $\tau$, $\mathcal{E}_h$ the set of edges of $\mathcal{T}_h$, and let $0 = t_0 < t_1 < ... < t_M = T$ be a partition of the time interval $I$ into subintervals $I_m = (t_{m-1}, t_m)$, $m = 1, \dots, M$. Let $\mathcal{C}_h = \{K\}$ be the corresponding subdivision of $Q$ into elements $K = \tau \times I_m$ with $h$ representing the maximum of the diameters of the $K \in \mathcal{C}_h$, and let $P_k(K)$ be the set of all polynomials in $(x, y)$ and $t$ of degree at most $k$ on $K$, and define for $k \geq 0$, the function spaces

$$W_h = \{v \in L_2(Q) : v|_K \in P_k(K) \quad \forall K \in \mathcal{C}_h\},$$

$$\mathbf{W}_h = \left\{\mathbf{w} \in [L_2(Q)]^2 : \mathbf{w}|_K \in [P_k(K)]^2 \quad \forall K \in \mathcal{C}_h\right\}.$$

We shall assume that $\mathcal{C}_h$ is a quasi-uniform subdivision of $Q$, i.e., for each $K \in \mathcal{C}_h$ there is an inscribed sphere in $K$ such that the ratio of the diameter of this sphere and the diameter of $K$ is bounded below, independently of $K$ and $h$. We shall use the following notation: Given a domain $G$, let $(\cdot, \cdot)_G$ denote the usual $L_2(G)$ scalar product and $||\cdot||_G$ the corresponding $L_2$-norm. Also, for a positive integer $s$, $H^s(G)$ will denote the usual Sobolev space of functions with square integrable derivatives of order less than or equal to $s$, with norm $||\cdot||_{s,G}$, defined by

$$||f||_{s,G} = \Big(\sum_{|\alpha| \leq s} \int_G |\mathcal{D}^\alpha f|^2\Big)^{1/2}.$$

To define a finite element method using discontinuous functions, we introduce the following notation: If $\beta = (\beta_1, \beta_2)$ is a given smooth vector field on $Q$, we define

$$\partial Q_{\mp}(\beta) = \{(\mathbf{x}, t) \in \partial Q : n_t(\mathbf{x}, t) + n(\mathbf{x}, t) \cdot \beta(\mathbf{x}, t) \lesseqgtr 0\},$$

where $\partial Q = \Omega \times \{0\} \cup \Omega \times \{T\} \cup \Gamma \times I$, and $(\mathbf{n}, n_t)$ is the outward unit normal to $\partial Q$. Similarly we define for $K \in \mathcal{C}_h$,

$$\partial K_{\mp}(\beta) = \{(\mathbf{x}, t) \in \partial K : n_t(\mathbf{x}, t) + n(\mathbf{x}, t) \cdot \beta(\mathbf{x}, t) \lesseqgtr 0\},$$

and write

$$\langle w, v \rangle_m = (w(\cdot, t_m), v(\cdot, t_m))_\Omega, \qquad |v|_m = \langle v, v \rangle_m^{1/2},$$

and

$$w_{\mp}(\mathbf{x}, t) = \lim_{s \to 0^{\mp}} w(\mathbf{x} + s\beta, t + s), \qquad [w] = w_+ - w_-.$$

In the sequel we suppress the domains from the subscript of the scalar products, unless it is necessary for the context and denote the inner product in $L_2$ over the actual domain, simply, by $(\cdot, \cdot)$. For notational convenience we shall use $\tilde{u}^h$ for $(u^h, u^h)$, unless when we specifically single out the variables, separately.

## 3.3 Stability Estimate

We write the weak discontinuous Galerkin (DG) variational formulation of the equation (3.1.1) as follows: find $u^h \in W^h$ such that

$$\begin{aligned}(u_t^h + \tilde{u}^h \cdot \nabla u^h, v + \delta(v_t + \tilde{u}^h \cdot \nabla v)) - (\varepsilon \Delta u^h, v + \delta(v_t + \tilde{u}^h \cdot \nabla v))& \\ + \sum_K \int_{\partial K_-(\tilde{u}^h)} [u^h] v_+ |n_t + \tilde{u}^h \cdot \mathbf{n}| ds + \langle u_+^h, v_+ \rangle_0& \\ = \langle u_0, v_+ \rangle_0.&\end{aligned} \tag{3.3.1}$$

If we set $v = u^h$ in variational equation we obtain,

$$
\begin{aligned}
(u^h_t+\tilde{u}^h\cdot\nabla\, u^h, u^h+\delta(u^h_t+\tilde{u}^h\cdot\nabla u^h)) - (\varepsilon\Delta u^h, u^h+\delta(u^h_t+\tilde{u}^h\cdot\nabla\, u^h))\\
+\sum_K\int_{\partial K_-(\tilde{u}^h)}[u^h]u^h_+|n_t+\tilde{u}^h\cdot\mathbf{n}| + \langle\, u^h_+, u^h_+\,\rangle_0 = \langle\, u_0, u^h_+\,\rangle_0,
\end{aligned}
\tag{3.3.2}
$$

Our main objective in stability estimate is to extract positive terms that appear in variational formulation so that we can define a new triple norm to prove the coercivity of DG-scheme that we introduce below. Now we compute each term in (3.3.2) separately. The first term can be considered as follows.

**Lemma 3.1.** *The first term in* (3.3.2) *can be identified viz,*

$$
\begin{aligned}
(u^h_t+\tilde{u}^h\cdot\nabla\, u^h, u^h+\delta(u^h_t+\tilde{u}^h\cdot\nabla\, u^h)) = \delta||u^h_t+\tilde{u}^h\cdot\nabla\, u^h||^2\\
+\frac{1}{2}\sum_K\int_{\partial K}(u^h)^2(n_t)\,ds+\sum_K\frac{1}{3}\int_{\partial K}(u^h)^2(\tilde{u}^h\cdot\mathbf{n})\,ds.
\end{aligned}
$$

**Proof.** We split the inner product as

$$
\begin{aligned}
&(u^h_t+\tilde{u}^h\cdot\nabla\, u^h, u^h)+(u^h_t+\tilde{u}^h\cdot\nabla\, u^h, \delta(u^h_t+\tilde{u}^h\cdot\nabla\, u^h))\\
&\quad=(u^h_t+\tilde{u}^h\cdot\nabla\, u^h, u^h)+\delta||u^h_t+\tilde{u}^h\cdot\nabla\, u^h||^2,
\end{aligned}
\tag{3.3.3}
$$

note that using integration by parts we may write,

$$
(u^h_t, u^h)_Q=\sum_K(u^h_t, u^h)_K=\sum_K\int_{\partial K}(u^h)^2(n_t)\,ds-\sum_K(u^h, u^h_t)_K.
$$

Therefore we have

$$
(u^h_t, u^h)_Q=\frac{1}{2}\sum_K\int_{\partial K}(u^h)^2(n_t)\,ds. \tag{3.3.4}
$$

Further by Green's formula we can write

$$
\sum_K(\tilde{u}^h\cdot\nabla\, u^h, u^h)_K=\sum_K\int_{\partial K}(u^h)^2(\tilde{u}^h\cdot\mathbf{n})\,ds-\sum_K(u^h, \tilde{u}^h\cdot\nabla\, u^h)_K-\sum_K(u^h, u^h\, div\,\tilde{u}^h)_K,
$$

since $(u^h, \tilde{u}^h \cdot \nabla u^h) = (u^h, u^h\, div\, \tilde{u}^h)$, we have

$$\sum_K (\tilde{u}^h \cdot \nabla u^h, u^h)_K = \sum_K \frac{1}{3} \int_{\partial K} (u^h)^2 (\tilde{u}^h \cdot \mathbf{n})\, ds. \tag{3.3.5}$$

Combining (3.3.3)- (3.3.5) we complete the proof. □

Next we identify the boundary terms appeared in the lemma:

$$\int_{\partial K} (u^h)^2 (n_t)\, ds = \int_{\partial K_+(\tilde{u}^h)} (u^h_-)^2 (n_t)\, ds + \int_{\partial K_-(\tilde{u}^h)} (u^h_+)^2 (n_t)\, ds. \tag{3.3.6}$$

Similarly,

$$\int_{\partial K} (u^h)^2 (\tilde{u}^h \cdot \mathbf{n})\, ds = \int_{\partial K_+(\tilde{u}^h)} (u^h_-)^2 (\tilde{u}^h \cdot \mathbf{n})\, ds + \int_{\partial K_-(\tilde{u}^h)} (u^h_+)^2 (\tilde{u}^h \cdot \mathbf{n})\, ds. \tag{3.3.7}$$

Observe that

$$\begin{aligned}
&\sum_K \left[ \int_{\partial K_+(\tilde{u}^h)} (u^h_-)^2 (n_t)\, ds + \int_{\partial K_+(\tilde{u}^h)} (u^h_-)^2 (\tilde{u}^h \cdot \mathbf{n})\, ds \right] \\
&\quad = \sum_K \int_{\partial K_+(\tilde{u}^h)} (u^h_-)^2 (n_t + \tilde{u}^h \cdot \mathbf{n})\, ds \\
&\quad = \int_{\partial Q_+} (u^h_-)^2\, ds + \sum_K \int_{\partial K_+(\tilde{u}^h)'} (u^h_-)^2 (n_t + \tilde{u}^h \cdot \mathbf{n})\, ds \\
&\quad = \int_{\partial \Omega_+ \times I} (u^h)^2 (\tilde{u}^h \cdot \mathbf{n})\, ds + |u^h|^2_M - \sum_K \int_{\partial K'_-(\tilde{u}^h)} (u^h_-)^2 (n_t + \tilde{u}^h \cdot \mathbf{n})\, ds \\
&\quad = \int_{\partial \Omega_+ \times I} (u^h)^2 |\tilde{u}^h \cdot \mathbf{n}|\, ds + |u^h|^2_M + \sum_K \int_{\partial K_-(\tilde{u}^h)'} (u^h_-)^2 |n_t + \tilde{u}^h \cdot \mathbf{n}|\, ds,
\end{aligned} \tag{3.3.8}$$

and

$$\begin{aligned}
&\sum_K \left[\int_{\partial K_-(\tilde{u}^h)} (u_+^h)^2(\tilde{u}^h \cdot \mathbf{n})\, ds + \int_{\partial K_-(\tilde{u}^h)} (u_+^h)^2(n_t)\, ds\right] \\
&\qquad = \sum_K \int_{\partial K_-(\tilde{u}^h)} (u_+^h)^2(n_t + \tilde{u}^h \cdot \mathbf{n})\, ds \\
&\qquad = \int_{\partial Q_-} (u_+^h)^2\, ds + \sum_K \int_{\partial K'_-(\tilde{u}^h)} (u_+^h)^2(n_t + \tilde{u}^h \cdot \mathbf{n})\, ds \\
&\qquad = -|u^h|_0^2 - \sum_K \int_{\partial K_-(\tilde{u}^h)'} (u_+^h)^2 |n_t + \tilde{u}^h \cdot \mathbf{n}|\, ds.
\end{aligned} \tag{3.3.9}$$

where $\partial K_+(\tilde{u}^h)' = \partial K_+(\tilde{u}^h) \setminus \partial Q_+$, and $\partial K_-(\tilde{u}^h)' = \partial K_-(\tilde{u}^h) \setminus \Omega \times \{0\}$.

**Lemma 3.2.** *Combining* (3.3.6)-(3.3.9)*, we can derive the following inequality*

$$\begin{aligned}
&\sum_K \Big[\frac{1}{2}\int_{\partial K} (u^h)^2(n_t)\, ds + \frac{1}{3}\int_{\partial K} (u^h)^2(\tilde{u}^h \cdot \mathbf{n})\, ds + \int_{\partial K_-(\tilde{u}^h)} [u^h]^2 |n_t + \tilde{u}^h \cdot \mathbf{n}|\, ds\Big] + |u_h|_0^2 \\
&\quad \geq \frac{1}{4}\left[|u^h|_0^2 + |u^h|_M^2 + \sum_K \int_{\partial K_-(\tilde{u}^h)'} [u^h]^2 |n_t + \tilde{u}^h \cdot \mathbf{n}|\, ds + \int_{\partial \Omega_+ \times I} (u^h)^2 |\tilde{u}^h \cdot \mathbf{n}|\, ds\right].
\end{aligned}$$

**Proof.** We have that

$$
\begin{aligned}
&\sum_{K\in\mathcal{C}_h}\left[(u_t^h,u^h)_K+(\tilde{u}^h\cdot\nabla u^h,u^h)_K+\int_{\partial K_-(\tilde{u}^h)'}[u^h]u_+^h|n_t+\tilde{u}^h\cdot\mathbf{n}|\,ds\right]+|u^h|_0^2\\
&\quad=\sum_K\Bigg[\int_{\partial K_-(\tilde{u}^h)'}(u_+^h-u_-^h)u_+^h|n_t+\tilde{u}^h\cdot\mathbf{n}|\,ds\\
&\qquad+\frac{1}{2}\left(\int_{\partial K_-(\tilde{u}^h)}(u_+^h)^2(n_t)\,ds+\int_{\partial K_-(\tilde{u}^h)}(u_-^h)^2(n_t)\,ds\right)\\
&\qquad+\frac{1}{3}\left(\int_{\partial K_-(\tilde{u}^h)}(u_-^h)^2(\tilde{u}^h\cdot\mathbf{n})\,ds-\int_{\partial K_-(\tilde{u}^h)}(u_+^h)^2(\tilde{u}^h\cdot\mathbf{n})\,ds\right)\Bigg]+|u^h|_0^2\\
&\quad\geq\sum_K\Bigg[\int_{\partial K_-(\tilde{u}^h)'}(u_+^h)^2|n_t+\tilde{u}^h\cdot\mathbf{n}|\,ds-\int_{\partial K_-(\tilde{u}^h)'}u_+^hu_-^h|n_t+\tilde{u}^h\cdot\mathbf{n}|\,ds\\
&\qquad-\frac{1}{2}\int_{\partial K_-(\tilde{u}^h)'}(u_+^h)^2|n_t+\tilde{u}^h\cdot\mathbf{n}|\,ds+\frac{1}{3}\int_{\partial K_-(\tilde{u}^h)'}(u_-^h)^2|n_t+\tilde{u}^h\cdot\mathbf{n}|\,ds\Bigg]\\
&\quad-\frac{1}{2}\int_{\partial Q_-}(u_+^h)^2\,ds+\frac{1}{4}\int_{\partial Q_+}(u_-^h)^2\,ds+|u^h|_0^2.
\end{aligned}
\tag{3.3.10}
$$

Since $\int_{\partial Q_-}(u_+^h)^2\,ds=|u^h|_0^2$, and $\int_{\partial Q_+}(u_-^h)^2\,ds=\int_{\partial\Omega_+\times I}(u^h)^2|\tilde{u}^h\cdot\mathbf{n}|\,ds+|u^h|_M^2$, we can derive

$$
\begin{aligned}
&\sum_K\int_{\partial K_-(\tilde{u}^h)'}[u^h]^2|n_t+\tilde{u}^h\cdot\mathbf{n}|\,ds+\int_{\partial Q_-}(u_+^h)^2\,ds+\int_{\partial Q_+}(u_-^h)^2\,ds\\
&\quad=|u^h|_0^2+|u^h|_M^2+\sum_K\int_{\partial K_-(\tilde{u}^h)'}[u^h]^2|n_t+\tilde{u}^h\cdot\mathbf{n}|\,ds+\int_{\partial\Omega_+\times I}(u^h)^2|\tilde{u}^h\cdot\mathbf{n}|\,ds,
\end{aligned}
$$

which gives the desired result. □

To derive a variational formulation, for the diffusive part of (3.1.1) based on discontinuous trial functions we need to introduce the operator $R:W_h\to\mathbf{W}_h$. For this we consider the homogenous Laplace equation, i.e., $-\Delta u=0$. Introducing the auxiliary vector variable $\theta=\nabla u$, the problem can be rewritten as

$$
\begin{cases}
\theta-\nabla u=0\\
-div\,\theta=0.
\end{cases}
$$

To formulate a discrete variational formulation for this equation, it is convenient to introduce some notation: Let $\tau$ be an element, and let $e$ be an edge of $\tau$; let $\tau^{ext}$ be the other element having $e$ as an edge, and let $u_h^{ext}$, and $\theta_h^{ext}$, denote the values of $u^h$, and $\theta_h$, respectively, in $\tau^{ext}$. Then define

$$(u^h)^0 = \frac{u^h + u_h^{ext}}{2},$$

$$(\theta^h)^0 = \frac{\theta^h + \theta_h^{ext}}{2},$$

i.e., the average value of the variable. Now we have the following variational formulation (see [9]-[12]):

$$\begin{cases} \text{find } (u^h, \theta^h) \in W_h \times \mathbf{W}_h \text{ such that for } \; m = 1, \dots, M, \\ \sum_m \int_{I_m} \sum_{\tau \in \mathcal{T}_h} \Big[ \int_\tau (\theta^h - \nabla h^h) \cdot \tau^h - \int_{\partial \tau} ((u^h)^0 - u^h) \tau^h \cdot \mathbf{n} \Big] = 0 \text{ for all } \tau^h \in \mathbf{W}_h, \\ \sum_m \int_{I_m} \sum_{\tau \in \mathcal{T}_h} \Big[ \int_\tau \theta^h \cdot \nabla v^h - \int_{\partial \tau} (\theta^h)^0 \cdot \mathbf{n} \; v^h \Big] = 0 \text{ for all } v^h \in W_h. \end{cases} \tag{3.3.11}$$

Notice that the first equation in (3.3.11) corresponds to the condition $\theta = \nabla u$, and the second one to the condition $-div\,\theta = 0$.

After some manipulations, equations (3.3.11) can be written in the following form:

$$\begin{cases} \sum_m \int_{I_m} \sum_{\tau \in \mathcal{T}_h} \int_\tau (\theta^h - \nabla h^h) \cdot \tau^h + \sum_{e \in \mathcal{E}_h} \int_e [u^h] \cdot (\tau^h)^0 = 0, \\ \sum_m \int_{I_m} \sum_{\tau \in \mathcal{T}_h} \int_\tau \theta^h \cdot \nabla v^h - \sum_{e \in \mathcal{E}_h} \int_e (\theta^h)^0 \cdot [v^h] = 0. \end{cases} \tag{3.3.12}$$

Actually, introducing the bilinear forms $a(\cdot,\cdot)$ on $\mathbf{W}_h \times \mathbf{W}_h$ and $b(\cdot,\cdot)$ on $W_h \times \mathbf{W}_h$ as

$$a(\theta^h, \tau^h) = \sum_m \int_{I_m} \int_\Omega \theta^h \cdot \tau^h,$$

$$b(u^h,\tau^h) = -\sum_m \int_{I_m} \sum_{\tau\in\mathcal{T}_h} \int_\tau \nabla u^h \cdot \tau^h + \sum_{e\in\mathcal{E}_h} \int_e [u^h]\cdot(\tau^h)^0,$$

the problem (3.3.12) can be written in a concise form viz,

$$\begin{cases} a(\theta^h,\tau^h) + b(u^h,\tau^h) = 0, \\ -b(v^h,\theta^h) = 0. \end{cases} \tag{3.3.13}$$

Now, we can derive a single variational equation by solving the first equation of (3.3.12) for $\theta^h$.

We define the linear operator $R: W_h \to \mathbf{W}_h$ by

$$(R(v),\tau^h)_Q = -\sum_m \int_{I_m} \sum_{e\in\mathcal{E}_h} \int_e [v]\mathbf{n}\cdot(\tau_0^h) \text{ for all } \tau^h \in \mathbf{W}_h.$$

From the first equation of (3.3.12) we have

$$R(u^h) = \theta^h - \nabla u^h.$$

Using this (with $\tau^h = \theta^h$ and $v = v^h$), in the second equation of (3.3.12), the scheme becomes

$$\sum_m \int_{I_m} \sum_{\tau\in\mathcal{T}_h} \int_\tau \big(\nabla u^h + R(u^h)\big)\cdot\big(\nabla v^h + R(v^h)\big) = 0. \tag{3.3.14}$$

We are interested in study of system (3.3.12) for discontinuous piecewise polynomials of degree $k \geq 1$, for both $v^h \in W_h$ and $\tau^h \in \mathbf{W}_h$. In order to guarantee the nonsingularity of the matrix associated with (3.3.13) an *Inf-Sup* condition is needed.

The system of equations (3.3.12) can be recognized as a saddle-point type problem and it is clear that an *Inf-Sup* condition should be satisfied in order to ensure existence and uniqueness of solution of (3.3.12). Unfortunately, the *Inf-Sup* condition does not hold for this choice of spaces, as in shown in [20] through a counter-example. Since an *a priori* control over the $u^h$ variable cannot be provided, it is natural to add the second equation of (3.3.13) a suitable stabilizing term. We describe here the modification proposed in [12]. Mimicking the definition of $R$, for $e \in \mathcal{E}_h$ we define the operator $r_e : W_h \to \mathbf{W}_h$ to be the restriction of $R$ to the elements sharing the edge $e \in \mathcal{E}_h$, i.e.,

$$(r_e(v), \tau^h)_Q = -\sum_m \int_{I_m} \int_e [v]\mathbf{n} \cdot (\tau^h)^0 \text{ for all } \tau^h \in \mathbf{W}_h. \tag{3.3.15}$$

It can be seen that the following relationship between $R$ and $r_e$ holds: for any triangle $\tau \in \mathcal{T}_h$, we have

$$\sum_{e \subset \partial\tau} r_e = R \text{ on } \tau. \tag{3.3.16}$$

If $e$ is an internal edge, it is clear from definition (3.3.15) that the support of $r_e$ is contained in the union of the two triangles sharing the edge $e$. The modification that we use here, proposed by Bassi and Rebay consists in replacing in (3.3.14), for each $\tau \in \mathcal{T}_h$,

$$\text{the term } \int_\tau R(u^h) \cdot R(v^h) \text{ by } \sum_{e \subset \partial\tau} \int_\Omega r_e(u^h) \cdot r_e(v^h).$$

This procedure can be interpreted in the following way. The quantity $r_e(v^h)$ allows to control the jump of $v^h$ on $e$; hence, a natural stabilization of (3.3.13) consists in adding to the left-hand side of the second equation in (3.3.14), the term

$$\lambda \sum_{e \in \mathcal{E}_h} \int_\Omega r_e(u^h) \cdot r_e(v^h),$$

where $\lambda > 0$ is a parameter, thus obtaining the new scheme

$$\sum_m \int_{I_m} \sum_{\tau \in \mathcal{T}_h} \int_\tau \big(\nabla u^h + R(u^h)\big) \cdot \big(\nabla v^h + R(v^h)\big) + \sum_m \int_{I_m} \lambda \sum_{e \in \mathcal{E}_h} \int_\Omega r_e(u^h) \cdot r_e(v^h) = 0.$$

If $\lambda$ is large enough, by definition of $R$ and $r_e$ we can also suppress the term $\int_\tau R(u^h) \cdot R(v^h)$ in last equation, obtaining the following scheme, equivalent to the formulation of Bassi and Rebay [12]:

$$\sum_m \int_{I_m} \sum_{\tau \in \mathcal{T}_h} \int_\tau (\nabla u^h \cdot \nabla v^h + \nabla u^h \cdot R(v^h) + R(u^h) \cdot \nabla v^h) \\ + \sum_m \int_{I_m} \lambda \sum_{e \in \mathcal{E}_h} \int_\Omega r_e(u^h) \cdot r_e(v^h) = 0. \tag{3.3.17}$$

As a consequence of (3.3.16) we have the following estimate

$$||R(v^h)||_K^2 \leq \gamma \sum_{e \subset \mathcal{E}_h \cap \partial \tau_K} ||r_e(v^h)||_K^2, \tag{3.3.18}$$

where $\tau_K$ corresponds to the element $K$.

Since the support of each $r_e$ is the union of elements sharing the edge $e$, we can write

$$\sum_{e \in \mathcal{E}_h} ||r_e(v^h)||_Q^2 = \sum_{K \in \mathcal{C}_h} \sum_{e \subset \mathcal{E}_h \cap \partial \tau_K} ||r_e(v^h)||_K^2. \tag{3.3.19}$$

For the remaining term, i.e., $-\varepsilon\delta\left(\Delta u^h, u_t^h + \tilde{u}^h \cdot \nabla\, u^h\right)$ we use from Cauchy-Schwarz inequality and inverse estimate to have an estimate for this term. We assume that $\delta = h$ and $\varepsilon = h$. Thus

$$-\varepsilon\delta\left(\Delta u^h, u_t^h + \tilde{u}^h \cdot \nabla\, u^h\right) \geq -\varepsilon\delta h^{-1}||\nabla u^h||\ ||u_t^h + \tilde{u}^h \cdot \nabla\, u^h|| \\ \geq -\frac{1}{4}\delta||\nabla u^h||^2 - 4\delta||u_t^h + \tilde{u}^h \cdot \nabla u^h||^2 \geq -\sigma|||u^h|||^2. \tag{3.3.20}$$

where $\sigma > 0$ is a constant depending on $h$ and determine later.

Using these notations we are now ready to reformulate the variational formulation for the discontinuous Galerkin approximation of (3.1.1) as:

*find $u^h \in W_h$ such that for $m = 0, 1, ..., M-1$ and for all $v^h \in W_h$*

$$\begin{aligned}
&(u_t^h + \tilde{u}^h \cdot \nabla u^h, v^h + \delta(v_t^h + \tilde{u}^h \cdot \nabla v^h))_Q + \varepsilon(\nabla u^h, \nabla v^h)_Q \\
&+ \sum_{K \in \mathcal{C}_h} \int_{\partial K_-(\tilde{u}^h)} [u^h] v_+^h |n_t + \tilde{u}^h \cdot \mathbf{n}| \, ds + \varepsilon(\nabla u^h, R(v^h))_Q + \varepsilon(R(u^h), \nabla v^h)_Q \\
&+ \lambda\varepsilon \sum_{e \in \mathcal{E}_h} (r_e(u^h), r_e(v^h))_Q - \delta\varepsilon(\Delta u^h, v_t^h + \tilde{u}^h \cdot \nabla v^h)_Q = 0.
\end{aligned} \tag{3.3.21}$$

To proceed and to write (3.3.21) on more compact form we define the discontinuous Galerkin trilinear form $B_{DG}$ by

$$\begin{aligned}
B_{DG}(\tilde{u}; u, v^h) = {}& (u_t + \tilde{u} \cdot \nabla u, v^h + \delta(v_t^h + \tilde{u}^h \cdot \nabla v^h))_Q + \langle\, u_+, v_+^h \,\rangle + \varepsilon(\nabla u, R(v^h))_Q \\
&+ \sum_{K \in \mathcal{C}_h} \int_{\partial K_-(\tilde{u}^h)} [u] v_+^h |n_t + \tilde{u}^h \cdot \mathbf{n}| \, ds + \varepsilon(\nabla u, R(v^h))_Q + \varepsilon(R(u), \nabla v^h)_Q \\
&+ \lambda\varepsilon \sum_{e \in \mathcal{E}_h} (r_e(u), r_e(v^h))_Q - \delta\varepsilon(\Delta u, v_t^h + \tilde{u}^h \cdot \nabla v^h)_Q,
\end{aligned} \tag{3.3.22}$$

moreover we define the linear form $L$ as

$$L(v^h) = \langle\, u_0, v_+^h \,\rangle_0.$$

Now using these notations we can formulate the problem (3.3.21) in the following form:

*find $u^h \in W_h$ such that*

$$B_{DG}(\tilde{u}^h; u^h, v^h) = L(v^h) \qquad \forall v^h \in W_h. \tag{3.3.23}$$

We derive our stability estimate and prove convergence rates for DG-scheme

(3.3.23) in the triple norm

$$|||u^h|||^2 = \delta||u_t^h + \tilde{u}^h \cdot \nabla u^h||_Q^2 + \varepsilon||\nabla u^h||_Q^2 + \varepsilon \sum_{e\in\mathcal{E}_h} ||r_e(u^h)||_Q^2$$
$$+ \frac{1}{4}\left[|u^h|_0^2 + |u^h|_M^2 + \sum_K \int_{\partial K_-(u^h)'} |[u^h]|^2|n_t + \tilde{u}^h \cdot \mathbf{n}|\, ds + \int_{\partial\Omega_+\times I} (u^h)^2|\tilde{u}^h \cdot \mathbf{n}|\, ds\right]. \tag{3.3.24}$$

Before the prove of stability estimate we need some assumptions that we introduce below. We assume that $\delta$ is small enough.

**Lemma 3.3.** *There exists a constant $\alpha > 0$ independent of $h$ such that*

$$\forall u^h \in W_h \qquad B_{DG}\left(\tilde{u}^h; u^h, u^h\right) \geq \alpha|||u^h|||^2.$$

**Proof.** Using the definition of $B_{DG}$ and (3.3.19) we have that

$$\begin{aligned} B_{DG}(\tilde{u}^h; u^h, u^h) =& |u^h|_0^2 + \sum_{K\in\mathcal{C}_h} \Big[(u_t^h, u^h)_K + (\tilde{u}^h \cdot \nabla u^h, u^h)_K \\ &+ \int_{\partial K_-(\tilde{u}^h)'} [u^h]u_+^h|n_t + \tilde{u}^h \cdot \mathbf{n}|\, ds \\ &+ h||u_t^h + \tilde{u}^h \cdot \nabla u^h||_K^2 - h\varepsilon(\Delta u^h, u_t^h + \tilde{u}^h \cdot \nabla u^h)_K \\ &+ \varepsilon||\nabla u^h||_K^2 + 2\varepsilon(\nabla u^h, R(u^h))_K + \lambda\varepsilon \sum_{e\subset\partial\tau_K} ||r_e(u^h)||_K^2\Big] \\ &:= \sum_{i=1}^{9} T_i. \end{aligned} \tag{3.3.25}$$

Now we estimate the terms $T_1, ..., T_9$, separately. From (3.3.10) we can write

$$\begin{aligned} T_1 + T_2 + T_3 + T_4 \geq \frac{1}{4}\Bigg[&|u^h|_0^2 + |u^h|_M^2 + \int_{\partial\Omega_+\times I} (u^h)^2|\tilde{u}^h \cdot \mathbf{n}|\, ds \\ &+ \sum_K \int_{\partial K_-(\tilde{u}^h)'} [u^h]^2|n_t + \tilde{u}^h \cdot \mathbf{n}|\, ds\Bigg]. \end{aligned} \tag{3.3.26}$$

Now we estimate $T_8$ using (3.3.16) and (3.3.18) and for some $\epsilon > 0$ we have

$$T_8 \geq -\varepsilon \left[\epsilon ||\nabla u^h||_K^2 + \frac{1}{\epsilon}||R(u^h)||_K^2\right] \geq -\varepsilon \left[\epsilon ||\nabla u^h||_K^2 + \frac{\gamma}{\epsilon} \sum_{e \subset \partial\tau_K} ||r_e(u^h)||_K^2\right].$$

thus we deduce that

$$\begin{aligned} T_7 + T_8 + T_9 &\geq \varepsilon \sum_{K \in \mathcal{C}_h} \left[(1-\epsilon)||\nabla u^h||_K^2 - \frac{1}{\epsilon}||R(u^h)||_K^2 + \lambda \sum_{e \subset \partial\tau_K} ||r_e(u^h)||_K^2\right] \\ &\geq \varepsilon \sum_{K \in \mathcal{C}_h} \left[(1-\epsilon)||\nabla u^h||_K^2 + (\lambda - \frac{\gamma}{\epsilon}) \sum_{e \subset \partial\tau_K} ||r_e(u^h)||_K^2\right]. \end{aligned} \tag{3.3.27}$$

As for the term $T_6$ we use from (3.3.20) where we assume that $0 < \sigma < 1 - \epsilon$, and all the constants depending on $h$, and as well as $h$ itself are assumed to be sufficiently small. Finally combining (3.3.26),(3.3.27) and (3.3.20) including the term $T_5$, and taking $\alpha = min(1 - \varepsilon - \sigma, \lambda - \frac{\gamma}{\epsilon})$, which is positive for $\frac{\gamma}{\lambda} < \epsilon < 1$ and $0 < \sigma < 1 - \epsilon$, the proof is complete. □

**Lemma 3.4.** *For any constant $C_1 > 0$ we have for $\beta = \tilde{u}^h$ and $u^h \in W_h$*

$$\begin{aligned} ||u^h||_Q^2 \leq \Big[\frac{1}{C_1}||u_t^h + \beta \cdot \nabla u^h||_Q^2 + \sum_{m=1}^{M} |u_-^h|_m^2 + C \sum_{K \in \mathcal{C}_h} \int_{\partial K_-(\beta)''} [u^h]^2 |n \cdot \beta| \, d\nu \\ + \int_{\partial\Omega_+ \times I} (u^h)^2 |n \cdot \beta| \, d\nu \, ds\Big] \, h \exp(C_1 h), \end{aligned}$$

*where*

$$\partial K_-(\beta)'' = \{(\mathbf{x}, t) \in \partial K_-(\beta)' : n_t(\mathbf{x}, t) = 0\}.$$

**Proof.** First note that by Green's formula and noting that $(\beta \cdot \nabla u^h, u^h) = (u^h, u^h \, div \, \beta)$ we have the following relation

$$(\beta \cdot \nabla u^h, u^h)_\tau = \frac{1}{3} \int_{\partial\tau} (u^h)^2 \, \mathbf{n} \cdot \beta \, d\sigma$$

thus we can conclude that

$$\begin{aligned}(\beta\cdot\nabla u^h,u^h)_\tau &= \frac{1}{3}\left[\int_{\partial\tau_-}(u^h)^2\,\mathbf{n}\cdot\beta\,d\sigma+\int_{\partial\tau_+}(u^h)^2\,\mathbf{n}\cdot\beta\,d\sigma\right]\\
&= \frac{1}{3}\left[\int_{\partial\tau_+}(u^h)^2\,|\mathbf{n}\cdot\beta|\,d\sigma-\int_{\partial\tau_-}(u^h)^2\,|\mathbf{n}\cdot\beta|\,d\sigma\right]\\
&= \frac{1}{3}\left[\int_{\partial\tau_+}(u^h)^2\,|\mathbf{n}\cdot\beta|\,d\sigma+\int_{\partial\tau_-}(u^h)^2\,|\mathbf{n}\cdot\beta|\,d\sigma\right]-\frac{2}{3}\int_{\partial\tau_-}(u^h)^2\,|\mathbf{n}\cdot\beta|\,d\sigma\\
&= \frac{1}{3}\left[\int_{\partial\tau_+}(u^h)^2\,|\mathbf{n}\cdot\beta|\,d\sigma+\int_{\partial\tau_-}(u^h)^2\,|\mathbf{n}\cdot\beta|\,d\sigma\right]+\frac{2}{3}\int_{\partial\tau_-}(u^h)^2\,\mathbf{n}\cdot\beta\,d\sigma\\
&= \frac{2}{3}\int_{\partial\tau_-}(u^h)^2\,\mathbf{n}\cdot\beta\,d\sigma+\frac{1}{3}\int_{\partial\tau}(u^h)^2|\mathbf{n}\cdot\beta|\,d\sigma,\end{aligned}$$

so we can write

$$\begin{aligned}\int_t^{t_m}\frac{d}{dt}||u^h(s)||_\tau^2ds &= 2\int_t^{t_m}(u_t^h,u^h)_\tau = 2\int_t^{t_m}\left[(u_t^h+\beta\cdot\nabla u^h,u^h)_\tau-(\beta\cdot\nabla u^h,u^h)_\tau\right]\\
&= 2\int_t^{t_m}\left[(u_t^h+\beta\cdot\nabla u^h,u^h)_\tau-\frac{2}{3}\int_{\partial\tau_-}(u^h)^2\,\mathbf{n}\cdot\beta\,d\sigma-\frac{1}{3}\int_{\partial\tau}(u^h)^2|\mathbf{n}\cdot\beta|\,d\sigma\right]\end{aligned}$$

thus we have for $t_m<t<t_{m+1},\ K=\tau\times I_m$,

$$\begin{aligned}||u^h(t)||_\tau^2 &= |u^h|_{m+1,\tau}^2-\int_t^{t_{m+1}}\frac{d}{dt}||u^h(t)||_\tau^2\,ds\\
&= |u^h|_{m+1,\tau}^2-2\int_t^{t_{m+1}}\Big[(u_t^h+\beta\cdot\nabla u^h,u^h)_\tau-\frac{2}{3}\int_{\partial\tau_-}(u^h)^2\mathbf{n}\cdot\beta\,d\nu\\
&\qquad -\frac{1}{3}\int_{\partial\tau}(u^h)^2|\mathbf{n}\cdot\beta|\,d\nu\Big],\end{aligned}$$

where $|u_-^h|_{m+1,\tau}$ is the obvious restriction of $|u_-^h|_{m+1}$ to $\tau$. Summing over $\tau\in\mathcal{T}_h$, we obtain

$$
\begin{aligned}
||u^h(t)||_\Omega^2 =& |u_-^h|_{m+1}^2 - 2\int_t^{t_{m+1}} (u_t^h + \beta\cdot\nabla u^h, u^h)_\Omega \\
&+ \frac{2}{3}\sum_K \int_{\partial K_-(\beta)''\cap\{s\,:\,t<s<t_{m+1}\}} [(u^h)^2]|\mathbf{n}\cdot\beta|\,d\nu \\
&+ \frac{2}{3}\int_{\partial\Omega_+\times\{s\,:\,t<s<t_{m+1}\}} (u^h)^2|\mathbf{n}\cdot\beta|\,d\nu \\
&\le |u_-^h|_{m+1}^2 + \frac{1}{C_1}||u_t^h + \beta\cdot\nabla u^h||_m^2 + C_1\int_t^{t_{m+1}} ||u^h(s)||_\Omega^2 \\
&+ C\sum_K \int_{\partial K_-(\beta)''\cap I_m} [u^h]^2|\mathbf{n}\cdot\beta|\,d\nu + \int_{\partial\Omega_+\times I_m} (u^h)^2|\mathbf{n}\cdot\beta|\,d\nu,
\end{aligned}
$$

where in above we use form the following argument to derive inequality

$$
[(u^h)^2] = (u_+^h)^2 - (u_-^h)^2 = (u_+^h - u_-^h)(u_+^h + u_-^h) = [u^h](u_+^h + u_-^h) \le C[u^h]^2 + \frac{1}{C}(u_+^h + u_-^h)^2,
$$

where we can take $C$ sufficiently large and hide the contribution from the $\frac{1}{C}(u_+^h + u_-^h)^2$ term in the norm on the left hand side, $||u^h(t)||_\Omega^2$.

Now using Grönwall's inequality we find that

$$
\begin{aligned}
||u^h(t)||_\Omega^2 \le \Bigg[ & |u_-^h|_{m+1}^2 + \frac{1}{C_1}||u_t^h + \beta\cdot\nabla u^h||_m^2 \\
&+ C\sum_K \int_{\partial K_-(\beta)''\cap I_m} [u^h]^2|\mathbf{n}\cdot\beta|\,d\nu + \int_{\partial\Omega_+\times I_m} (u^h)^2|\mathbf{n}\cdot\beta|\,d\nu \Bigg] \exp(C_1 h).
\end{aligned}
$$

Integrating over $I_m$ and summation for $m = 0, ..., M-1$, complete the proof.

□

# Chapter 4

# Error Estimation and Convergence

We now turn to error estimates. Let $\hat{u}^h \in H_0^1(Q)$ be an interpolant of exact solution $u$ with the interpolant error denoted by $\eta = u - \hat{u}^h$ and set $\xi = u^h - \hat{u}^h$. Then we have

$$e :\equiv u - u^h = (u - \hat{u}^h) - (u^h - \hat{u}^h) = \eta - \xi.$$

The objective in error estimates is to dominate $|||\,\xi\,|||$ by the known interpolation estimates for $|||\eta|||$. Our main result in this chapter is as follows:

**Theorem 4.1.** *Assume $u^h \in W_h$ and $u \in H^{k+1}(Q) \cap W^{k+1,\infty}(Q)$ with $k \geq 1$ are the solutions of* (3.3.23) *and (3.1.1), respectively, such that*

$$||\nabla u||_\infty + ||\tilde{u}||_\infty + ||div\,\tilde{u}||_\infty + ||\nabla \eta||_\infty \leq C. \tag{4.0.1}$$

*Then there exists a constant $C$ such that*

$$|||u - u^h||| \leq Ch^{k+\frac{1}{2}}||u||_{H^{k+1}(Q)}.$$

Before we prove our main result we state the following results for estimating the trilinear form $B$.

Since $u$ satisfies (3.1.1), from (3.3.23) we have for $v^h \in W_h$

$$B_{DG}(\tilde{u}; u, v^h) = L(v^h) = B_{DG}(\tilde{u}^h; u^h, v^h),$$

so that by definition of $\eta$ we have $\hat{u}^h = u - \eta$ and by Lemma 3.3 we can write

$$\begin{aligned}
\alpha ||| \xi |||^2 \leq B_{DG}(\tilde{u}^h; \xi, \xi) &= B_{DG}(\tilde{u}^h; u^h - \hat{u}^h, \xi) \\
&= B_{DG}(\tilde{u}; u, \xi) - B_{DG}(\tilde{u}^h; \hat{u}^h, \xi) \\
&= B_{DG}(\tilde{u}; u, \xi) - B_{DG}(\tilde{u}^h; u - \eta, \xi) \\
&= B_{DG}(\tilde{u}^h; \eta, \xi) + \left[ B_{DG}(\tilde{u}; u, \xi) - B_{DG}(\tilde{u}^h; u, \xi) \right] \\
&:= T_1 + T_2 - T_3,
\end{aligned} \tag{4.0.2}$$

that we consider them at the following Lemmas.

**Lemma 4.2.** *If we assume that the assumptions of Theorem 4.1 hold then we can derive the following estimation for the term* $T_1$

$$\begin{aligned}
|T_1| \leq c ||| \xi |||^2 &+ C h^{-1} || \eta ||_Q^2 + C h || \eta ||_{1,Q}^2 + C h^{2k+1} \\
&+ C \sum_{m=0}^{M} | \eta |_m^2 + \int_{\partial\Omega_+ \times I} | \eta |^2 \, |\mathbf{n} \cdot \beta| \, d\nu \, ds \\
&+ C h || \nabla \eta ||_\infty (|| \xi ||_Q + || \eta ||_Q)^2 + C h || \xi ||_Q^2 + C_2 h^k || \xi ||_Q^2.
\end{aligned}$$

**Proof.** For the term $T_1$ we have

$$\begin{aligned}
T_1 =& \langle \eta_+, \xi_+ \rangle_0 + \sum_{K\in\mathcal{C}_h} \Big[ (\eta_t + \tilde{u}^h \cdot \nabla\eta, \xi + h(\xi_t + \tilde{u}^h \cdot \nabla\xi))_K \\
&+ \int_{\partial K_-(\tilde{u}^h)'} [\eta]\xi_+ |n_t + \tilde{u}^h \cdot \mathbf{n}|\, d\nu - h\varepsilon(\Delta\eta, \xi_t + \tilde{u}^h \cdot \nabla\xi)_K \\
&+ \varepsilon(\nabla\eta, \nabla\xi)_K + \lambda\varepsilon \sum_{e\in\mathcal{E}_h} (r_e(\eta), r_e(\xi))_K + \varepsilon(R(\eta), \nabla\xi)_K + \varepsilon(\nabla\eta, R(\xi))_K \Big] \\
&:= \sum_{i=1}^{8} S_i.
\end{aligned} \tag{4.0.3}$$

Thus we need to estimate $S_i,\ 1 \le i \le 8$. For the term $S_1$ we have

$$|S_1| \le C|\eta_+|_0^2 + \frac{1}{C}|\xi_+|_0^2. \tag{4.0.4}$$

First we split the term $S_2$ into two parts, and for the first term we use integration by parts and for the second term we will use the Cauchy-Schwarz inequality and then try to hide some of them into triple norm of $\xi$,

$$S_2 = \sum_{K\in\mathcal{C}_h} \left[ (\eta_t + \tilde{u}^h \cdot \nabla\eta, \xi)_K + (\eta_t + \tilde{u}^h \cdot \nabla\eta, +h(\xi_t + \tilde{u}^h \cdot \nabla\xi))_K \right],$$

a similar argument as in proof of stability estimate using integration by parts gives

$$(\eta_t, \xi)_K = \int_{\partial K} \eta\, \xi(n_t) - (\eta, \xi_t)_K,$$

and using Green's formula yields

$$(\tilde{u}^h \cdot \nabla\eta, \xi)_K = \int_{\partial K} \eta\, \xi(\mathbf{n} \cdot \beta) - (\eta, \beta \cdot \nabla\xi)_K - (\eta, \xi\, div\, \beta)_K,$$

where by $\beta$ we mean $\tilde{u}^h$. So if we combine these equation we have

$$\begin{aligned}(\eta_t + \tilde{u}^h \cdot \nabla\eta, \xi)_K &= \int_{\partial K} \eta\, \xi(n_t + \beta\cdot\mathbf{n}) - (\eta, \xi_t + \beta\cdot\nabla\xi)_K - (\eta, \xi\, div\,\beta) \\ &= \int_{\partial K_-} \eta_+\, \xi_+(n_t + \beta\cdot\mathbf{n}) + \int_{\partial K_+} \eta_-\, \xi_-(n_t + \beta\cdot\mathbf{n}) \\ &\quad - (\eta, \xi_t + \beta\cdot\nabla\xi)_K - (\eta, \xi\, div\,\beta),\end{aligned} \tag{4.0.5}$$

summing over $K$ we can write

$$\sum_{K\in\mathcal{C}_h} \int_{\partial K_-} \eta_+\, \xi_+(n_t + \beta\cdot\mathbf{n}) = -\sum_K \int_{\partial K_-(\beta)'} \eta_+\, \xi_+|n_t + \beta\cdot\mathbf{n}|,$$

and

$$\begin{aligned}\sum_{K\in\mathcal{C}_h} \int_{\partial K_+} \eta_-\, \xi_-(n_t + \beta\cdot\mathbf{n}) &= \int_{\partial\Omega_+\times I} \eta_-\, \xi_-|\mathbf{n}\cdot\beta| - \sum_K \int_{\partial K_-(\beta)'} \eta_-\, \xi_-(n_t + \mathbf{n}\cdot\beta) \\ &= \int_{\partial\Omega_+\times I} \eta_-\, \xi_-|\mathbf{n}\cdot\beta| + \sum_K \int_{\partial K_-(\beta)'} \eta_-\, \xi_-|n_t + \mathbf{n}\cdot\beta|,\end{aligned}$$

and then using from $S_3$ with above equalities we have

$$\begin{aligned}&\sum_K \left[\int_{\partial K_-(\beta)'} \eta_-\, \xi_-|n_t + \mathbf{n}\cdot\beta| - \int_{\partial K_-(\beta)'} \eta_+\, \xi_+|n_t + \mathbf{n}\cdot\beta| + \int_{\partial K_-(\beta)'} [\eta]\xi_+|n_t + \mathbf{n}\cdot\beta|\right] \\ &= \sum_K \left[\int_{\partial K_-(\beta)'} \eta_-\, \xi_-|n_t + \mathbf{n}\cdot\beta| - \int_{\partial K_-(\beta)'} \eta_-\, \xi_+|n_t + \mathbf{n}\cdot\beta|\right] \\ &= -\sum_K \int_{\partial K_-(\beta)'} \eta_-\, [\xi]\, |n_t + \mathbf{n}\cdot\beta|.\end{aligned} \tag{4.0.6}$$

To bound the last term that appear in right hand side of (4.0.6), the crucial part is to estimate a term of the form

$$T = \sum_{K\in\mathcal{C}_h} \int_{\partial K_-(\beta)''} \eta_-\, [\xi]\, |\mathbf{n}\cdot\beta|\, d\nu,$$

where again by $\partial K_-(\beta)''$ we mean

$$\partial K_-(\beta)'' = \{(\mathbf{x}, t) \in \partial K_-(\beta)' : n_t(\mathbf{x}, t) = 0\}.$$

To this approach using Cauchy-Schwarz inequality we have for $\delta > 0$ that

$$\begin{aligned} |T| &\leq \frac{C}{\delta} \sum_{K \in \mathcal{C}_h} \int_{\partial K_-(\beta)''} |\eta_-|^2 \, |\mathbf{n} \cdot \beta| \, d\nu + C\delta \sum_{K \in \mathcal{C}_h} \int_{\partial K_-(\beta)''} [\xi]^2 \, |\mathbf{n} \cdot \beta| \, d\nu \\ &\leq \frac{C}{\delta} \sum_{K \in \mathcal{C}_h} \int_{\partial K_-(\beta)''} |\eta_-|^2 \, |\mathbf{n} \cdot \beta| \, d\nu + c||| \, \xi \, |||^2, \end{aligned} \tag{4.0.7}$$

where we assume that $C\delta$ is sufficiently small such that $C\delta < c << 1$. Here the last term can be hidden in $||| \, \xi \, |||^2$, and we estimate the first one as follows

$$\begin{aligned} \sum_{K \in \mathcal{C}_h} \int_{\partial K_-(\beta)''} |\eta_-|^2 \, |\mathbf{n} \cdot \beta| \, d\nu &\leq || \, \eta \, ||_\infty^2 \sum_{K \in \mathcal{C}_h} \int_{\partial K_-(\beta)''} |\tilde{u}^h \cdot \mathbf{n}| \, d\nu \\ &\leq || \, \eta \, ||_\infty^2 \sum_{K \in \mathcal{C}_h} \left[ \int_{\partial K_-(\beta)''} |\tilde{u}^h \cdot \mathbf{n}|^2 \, d\nu + \int_{\partial K_-(\beta)''} d\nu \right] \\ &\leq C|| \, \eta \, ||_\infty^2 \sum_{K \in \mathcal{C}_h} \left[ Ch^{-1} ||\tilde{u}^h||_K^2 + Ch^2 \right], \end{aligned} \tag{4.0.8}$$

where in the second inequality we use the Cauchy-Schwarz inequality and in the last one we have used the Trace estimate:

$$\int_{\partial K} v^2 \, d\nu \leq Ch^{-1} \int_K v^2 \, dx \,, \qquad \forall \, v \in P_k(K),$$

and the fact that

$$V(K) = \int_K 1 \; dx \leq Ch^3,$$

where by $V(K)$ we mean the volume of the time-space triangle $K$. To bound the term $\sum_{K \in \mathcal{C}_h} ||\tilde{u}^h||_K = ||\tilde{u}^h||_Q$ first observe that we can write $\tilde{u}^h = \tilde{u}^h - \tilde{u} + \tilde{u}$, thus

$$||\tilde{u}^h|| \leq ||\tilde{u}^h - \tilde{u}|| + ||\tilde{u}||.$$

On the other hand from definition of $\tilde{u}$ we can easily see that

$$||\tilde{u}^h - \tilde{u}||_Q \leq C||u - u^h||_Q \leq C \left( || \, \xi \, ||_Q + || \, \eta \, ||_Q \right), \tag{4.0.9}$$

and consequently we have

$$||\beta||_Q = ||\tilde{u}^h||_Q \le C\,(||\,\xi\,||_Q + ||\,\eta\,||_Q) + ||\tilde{u}||_Q. \qquad (4.0.10)$$

Moreover from the standard interpolation theory we know that the interpolation error $\eta$ satisfies

$$||\,\eta\,||_\infty = ||u - \hat{u}^h||_\infty \le Ch^{k+1}||u||_{k+1,\infty}. \qquad (4.0.11)$$

Combining (4.0.8)-(4.0.10) we can write

$$\sum_{K\in\mathcal{C}_h}\int_{\partial K_-(\beta)''}|\eta_-|^2\,|\mathbf{n}\cdot\beta|\,d\nu \le Ch^{2k+2}||u||^2_{k+1,\infty}\times\left[\,h^{-1}\left(||\,\xi\,||^2_Q + ||\,\eta\,||^2_Q + ||\tilde{u}||^2_Q\right) + h^2\,\right]. \qquad (4.0.12)$$

Thus (4.0.7) and (4.0.11) imply that

$$\begin{aligned}|T| &\le c|||\,\xi\,|||^2 + Ch^{2k+2}||u||^2_{k+1,\infty}\times\left[\,h^{-1}\left(||\,\xi\,||^2_Q + ||\,\eta\,||^2_Q + ||\tilde{u}||^2_Q\right) + h^2\,\right]\\ &= c|||\,\xi\,|||^2 + Ch^{2k+1}||u||^2_{k+1,\infty}\times\left(\,||\,\xi\,||^2_Q + ||\,\eta\,||^2_Q + ||\tilde{u}||^2_Q\,\right) + Ch^{2k+4}||u||^2_{k+1,\infty}.\end{aligned} \qquad (4.0.13)$$

Estimating $||\,\xi\,||^2$ from the Lemma 3.4

$$\begin{aligned}||\,\xi\,||^2_Q \le \Big[\frac{1}{C_1}||\xi_t + \beta\cdot\nabla\xi||^2_Q + \sum_{m=1}^{M}|\xi_-|^2_m + \sum_{K\in\mathcal{C}_h}\int_{\partial K_-(\beta)''}[\xi]^2|n\cdot\beta|\,d\nu\\ + \int_{\partial\Omega_+\times I}\xi^2|n\cdot\beta|\,d\nu\,ds\Big]\,h\exp(C_1h),\end{aligned}$$

and since the coefficient of $||\,\xi\,||^2$ is of order $h^{2k+1}$ and thus very small in comparison to the $||\,\xi\,||^2$ term in triple norm of $\xi$, $|||\,\xi\,|||^2$. We rearrange the higher order (small) terms and hide them in the correspondig terms in $|||\,\xi\,|||^2$. As for $||\,\eta\,||^2$ we can see that

$$||\,\eta\,||^2_Q \le ||\,\eta\,||^2_\infty\int_Q dx \le Ch^{2k+2}||\,u\,||^2_{k+1,\infty},$$

and again we can ignore this term because of its small coefficient. The last term in (4.0.13) has a $h$ coefficient of order $2k+4$ and can be ignored. So by (4.0.12) and by assumptions of the lemma we obtain

$$|T| \leq Ch^{2k+1} + \frac{1}{C_1} |||\, \xi \,|||^2, \tag{4.0.14}$$

where $C_1$ is a sufficiently large constant. So we have estimated the crucial part of the term

$$-\sum_K \int_{\partial K_-(\beta)'} \eta_- \, [\xi] \, |n_t + \mathbf{n} \cdot \beta| = T + T',$$

which is harder and request more attention in error estimate and now remains another part of this term, i.e.,

$$T' = -\sum_{m=1}^{M-1} \langle \, \eta_- \, , \, [\xi] \, \rangle + \langle \, \eta_- \, , \, \xi_- \, \rangle_M + \int_{\partial\Omega_+ \times I} \eta^2 \, (\mathbf{n} \cdot \beta) \, d\nu \, ds,$$

that once again using the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} |T'| &\leq \frac{1}{C} \left[ \sum_{m=1}^{M-1} |\, [\, \xi \,] \,|_m^2 + |\, \xi \,|_M^2 \right] + C \sum_{m=1}^{M} |\, \eta_- \,|_m^2 \\ &\leq \frac{1}{C} |||\, \xi \,|||^2 + C \sum_{m=1}^{M} |\, \eta_- \,|_m^2 + \int_{\partial\Omega_+ \times I} |\, \eta \,|^2 \, |\mathbf{n} \cdot \beta| \, d\nu \, ds. \end{aligned} \tag{4.0.15}$$

For the last two terms appeared in (4.0.5) we use to derive Cauchy-Schwarz inequality

$$\sum_K (\eta, \xi_t + \beta \cdot \nabla \xi) \leq \sum_K \left[ \frac{C}{h} ||\, \eta \,||_K^2 + \frac{h}{C} ||\xi_t + \beta \cdot \nabla \xi||_K^2 \right] \tag{4.0.16}$$

where $C > 1$, and we hide the term $\frac{h}{C} ||\xi_t + \beta \cdot \nabla \xi||^2$ in triple norm of $|||\, \xi \,|||^2$. Similarly for the other term we have

$$\begin{aligned} \sum_K (\eta, \xi \, div \, \beta) &= (\eta, \xi \, div \, \tilde{u}) + (\eta, \xi \, div \, (\beta - \tilde{u})) \\ &\leq ||\, \eta \,||_Q ||\, \xi \,||_Q ||\, div \, \tilde{u} \,||_\infty + ||\, \eta \,||_Q ||\, \xi \,||_Q ||\, div \, (\beta - \tilde{u}) \,||, \end{aligned}$$

and from the (4.0.9) we conclude that

$$(\eta, \xi\, div\, \beta)_Q \leq ||\, \eta\, ||_Q ||\, \xi\, ||_Q ||\, div\, \tilde{u}\, ||_\infty + ||\, \eta\, ||_Q ||\, \xi\, ||_Q (||\, \nabla \eta\, ||_Q + ||\, \nabla \xi\, ||_Q), \tag{4.0.17}$$

so we can then see by Cauchy-Schwarz inequality that

$$||\, \eta\, ||_Q ||\, \xi\, ||_Q ||\, div\, \tilde{u}\, ||_\infty \leq ||\, div\, \tilde{u}\, ||_\infty^2 (h^{-1} ||\, \eta\, ||_Q^2 + h ||\, \xi\, ||_Q^2), \tag{4.0.18}$$

and

$$||\, \eta\, ||_Q ||\, \xi\, ||_Q ||\, \nabla \eta\, ||_Q \leq C ||\, \nabla \eta\, ||_\infty^2 (h^{-1} ||\, \eta\, ||_Q^2 + h ||\, \xi\, ||_Q^2). \tag{4.0.19}$$

Further by inverse inequality and standard interpolation theory we can derive

$$\begin{aligned} ||\, \eta\, ||_Q ||\, \xi\, ||_Q (||\, \nabla \xi\, ||_Q &\leq C h^{-1} ||\, \eta\, ||_Q ||\, \xi\, ||_Q^2 \\ &\leq C ||\, \eta\, ||_\infty h^{-1} ||\, \xi\, ||_Q^2 \\ &\leq C h^{k+1} ||\, u\, ||_{\infty, k+1} h^{-1} ||\, \xi\, ||_Q^2 \\ &\leq C h^k ||\, \xi\, ||_Q^2, \end{aligned} \tag{4.0.20}$$

so by (4.0.17)-(4.0.20) we obtain

$$(\eta, \xi\, div\, \beta)_Q \leq C h^{-1} ||\, \eta\, ||_Q^2 + C_1 ||\, \xi\, ||_Q^2 + C_2 h^k ||\, \xi\, ||_Q^2. \tag{4.0.21}$$

For the control on the remaining term in $S_2$ also we use Cauchy-Schwarz inequality

$$\begin{aligned} \sum_K h(\eta_t + \beta \cdot \nabla \eta, \xi_t + \beta \cdot \nabla \xi)_K &\leq \sum_k \left[ Ch ||\eta_t + \beta \cdot \nabla \eta||_K^2 + \frac{h}{C} ||\xi_t + \beta \cdot \nabla \xi||_K^2 \right] \\ &\leq \sum_k Ch ||\eta_t + \beta \cdot \nabla \eta||_K^2 + \frac{1}{C} |||\, \xi\, |||^2, \end{aligned} \tag{4.0.22}$$

where again $C > 1$, and we hide the second term appear in (4.0.22) in triple norm of $\xi$. From (4.0.9) we can derive for the first term that appear in last inequality of (4.0.22) the following estimate

$$\begin{aligned} \|\eta_t + \tilde{u}^h \cdot \nabla\eta\|_Q &\le \|\eta_t + \tilde{u}\cdot\nabla\eta\|_Q + \|(\tilde{u}^h - \tilde{u})\cdot\nabla\eta\|_Q \\ &\le \|\eta_t\|_Q + \|\tilde{u}\|_\infty \|\nabla\eta\|_Q + C\|\nabla\eta\|_\infty (\|\xi\|_Q + \|\eta\|_Q) \\ &\le C\|\eta\|_{1,Q} + C\|\nabla\eta\|_\infty (\|\xi\|_Q + \|\eta\|_Q)\,, \end{aligned} \tag{4.0.23}$$

therefore by (4.0.22) and (4.0.23)we have

$$\sum_K h(\eta_t + \beta\cdot\nabla\eta, \xi_t + \beta\cdot\nabla\xi)_K \le Ch\|\eta\|_{1,Q}^2 + Ch\|\nabla\eta\|_\infty (\|\xi\|_Q + \|\eta\|_Q)^2 + \frac{1}{C}|||\xi|||^2. \tag{4.0.24}$$

And by this we can finish estimating the terms $|S_1|+|S_2+S_3|$ from the (4.0.4), (4.0.14)-(4.0.16), (4.0.21), (4.0.22) and (4.0.24) by following relation

$$\begin{aligned} |S_1| + |S_2+S_3| \le C|||\xi|||^2 + Ch^{2k+1} + C\sum_{m=0}^{M} |\eta|_m^2 + \frac{C}{h}\|\eta\|_Q^2 + ch\|\eta\|_{1,Q}^2 \\ + Ch\|\nabla\eta\|_\infty(\|\xi\|_Q + \|\eta\|_Q)^2 + Ch\|\xi\|_Q^2 + C_2 h^k\|\xi\|_Q^2. \end{aligned} \tag{4.0.25}$$

Now we consider the term $S_4$ as follows

$$\begin{aligned} |S_4| = h\varepsilon|(\Delta\eta, \xi_t + \beta\cdot\nabla\xi)_Q| &\le h\varepsilon\|\Delta\eta\|_Q\|\xi_t + \beta\cdot\nabla\xi\|_Q \\ &\le C\varepsilon\|\nabla\eta\|_Q\|\xi_t + \beta\cdot\nabla\xi\|_Q \\ &\le C\|\eta\|_Q\|\xi_t + \beta\cdot\nabla\xi\|_Q \\ &\le Ch^{-1}\|\eta\|_Q^2 + \frac{h}{C}\|\xi_t + \beta\cdot\nabla\xi\|_Q^2 \\ &\le Ch^{-1}\|\eta\|_Q^2 + c|||\xi|||^2, \end{aligned} \tag{4.0.26}$$

where we use the inverse inequality twice and from the fact that $\varepsilon = h$ and Cauchy-Schwarz inequality. For the term $S_5$ also we use from the inverse inequality to obtain

$$\begin{aligned}|S_5| = \varepsilon|(\nabla\eta, \nabla\xi)_Q| &\le \varepsilon||\,\nabla\eta\,||_Q||\,\nabla\xi\,||_Q\\ &\le C||\,\eta\,||_Q||\,\nabla\xi\,||_Q\\ &\le Ch^{-1}||\,\eta\,||_Q^2 + \frac{h}{C}||\,\nabla\xi\,||_Q^2\\ &\le Ch^{-1}||\,\eta\,||_Q^2 + c|||\,\xi\,|||^2.\end{aligned} \tag{4.0.27}$$

Moreover, from the definition of operators $R$ and $r_e$, and from the fact that $\eta$ is a continuous function, so the jump of $\eta$, $[\,\eta\,]$ in definitions of $R$ and $r_e$ will be equal to zero and we can easily deduce that $S_6 = 0$ and $S_7 = 0$. Thus it remains to estimate the term $S_8$. To this end we use (3.3.18), (3.3.19), and the inverse inequality to obtain

$$\begin{aligned}|S_8| = \varepsilon|(\nabla\eta, R(\xi))_Q| &\le \sum_{K\in\mathcal{C}_h} \varepsilon||\,\nabla\eta\,||_K||\,R(\xi)\,||_K\\ &\le \sum_{K\in\mathcal{C}_h}\Big(C\varepsilon||\,\nabla\eta\,||_K^2 + \frac{\varepsilon}{C_1}||\,R(\xi)\,||_K^2\Big)\\ &\le Ch^{-1}||\,\eta\,||_Q^2 + C_2\sum_{e\in\mathcal{E}_h}||\,r_e(\xi)\,||_Q^2\\ &\le Ch^{-1}||\,\eta\,||_Q^2 + c|||\,\xi\,|||^2\end{aligned} \tag{4.0.28}$$

where, as above, $C_1$ is taken to be large enough. So by (4.0.25)-(4.0.28) we can derive desired conclusion for $T_1$

$$\begin{aligned}|T_1| \le c|||\,\xi\,|||^2 &+ Ch^{-1}||\,\eta\,||_Q^2 + Ch||\,\eta\,||_{1,Q}^2 + Ch^{2k+1}\\ &+ C\sum_{m=0}^{M}|\,\eta\,|_m^2 + \int_{\partial\Omega_+\times I}|\,\eta\,|^2\,|\mathbf{n}\cdot\beta|\,d\nu\,ds\\ &+ Ch||\,\nabla\eta\,||_\infty(||\,\xi\,||_Q + ||\,\eta\,||_Q)^2 + Ch||\,\xi\,||_Q^2 + C_2h^k||\,\xi\,||_Q^2,\end{aligned} \tag{4.0.29}$$

and this complete the proof. □

We now turn to the term $T_2 - T_3$ and try to estimate this term.

**Lemma 4.3.** *Under assumption of Theorem 4.1 we have the following estimate for the $T_2 - T_3$*

$$
\begin{aligned}
|T_2 - T_3| \le\ & C(||\,\xi\,||_Q + ||\,\eta\,||_Q)||\,\nabla u\,||_\infty||\,\xi\,||_Q \\
& + Ch(||\,\xi\,||_Q + ||\,\eta\,||_Q)^2||\,\nabla u\,||_\infty^2 + c|||\,\xi\,|||^2.
\end{aligned}
$$

**Proof.** To estimate the term $T_2 - T_3$, we first note that

$$
T_2 - T_3 = ((\tilde{u} - \tilde{u}^h)\cdot\nabla u, \xi)_Q + h((\tilde{u} - \tilde{u}^h)\cdot\nabla u, \xi_t + \tilde{u}^h\cdot\nabla\xi)_Q,
$$

so by (4.0.9), and by Cauchy-Schwarz inequality we have

$$
\begin{aligned}
|T_2 - T_3| \le\ & C(||\,\xi\,||_Q + ||\,\eta\,||_Q)||\,\nabla u\,||_\infty||\,\xi\,||_Q \\
& + Ch(||\,\xi\,||_Q + ||\,\eta\,||_Q)^2||\,\nabla u\,||_\infty^2 + Ch||\,\xi_t + \tilde{u}^h\cdot\nabla\xi\,||_Q^2 \\
& \le C(||\,\xi\,||_Q + ||\,\eta\,||_Q)||\,\nabla u\,||_\infty||\,\xi\,||_Q \\
& + Ch(||\,\xi\,||_Q + ||\,\eta\,||_Q)^2||\,\nabla u\,||_\infty^2 + c|||\,\xi\,|||^2.
\end{aligned}
\tag{4.0.30}
$$

where we hide the terms as $Ch||\,\xi_t + \tilde{u}^h\cdot\nabla\xi\,||_Q^2$, in (4.0.30) in $|||\,\xi\,|||^2$, and complete the proof. □

**Lemma 4.4.** *Under the assumptions of Theorem 4.1 we have that*

$$
\begin{aligned}
|B_{DG}(\tilde{u}; u, \xi) - B_{DG}(\tilde{u}^h; \hat{u}^h, \xi)| \le\ & \tilde{C}|||\,\xi\,|||^2 + Ch^{2k+1} \\
& + C\left[\int_{\partial\Omega_+\times I}\eta^2 + h||\,\eta\,||_{1,Q}^2|\tilde{u}^h\cdot\mathbf{n}|d\nu\,ds + h^{-1}||\eta||_Q^2 + \sum_{m=0}^{M}|\eta|_m^2 + h||\,\eta\,||_{1,Q}^2\right] \\
& + C\Big(||\,\xi\,||_Q + ||\,\eta\,||_Q\Big)||\,\xi\,||_Q + Ch\Big(||\,\xi\,||_Q + ||\,\eta\,||_Q\Big)^2 \\
& + Ch||\,\xi\,||_Q^2 + C_2h^k||\,\xi\,||_Q^2.
\end{aligned}
$$

*where the constant $\tilde{C} < 1$.*

**Proof.** The proof follows using Lemmas 4.2 and 4.3 and the fact that $||\,\nabla u\,||_\infty < \infty$ and $||\,\nabla\eta\,||_\infty < \infty$. □

Now we are ready to prove the main theorem.

**Proof of Theorem 4.1.** Using Lemmas (4.2)- (4.4) and (4.0.2) we can see that

$$
\begin{aligned}
C|||\,\xi\,|||^2 \leq{}& Ch^{2k+1} \\
&+ C\left[\int_{\partial\Omega_+\times I} \eta^2|\tilde{u}^h\cdot\mathbf{n}|d\nu\,ds + h^{-1}||\eta||_Q^2 + \sum_{m=0}^{M}|\eta|_m^2 + h||\,\eta\,||_{1,Q}^2\right] \\
&+ C\Big(||\,\xi\,||_Q + ||\,\eta\,||_Q\Big)||\,\xi\,||_Q + Ch\Big(||\,\xi\,||_Q + ||\,\eta\,||_Q\Big)^2 \\
&+ Ch||\,\xi\,||_Q^2 + C_2h^k||\,\xi\,||_Q^2.
\end{aligned}
\tag{4.0.31}
$$

We have the following estimation for third term as

$$
\begin{aligned}
\Big(||\,\xi\,||_Q + ||\,\eta\,||_Q\Big)||\,\xi\,||_Q &= ||\,\xi\,||_Q^2 + ||\,\eta\,||_Q||\,\xi\,||_Q \\
&\leq ||\,\xi\,||_Q^2 + ch||\,\xi\,||_Q^2 + ch^{-1}||\,\eta\,||_Q^2,
\end{aligned}
$$

where we use the Cauchy-Schwarz inequality to derive this estimation. Now estimating $||\,\xi\,||_Q^2$ form Lemma 3.4 we have

$$
\begin{aligned}
||\,\xi\,||_Q^2 \leq \Big[\frac{1}{C_1}||\xi_t + \beta\cdot\nabla\xi||_Q^2 + \sum_{m=1}^{M}|\xi_-|_m^2 + \sum_{K\in\mathcal{C}_h}\int_{\partial K_-(\beta)''}[\xi]^2|n\cdot\beta|\,d\nu \\
+ \int_{\partial\Omega_+\times I}\xi^2|n\cdot\beta|\,d\nu\,ds\Big]\,h\exp(C_1h),
\end{aligned}
$$

hiding the terms as $||\xi_t+\beta\cdot\nabla\xi||_Q^2$, in (4.0.31) in $|||\,\xi\,|||^2$, and noting that we can ignore from the other terms in the form $||\,\xi\,||_Q^2$ since the coefficient $h$, make them ignorable in comparison with other terms and also for the term in form $h||\,\eta\,||_Q^2$ and $||\,\eta\,||_Q^2$ appear in (4.0.31) this argument is true from the standard interpolation theory. The other terms appear in estimating $||\,\xi\,||_Q^2$ also can be ignored in comparison of terms appear in $|||\,\xi\,|||^2$ except the term in the form

$h\sum_{m=1}^{M}|\xi_-|_m^2$ that is a term that don't appear in $|||\,\xi\,|||^2$. So we can conclude that

$$|||\,\xi\,|||^2 \le Ch^{2k+1} + C\left[\int_{\partial\Omega_+\times I}\eta^2|\tilde{u}^h\cdot\mathbf{n}|d\nu\,ds + h^{-1}||\eta||_Q^2 + \sum_{m=0}^{M}|\eta|_m^2 + h||\,\eta\,||_{1,Q}^2 + h\sum_{m=1}^{M}|\xi_-|_m^2\right].$$

Finally, by standard interpolation theory we have (see e.g. [21], p. 123)

$$\left[h\int_{\partial\Omega_+\times I}\eta^2|\tilde{u}^h\cdot\mathbf{n}|d\nu\,ds + ||\eta||_Q^2 + h\sum_{m=0}^{M}|\eta|_m^2 + h^2||\,\eta\,||_{1,Q}^2\right]^{1/2} \le Ch^{k+1}||\,u\,||_{k+1,Q}.$$

Thus by assumption of theorem that $||\,u\,||_{k+1,\infty} \le \infty$ we have

$$|||\,\xi\,|||^2 \le Ch^{2k+1} + C_1h\sum_{m=1}^{M}|\xi_-|_m^2. \tag{4.0.32}$$

We shall now use the following discrete Grönwall's estimate. If

$$y(\cdot,t_m) \le C + C_1h\sum_{m=1}^{M}|y(\cdot,t_j)|_m^2,$$

then

$$y(t_m) \le C\,e^{C_1t} \le C\,e^{C_1T}.$$

Obviously (4.0.32) implies that

$$|\xi_-|_m^2 \le Ch^{2k+1} + C_1h\sum_{m=1}^{M}|\xi_-|_m^2,$$

so that using discrete Grönwall's estimate

$$|\xi_-|_m^2 \le Ch^{2k+1}\,e^{C_1T}. \tag{4.0.33}$$

By (4.0.32) and (4.0.33)

$$|||\,\xi\,|||^2 \le Ch^{2k+1} + C_1h\sum_{m=1}^{M}(Ch^{2k+1}\,e^{C_1T}) \le C(T)h^{2k+1},$$

where

$$C(T) = C\, e^{C_1 T}.$$

So we can see that

$$|||\,\xi\,|||^2 \leq C(T)h^{2k+1}.$$

On the other hand, by definition of triple norm and recalling that interpolation error is of the order $h^{k+1/2}$, we can conclude that $|||\,\eta\,|||^2 \leq Ch^{2k+1}$ and noting that

$$|||\,e\,|||^2 \leq |||\,\xi\,|||^2 + |||\,\eta\,|||^2$$

we can write

$$|||\,e\,|||^2 \leq Ch^{2k+1}.$$

This complete the proof. $\square$

# Chapter 5

# Numerical Results and Implementation

## 5.1 Introduction

In this chapter we use the DG method in order to solve some examples numerically. The DG method has the combined advatage of the finite element and finite volume methods. In this method one may assume, for different elements, shape functions of different degrees. Also the method dose not need to rely on artificial diffusion. On the other hand, this method have the property of elementwise conservation. An advantage of the DG method is the ability of using the grids with non-matching elements. The mass matrix of the DG method is block diagonal matrix with independent blocks. Below we describe the implemented version of this method.

## 5.2 Description of the DG Method

We solve some examples in 1D space dimension and time. First we split our space interval into subintervals, $x_0 < x_1 < ... < x_m$ with the constat mesh (spatial step size) $h = \frac{x_m - x_0}{m}$, and also the time interval, $0 = t_0 < t_1 < ... < t_n = 1$ with the fixed time step $k = \frac{t_n - t_0}{n}$. We use the elementwise space-time basis functions. The representations for the elementwise solution are independent from each others, i.e., in each element $K \in \mathcal{C}_h$, where $K = [x_i, x_{i+1}] \times (t_n, t_{n+1})$ we have

$$\begin{aligned} u(x,t) =& \varphi_i \left( \Theta_1 \tilde{u}_i^n + \Theta_2 u_i^{n+1} \right) \\ &+ \varphi_{i+1} \left( \Theta_1 \tilde{u}_{i+1}^n + \Theta_2 u_{i+1}^{n+1} \right), \end{aligned} \tag{5.2.1}$$

where

$$\varphi_i = \frac{x_{i+1} - x}{h}, \qquad \varphi_{i+1} = \frac{x - x_i}{h}, \qquad \forall\, x \in [x_i, x_{i+1}],$$

and

$$\Theta_1 = \frac{t_{n+1} - t}{k}, \qquad \Theta_2 = \frac{t - t_n}{k}, \qquad \forall\, t \in [t_n, t_{n+1}],$$

where the functions $\varphi_i$ and $\Theta_i$ are called shape functions and the nodal values of $u^h$ for node $i$ at $t_+^n$ and $t_-^{n+1}$ are denoted by, respectively, $\tilde{u}_i^n$ and $u_i^{n+1}$, and for node $x_-^i$ at $t_n$ by $\hat{u}_i^n$. As for test functions, for each $K \in \mathcal{C}_h$, we introduce the following four test functions:

$$v_1 = \varphi_i \Theta_1, \qquad v_2 = \varphi_i \Theta_2,$$

and

$$v_3 = \varphi_{i+1} \Theta_1, \qquad v_4 = \varphi_{i+1} \Theta_2.$$

In the variational formulation, first we replace $u^h$ by (5.2.1), and then we replace $v^h$ with one of the four test functions $v_i$, $i = 1, 2, 3, 4$. So, we then have four equations and four unknowns, and we can solve the linear system of equations, $Ax = b$, for each element. This procedure is performed first in space for elements in one time step and then we proceed to the next time step.

For example, we perform this strategy for 1D inviscid Burgers equation, $u_t + uu_x = 0$, in one element $K$, with variational formulation introduced as follows

$$\int_K (u_t^h + uu_x^h((v^h + h(v_t^h + uv_x^h)) \, dx \, dt + \int_{\partial K_-(u)} [u^h] v_+^h \, |n_t + u \cdot n| \, ds = 0. \tag{5.2.2}$$

If we use $u$ form the equation (5.2.1) we have the following results

$$u_t = \varphi_i \left( \frac{u_i^{n+1} - \tilde{u}_i^n}{k} \right) + \varphi_{i+1} \left( \frac{u_{i+1}^{n+1} - \tilde{u}_{i+1}^n}{k} \right),$$

and

$$u_x = \Theta_1 \left( \frac{\tilde{u}_{i+1}^n - \tilde{u}_i^n}{h} \right) + \Theta_2 \left( \frac{u_{i+1}^{n+1} - u_i^{n+1}}{h} \right).$$

For $v_1$ we compute the terms appearing in variational formulation noting that $(v_1)_t = \frac{-1}{k}\varphi_i$ and $(v_1)_x = \frac{-1}{h}\Theta_1$, viz

$$\int_{t_n}^{t_{n+1}} \int_{x_i}^{x_{i+1}} u_t \, v_1 = \frac{h}{6}(u_i^{n+1} - \tilde{u}_i^n) + \frac{h}{12}(u_{i+1}^{n+1} - \tilde{u}_{i+1}^n),$$

and the next one

$$\int_{t_n}^{t_{n+1}} \int_{x_i}^{x_{i+1}} uu_x \, v_1 = \frac{uk}{6}(\tilde{u}_{i+1}^n - \tilde{u}_i^n) + \frac{uk}{12}(u_{i+1}^{n+1} - u_i^{n+1})$$

the third term is

$$\begin{aligned} \int_{t_n}^{t_{n+1}} \int_{x_i}^{x_{i+1}} hu_t((v_1)_t + u(v_1)_x) = &- \frac{h^2}{3k}(u_i^{n+1} - \tilde{u}_i^n) - \frac{h^2}{6k}(u_{i+1}^{n+1} - \tilde{u}_{i+1}^n) \\ &- \frac{hu}{4}(u_i^{n+1} - \tilde{u}_i^n) - \frac{hu}{4}(u_{i+1}^{n+1} - \tilde{u}_{i+1}^n), \end{aligned}$$

and for the last term in first integral we have

$$\begin{aligned} \int_{t_n}^{t_{n+1}} \int_{x_i}^{x_{i+1}} huu_x((v_1)_t + u(v_1)_x) = &- \frac{hu}{4}(\tilde{u}_{i+1}^n - \tilde{u}_i^n) - \frac{hu}{4}(u_{i+1}^{n+1} - u_i^{n+1}) \\ &- \frac{u^2k}{3}(\tilde{u}_{i+1}^n - \tilde{u}_i^n) - \frac{u^2k}{6}(u_{i+1}^{n+1} - u_i^{n+1}). \end{aligned}$$

As for the jumps, we note that there are two type of jumps: in the left side of space interval boundary, i.e., in $\{x_i\} \times (t_n, t_{n+1})$ we have $n_t = -1$, and $n = 0$, whereas in $\{t_n\} \times (x_i, x_{i+1})$, $n_t = 0$, and $n = -1$. Now we compute the jumps in the time boundary for $v_1$

$$\int_{x_i}^{x_{i+1}} \varphi_i \left[\varphi_i(\tilde{u}_i^n - u_i^n) + \varphi_{i+1}(\tilde{u}_{i+1}^n - u_{i+1}^n)\right] |-1| = \frac{h}{3}(\tilde{u}_i^n - u_i^n) + \frac{h}{6}(\tilde{u}_{i+1}^n - u_{i+1}^n),$$

and for jumps in the space boundary we have

$$\begin{aligned}\int_{t_n}^{t_{n+1}} &\Theta_1 \left[\Theta_1(\tilde{u}_i^n - \hat{u}_i^n) + \Theta_2(u_i^{n+1} - \hat{u}_i^{n+1})\right] \; |-u| \\ &= \left[\frac{k}{3}(\tilde{u}_i^n - \hat{u}_i^n) + \frac{k}{6}(u_i^{n+1} - \hat{u}_i^{n+1})\right] |-u|.\end{aligned}$$

Note that in the above relation $u_{i+1}^n$, $u_i^n$, $\hat{u}_i^{n+1}$ and $\hat{u}_i^n$ are known from previous steps and are transformed to the right hand side of equation when we replace these terms in variational formulation (5.2.2).

Rearranging coefficients of unknowns for $v_1$ we have

$$\Big[(h/6 - h^2/3k) + u(k/12 + uk/6)\Big] u_i^{n+1},$$

$$\Big[(h/12 - h^2/6k) + u(k/12 - h/2 - uk/6)\Big] u_{i+1}^{n+1},$$

$$\Big[(h^2/3k + h/6) + u(h/2 + k/6 + uk/3)\Big] \tilde{u}_i^n,$$

and

$$\Big[(h^2/6k + h/12) + u(k/6 - uk/3)\Big] \tilde{u}_{i+1}^n.$$

And on the right hand side we have

$$\frac{u_i^n h}{3} + \frac{u_{i+1}^n h}{6} + u\left(\frac{\hat{u}_i^n k}{3} + \frac{\hat{u}_i^{n+1} k}{6}\right).$$

Summing up the equation for $v_1$ becomes

$$\begin{aligned}
&\Big[(h/6 - h^2/3k) + u(k/12 + uk/6)\Big]u_i^{n+1} \\
&+ \Big[(h/12 - h^2/6k) + u(k/12 - h/2 - uk/6)\Big]u_{i+1}^{n+1} \\
&+ \Big[(h^2/3k + h/6) + u(h/2 + k/6 + uk/3)\Big]\tilde{u}_i^n \\
&+ \Big[(h^2/6k + h/12) + u(k/6 - uk/3)\Big]\tilde{u}_{i+1}^n \\
&\qquad\qquad = \frac{u_i^n h}{3} + \frac{u_{i+1}^n h}{6} + u(\frac{\hat{u}_i^n k}{3} + \frac{\hat{u}_i^{n+1} k}{6}).
\end{aligned} \tag{5.2.3}$$

Similarly, we can compute these coefficients for $v_2$, $v_3$ and $v_4$. It is worth mention that for $v_2$ we have the jump term only in space boundary, for $v_3$ only in time boundary, and for $v_4$ the jump term is equal to zero. In this way we have a four by four matrix (mass matrix) for each element, and then we can solve this linear system of equations to derive the unknowns $u_i^{n+1}$, $u_{i+1}^{n+1}$, $\tilde{u}_i^n$, and $\tilde{u}_{i+1}^n$, for each element.

## 5.3 Some Numerical Examples

**Example 1.** The first example is the linear advection equation

$$u_t + au_x = 0 \tag{5.3.1}$$

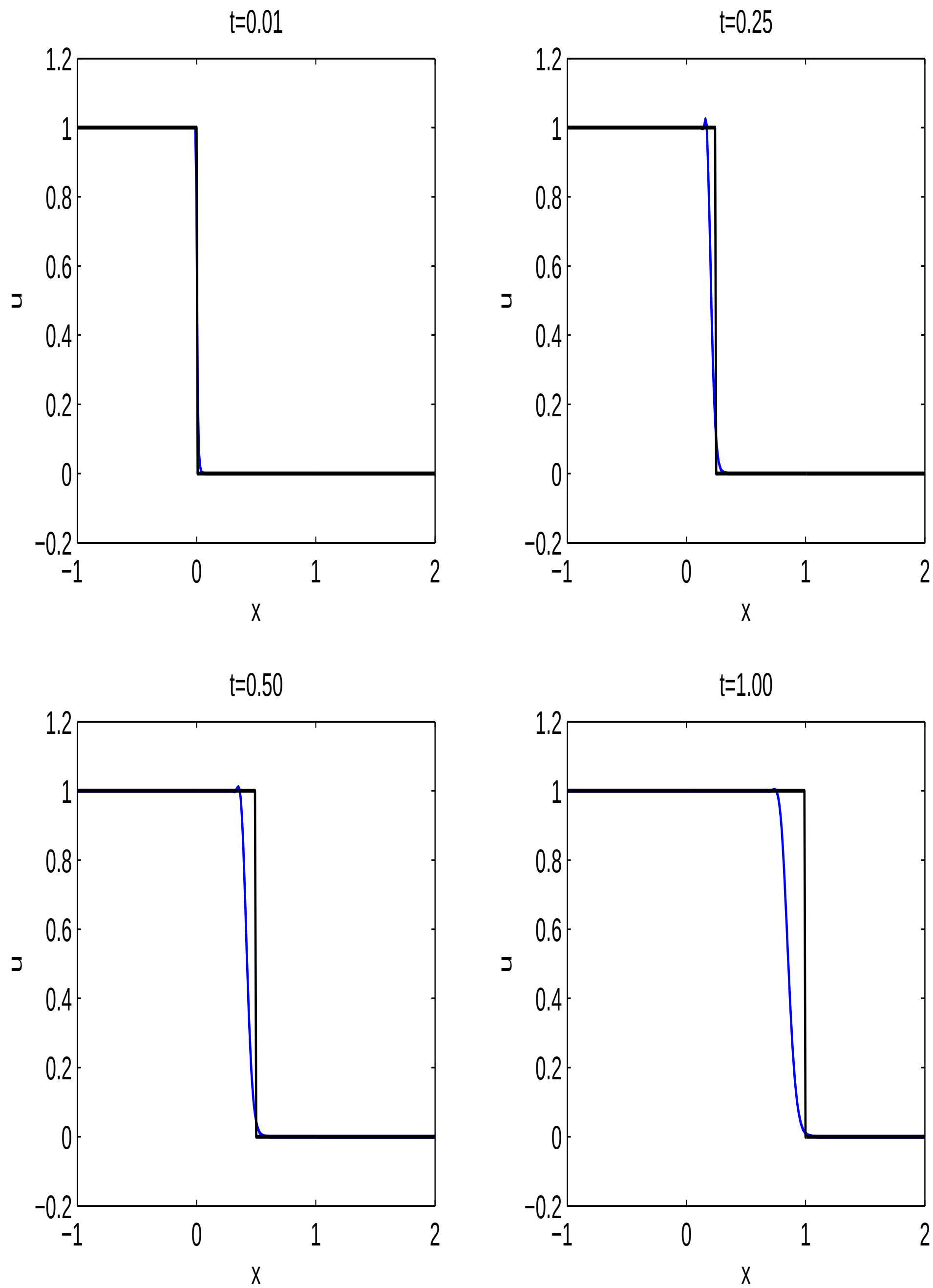
with $a = 1$ and initial data

$$u_0(x) = u(x,0) = \begin{cases} 1 & x < 0, \\ 0 & x > 0. \end{cases}$$

The exact solution is

$$u(x,t) = \begin{cases} 1 & x < t, \\ 0 & x > t. \end{cases}$$

and has a shock that propagates with shock speed 1. See the Figure 5.3 for numerical results and exact solution at the times 0.01, 0.25, 0.5 and 1.


t=0.01
t=0.25
t=0.50
t=1.00
u
x
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−1
0
1
2

**Example 2.** In this example we consider the equation (5.3.1), with $a = 1$ and initial data

$$u_0(x) = u(x,0) = \begin{cases} 0 & x < 0, \\ 1 & x > 0. \end{cases}$$

The exact solution is

$$u(x,t) = \begin{cases} 0 & x < t, \\ 1 & x > t. \end{cases}$$

See the exact and numerical solution in Figure 5.3.

**Example 3.** Now we consider the (5.3.1) with $a = 1$ and initial data

$$u(x,0) = \begin{cases} 0 & x < 0, \\ 1 & 0 < x < 2, \\ 0 & x > 2. \end{cases}$$

The exact solution is

$$u(x,t) = \begin{cases} 0 & x < t, \\ 1 & t < x < t+2, \\ 0 & x > t+2. \end{cases}$$

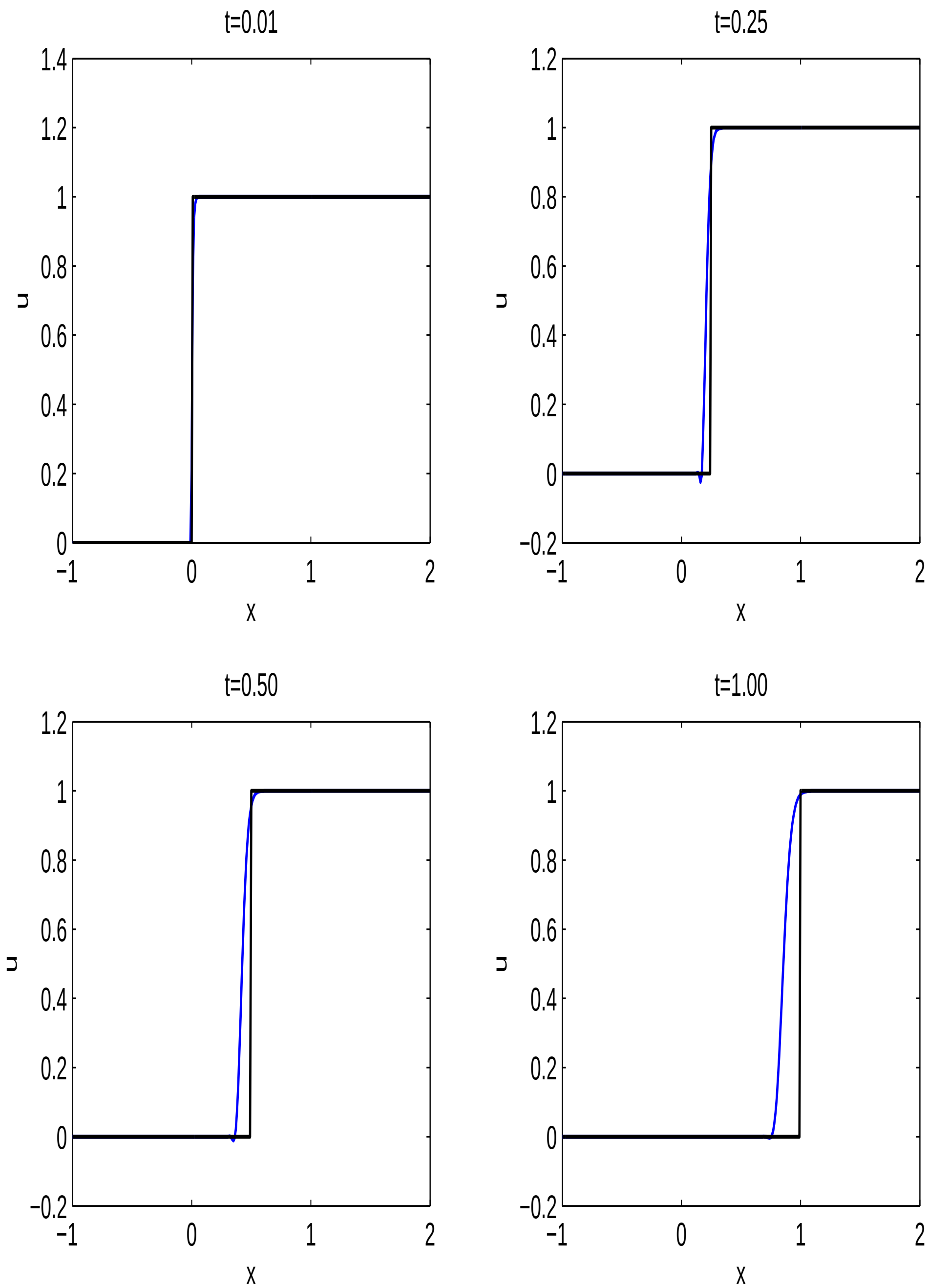
We compare the exact solution and the numerical solution in Figure 5.3.

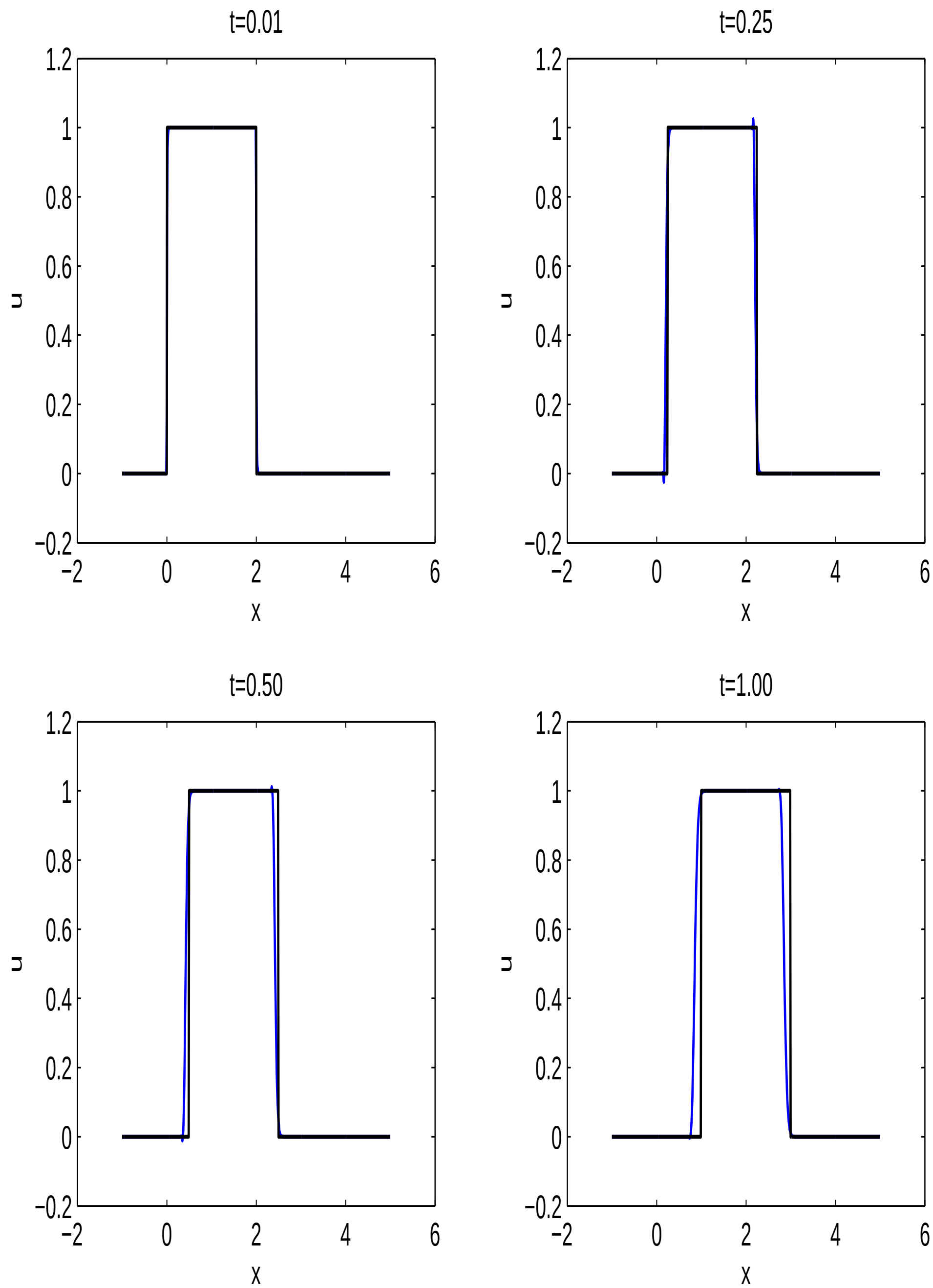
**Example 4.** In this example we consider the Burgers equation $u_t + uu_x = 0$, with initial data

$$u(x,0) = \begin{cases} 1 & x < 0, \\ 2 & x > 0. \end{cases}$$

The exact solution is a rarefaction wave of the form

$$u(x,t) = \begin{cases} 1 & x < t, \\ x/t & t \le x \le 2t, \\ 2 & x > 2t. \end{cases}$$


t=0.01
t=0.25
t=0.50
t=1.00
u
x


t=0.01
t=0.25
t=0.50
t=1.00
u
x

See the exact solution and numerical solution in Figure 5.3.

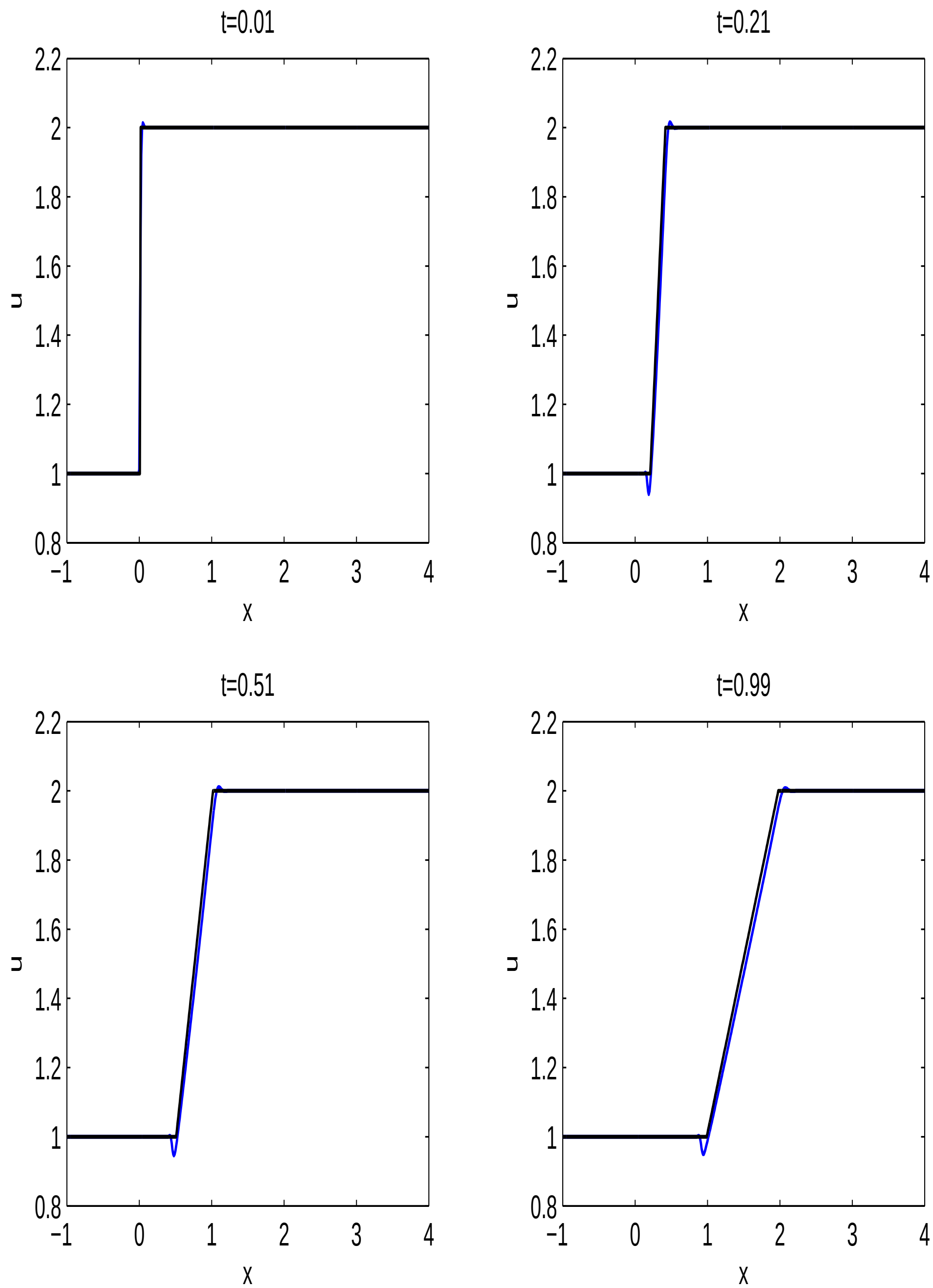
**Example 5.** Finally for Burgers equation with initial data

$$u(x,0) = \begin{cases} 2 & x < 0, \\ 1 & x > 0. \end{cases}$$

The exact solution is a shock wave that propagates with speed 1.5. The form of solution is

$$u(x,t) = \begin{cases} 2 & x < \frac{3}{2}t, \\ \\ 1 & x > \frac{3}{2}t. \end{cases}$$

As shown in the Figure 5.3, the approximation for this nonlinear case has a poor behavior, mainly due to the fact that we have used the non conservative form of Burgers equation instead of conservative form, i.e., $u_t + (\frac{u^2}{2})_x = 0$. The method is adequate for smooth solutions but will not, in general, converge to a discontinuous weak solution of Burgers equation as the grid refined. To derive the good results we need to use the conservative form of Burgers equation, and semi discretization methods, i.e., use DG method in space and then solve an ODE in time, for example, with Rung-Kutta method.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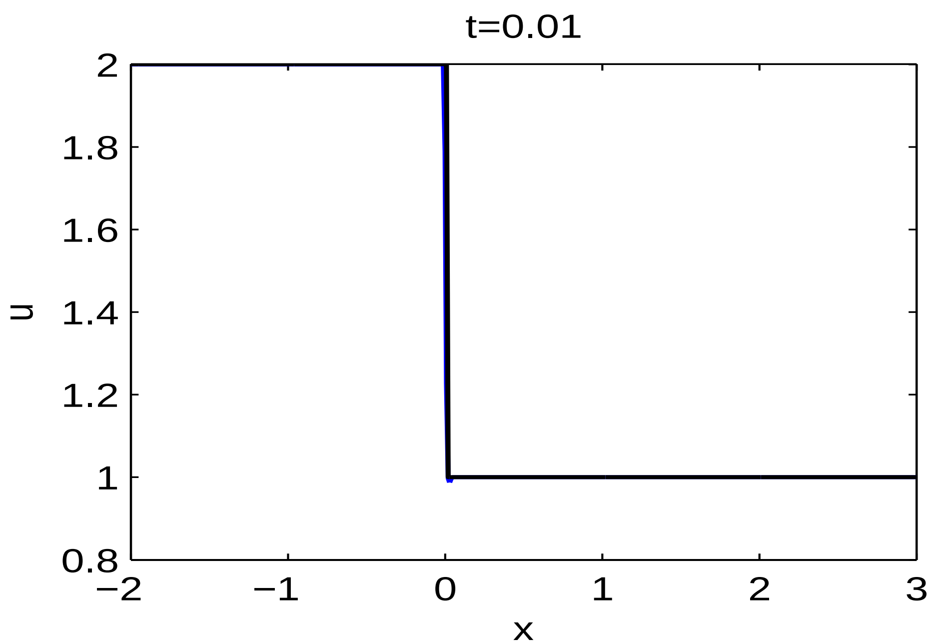
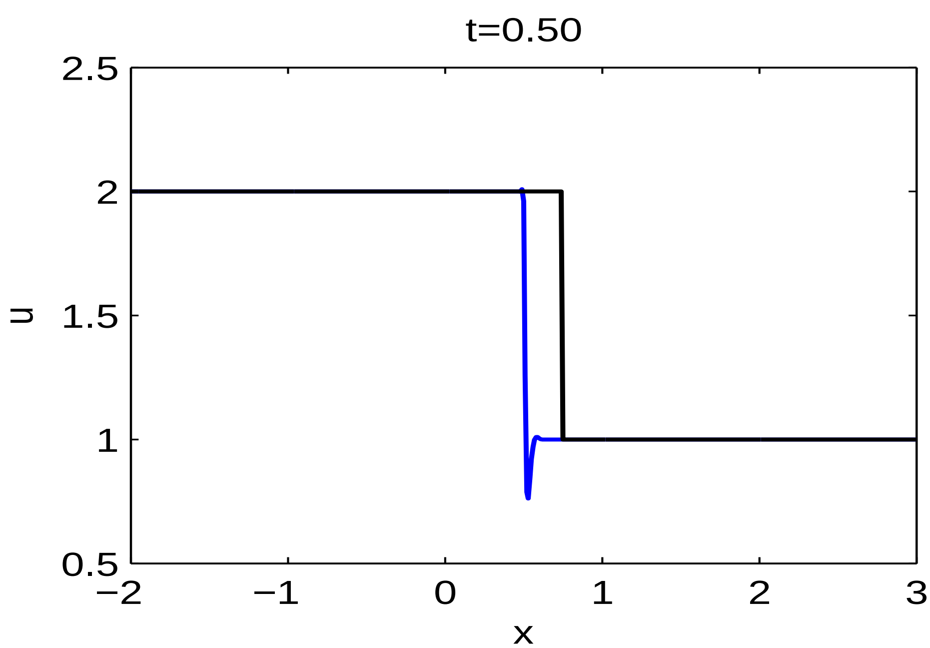
t=0.21
t=0.51
t=0.99
2.2
2
1.8
1.6
1.4
1.2
1
0.8
u
−1
0
1
2
3
4
x


t=0.01
2
1.8
1.6
1.4
1.2
1
0.8
u
−2
−1
0
1
2
3
x


t=0.50
2.5
2
1.5
1
0.5
u
−2
−1
0
1
2
3
x

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