AN ITERATIVE APPROACH TO A MESHLESS SCHEME FOR NUMERICAL SOLUTION OF MULTI-DIMENSIONAL CONVECTION-DIFFUSION PROBLEMS

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Abstract

This study is on the stability and convergence analysis of meshless and some finite element schemes for the numerical solution of a convection dominated convection diffusion equation. Here we have developed a meshless method that efficiently resolves the oscillatory behavior of this equation in the boundary layers, meanwhile keeping the order of convergence in the optimal level of the finite element method. We justify the theory by implementing several examples.

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Key words and phrases. Convection-diffusion, singular perturbation, numerical solution, meshless schemes, finite element schemes, convergence rates.

Contents

1. Chapter 1

Introduction

We consider numerical study of a convection diffusion problem in R^d, d=1,2. In the 1-dimensional case we study an equation with variable coefficients,

$$\begin{cases} -\varepsilon u'' + b(x)u' + c(x)u = f, \ x \in [x_0 \ x_1], \\ \quad u(x_0) = u_0, \ u(x_1) = u_1, \end{cases} \tag{1.1}$$

As an extension to higher dimensional case we consider a convection-diffusion problems of the form

$$\begin{cases} -\varepsilon \nabla.\nabla u + b.\nabla u = f(x,y), \qquad (x,y) \in \Omega = [0\ 1] \times [0\ 1] \\ u(x,y) = 0, \quad (x,y) \in \overline{\Omega} \end{cases} \tag{1.2}$$

Finally we study an initial boundary value problems viz,

$$\begin{cases} u_t - \varepsilon \nabla.\nabla u + b.\nabla u = f, \qquad X \in \Omega, \\ u(X,t) = 0, \qquad X \in \overline{\Omega} \\ u(X,0) = u_0(X) \qquad X \in \Omega. \end{cases} \tag{1.3}$$

The convection-diffusion problems arises in several areas such as the moisture transport in desiccated soil, the potential function of fluid injection through one side of a long vertical channel, the potential for a semiconductor device modeling and steady flow of a viscous, incompressible fluid.

There is a lot of literature dealing with the numerical solution of singularly perturbed problems, related to the problems of type (1.1)-(1.3). One of the main difficulties in the numerical solution of convection-diffusion problems is to find an approximation scheme which is uniformly accurate with a reasonable convergence rate (mostly of order ε).

with decreasing ε the difficulty in solving Problems (1.1)-(1.3) would raise. To see this phenomenon we consider a simple one dimension convection-diffusion problem

$$-\varepsilon u'' + bu' = 0, \quad x \in [x_0\ x_1], \qquad u(x_0) = u_0, \quad u(x_1) = u_1, \tag{1.4}$$

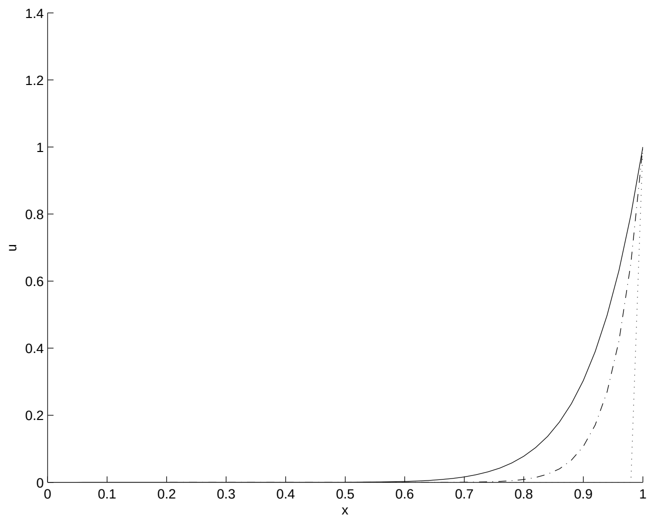

FIGURE 1. Exact solution of problem (1.4) for:
$\varepsilon = 0.1$ ____, $\varepsilon = 0.05$ _._._, $\varepsilon = 0.0001$ ....

With constant coefficient $b > 0$. This problem has an exact solution given by

$$u(x) = u_0 + (u_1 - u_0)\frac{e^{-b(1-x)/\varepsilon} - e^{(-b/\varepsilon)}}{1 - e^{-b/\varepsilon}}$$

Due to the exponential factor (1-x)/ε,the solution changes rapidly in the subinterval (1-ε , 1). That is there is a boundary layer of width ε around x=1 .
The Figure 1, is plotted by setting $u_0 = 0, u_1 = 1$ and b =1 and for the values of $\varepsilon = 1$, $\varepsilon = 10^{-1}$, $\varepsilon = 0.05$, $\varepsilon = 10^{-4}$ (see fig 1).
In fixed interval as (1-ε , 1) the solution has a fluctuating behavior and therefore an ε -uniform approach requires certain strategy in choice of the numerical mesh size h. For example, although the upwind difference scheme gives more stable result, Kellog and Tsan (Malley, 1991) have analyzed the behavior of the error of the standard upwind scheme on a uniform mesh and they show that, for instance, in the layer it is not $\varepsilon - uniform$ in the discrete maximum norm. In other words for the standard upwind and centred finite difference methods on a uniform mesh, local (pointwise) error is not necessarily reduced by successive uniform refinement of the mesh. Furthermore, although the standard centered scheme is of order $\mathcal{O}(h^2)$, it is numerically unstable and gives oscillatory solutions unless the mesh is overlay refined. Therefore, we need more efficient methods in order to capture numerical solutions which has the feature of $\varepsilon - uniform$ convergence.

Classical convergence results for numerical methods for problems (2.1) have the structure $\|U - u\| \leq Ch^k$, where C is a constant which tends to infinity as the perturbation parameter ε approaches zero. This means that the maximal step size h has to be chosen proportional to some positive power of ε which is not so practical. Therefor so many people are looking for so the called uniform or robust methods where the numerical costs are independent of the perturbation parameter ε. In general because using a grid (mesh) for this kinds of problems is associated with certain difficulties, we will introduce a meshless method in chapter 3, where we solve the convection-diffusion problems with an accuracy depending on ε, and drive a convergence rate of order ε

2. Chapter 2

Some methods to solve singularly perturbed convection-diffusion

Consider the 1D model problem

$$\begin{cases} -\varepsilon u'' + b(x)u' + c(x)u = f(x), \ x \in [0\ 1], \\ \qquad u(0) = u_0, \quad u(1) = u_1, \end{cases} \tag{2.1}$$

where $\varepsilon > 0, \quad b(x) > b_0 > 0, \quad c(x) \geq 0$. We choose a uniform mesh $x_j = jh, j = 0, .., J,$ with $x_0 = 0, \ x_J = 1,$ such that $b_j = b(x_j), \ c_j = c(x_j), \ f_j = f(x_j) \ and \ U_j \approx u(x_j)$. Standard discretizations of differential equations use a central difference approximation of the convective term. That is, one approximates $u'(x_i)$ by $(u_{i+1} - u_{i-1})/(2h)$. Using this discretization and the standard approximation $(u_{i-1} - 2u_i + u_{i+1})/h^2$ of $u''(x_i)$ yield a Central difference scheme, and if we use an approximation for $u'(x_i)$ as $(u_i - u_{i-1})/h$, we will get simple upwind scheme.

Below we use standard difference notation :

$$\Delta_+ U_j = U_{j+1} - U_j \quad \Delta_- U_j = U_j - U_{j-1} \quad \delta^2 = \Delta_+ - \Delta_- = \Delta_+ \Delta_-,$$
$$\Delta_0 = \tfrac{1}{2}(\Delta_+ + \Delta_-).$$

Then, the upwind scheme is given by

$$-\varepsilon \frac{\delta^2}{h^2} U_j + b_j \frac{\Delta_-}{h} U_j + c_j U_j = f_j,$$

and the central difference scheme can be written as

$$-\varepsilon \frac{\delta^2}{h^2} U_j + b_j \frac{\Delta_0}{h} U_j + c_j U_j = f_j.$$

Both schemes are desirable when ε is not a small number. The upwind scheme has stable solution and does not have oscillations .

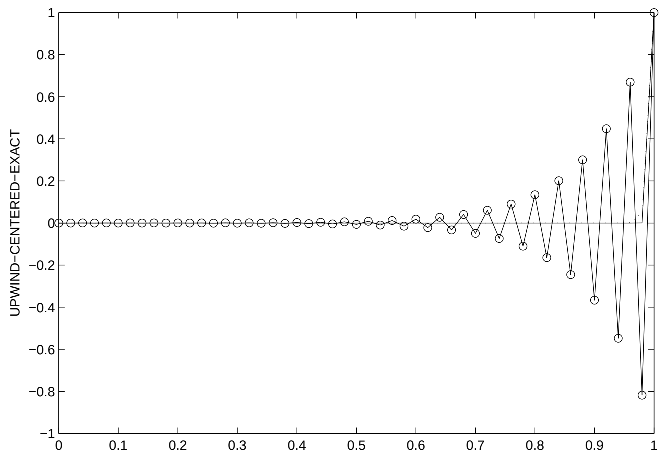

FIGURE 2. Exact solution of problem(1.4) for$\epsilon = 10^{-5}, b = 1, f = 0$

Theorem 2.1 (error bound for simple upwinding on an uniform mesh)**.**

Let $\{u_i^N\}_{i=0}^N$ *be the computed solution using simple up-winding on an equidistant mesh with N subintervals. Suppose that* $h \geq \varepsilon$. *Then there exists a constant C such that for i = 0, . . . , N,*

$$|u_i - u_i^N| \leq C[h + e^{\frac{-b_0(1-x_i)}{b_0 h + 2\varepsilon}}].$$

For a proof see Kellogg and Tsan (1978) or Roos et al. (1996).
If x_i is bounded away from 1, then the above theorem implies that $|u_i - u_i^N| \leq Ch$. That is, the upwind scheme yields an O(h)-accurate solution away from x = 1. But at interior mesh points, that lie close to or inside the layer, the scheme is only O(1)-accurate. So upwind performs weakly in a neighborhood of layers, see Fig 2.

3. CHAPTER 3

A meshless iterative approach to numerical solution of convection-diffusion problems

In the following section, we first consider equation (2.1) with $x_0 = 0, x_1 = 0,\ b(x) \geq b_0 > 0$ then we extend to results to the general form (2.1).

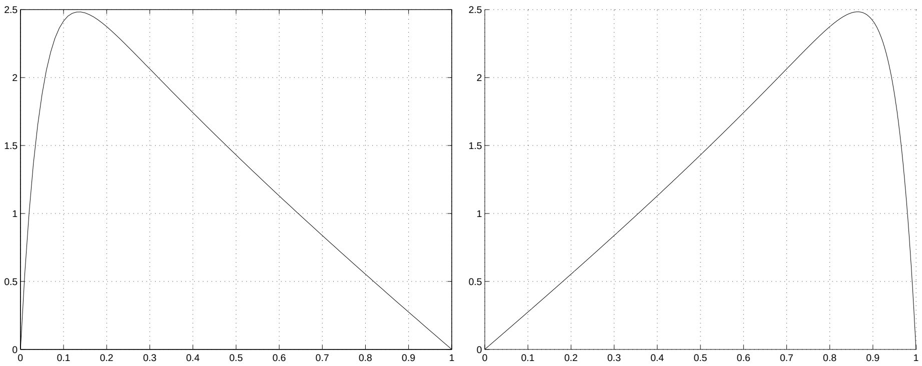

FIGURE 3. The layer region on the left (left figure), The layer on the right (right figure) with the change of variable $x \longrightarrow (-x+1)$

Establishing a solution form when c=0

We let c=0 at equation (2.1). This simplification does not effect our global results because we can use an iteration method and solve eq (2.1) in global case. Without loss of generality, we may assume that the boundary layer appears on the right boundary point. For the left boundary point we can use the result for the right layer by a simple change of variable, and then repeat the same procedure. See Fig 3.

To begin with, we consider the convection equation:

$$b(x)u' = f, \quad x \in [x_0\ x_1], \quad u(x_0) = u_0, \qquad b(x) \geq b_0 > 0. \tag{3.1}$$

The solution for (3.1) is close to the solution for (2.1) when $(c = 0)$, except at the boundary layer. We split U as follows ;

$$U = \begin{cases} u_1 - \gamma x^{\alpha} e^{\beta x} & \text{for } 0 \leq u_1(1) \\ u_1 + \gamma x^{\alpha} e^{\beta x} & \text{for } 0 \geq u_1(1) \end{cases} \tag{3.2}$$

where, u_1 is a solution for (3.1) and $\alpha, \beta, \gamma \in R, \quad \alpha \geq 2, \gamma \geq 0$.
U will approximate the exact solution u of problem (2.1) if
1. Out of layer U is almost equal to u_1.
2. Inside layer U is a good approximation for u_1. In other word, we try to find α, β ,and γ such that the term $\gamma x^{\alpha} e^{\beta x}$ corrects the solution u_1 in layer region.

First we discuss when $c = 0$ in (2.1) and we try to find parameters γ , α , β. Next we use an iterative method to solve (2.1) when $c \neq 0$.

Requiring that the approximate solution U satisfies the boundary data $U(1) = 0$, we have

$$u_1(1) - \gamma\ e^{\beta} = 0. \tag{3.3}$$

Let $R(U)$ be the residual function of (2.1). Since $-\varepsilon(U-u)''+b(x)(U-u)' = R(U)$. Letting

$$R(U(1)) = 0, \tag{3.4}$$

we get an approximate solution which is close to exact solution u at $x = 1$, because $u(1) = U(1)$, and from (3.4) we have almost $u'(1) \approx U'(1)$ (ε is small). So far, we have two equations and we would like to provide a third equation for the three unknown parameter α, β, γ .

Lemma 3.1. *If $\theta = \gamma x^{\alpha} e^{\beta x}$, where $\beta \leq 0, \alpha \geq 2$ and α, β and γ are real constants, then there exist a real $\hat{\theta}$, with $0 < \hat{\theta} < -\beta$ such that*

$$\|\theta\|_1 = \beta^{-1} u_1(1)(1 - e^{\hat{\theta}}). \tag{3.5}$$

Proof. We compute

$$\begin{aligned} \|\theta\|_1 &= \int_0^1 |\theta| dt = \gamma \frac{e^{\beta}}{\beta} - \gamma \frac{\alpha}{\beta} (-\beta^{-1})^{\alpha} \int_0^{-\beta} e^{-w} w^{\alpha-1} dw \\ &= \gamma \frac{e^{\beta}}{\beta} - \gamma \frac{\alpha}{\beta} (-\beta^{-1})^{\alpha} \Gamma(\alpha, -\beta), \end{aligned} \tag{3.6}$$

where $\Gamma(\alpha, -\beta)$ is lower incomplete gamma function. We use the following properties of gamma functions,

$$\Gamma(\alpha, -\beta) = (\alpha - 1)![1 - e^{\beta} \sum_{i=0}^{\alpha-1} (-\beta)^i/(i!)]$$

and

$$\gamma \frac{\alpha}{\beta} (-\beta^{-1})^{\alpha} \Gamma(\alpha, -\beta) = \alpha! \gamma \frac{\alpha}{\beta} (-\beta^{-1})^{\alpha} [1 - e^{\beta} \sum_{i=0}^{\alpha-1} (-\beta)^i/(i!)].$$

Then applying the reminder of Taylor theorem we get for $0 < \hat{\theta} < -\beta$,

$$\begin{aligned} &\alpha! \gamma \frac{\alpha}{\beta} (-\beta^{-1})^{\alpha} \Big[1 - e^{\beta} \sum_{i=0}^{\alpha-1} (-\beta)^i/(i!) \Big] \\ &\qquad = \alpha! \gamma \frac{\alpha}{\beta} (-\beta^{-1})^{\alpha} [1 - e^{\beta} (e^{-\beta} - e^{\hat{\theta}} (-\beta^{\alpha})/(\alpha!)] \\ &\qquad = (\alpha! \gamma \frac{\alpha}{\beta} (-\beta^{-1})^{\alpha}) (-\beta)^{\alpha} e^{\beta + \hat{\theta}}/(\alpha!) = \gamma e^{\beta + \hat{\theta}}/\beta, \end{aligned} \tag{3.7}$$

so that

$$\|\theta\|_1 = \frac{\gamma e^{\beta}}{\beta} - \frac{\gamma e^{\beta + \hat{\theta}}}{\beta} = \frac{u_1(1)[1 - e^{\hat{\theta}}]}{\beta}.$$

We may apply the Sandwich theorem to see that the above lemma is valid when α is a real number. □

we conclude that: If $u_1(1) = 0$ then $u_1 = U$.

The norm $\|\theta\|_1$ have a geometric meaning; it represents the area between U and u_1 (Fig 4)

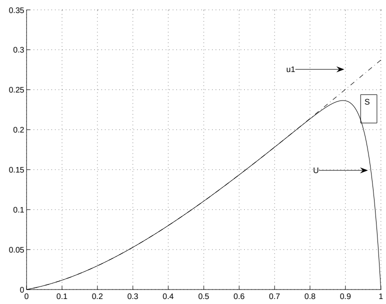

FIGURE 4. $S = \|\Theta\|_1$

As for the second equation, to find the unknown parameters, we use the equation (3.4): $(R(U(1)) = 0$. Let

$$L(u) = -\varepsilon u'' + b(x)u',$$

and suppose that $u_1(1) > 0$, so that
$L(U) = -\varepsilon U'' + bU'$ and from (3.2) and because u_1 is a solution of (3.1) we have,

$$L(U) = -\varepsilon u_1'' + f - L(\gamma x^\alpha e^{\beta x}).$$

Further $R(U) = L(U) - f$ implies that

$$R(U) = -\varepsilon u_1'' - L(\gamma x^\alpha e^{\beta x}).$$

Using (3.4) we set

$$[R(U)]_{x=1} = [-\varepsilon u_1'' - L(\gamma x^\alpha e^{\beta x})]_{x=1} = 0,$$

this yields the quadratic equation ;

$$A\alpha^2 + B\alpha + C = 0$$

with a solution of the form
(3.8)

$$A = \varepsilon, \quad B = -\varepsilon + 2\beta\varepsilon - b(1), \quad C = \frac{-\varepsilon u_1''(1)}{u_1(1)} + \varepsilon\beta^2 - b(1)\beta + f(1)$$

where

$$0 \le B^2 - 4AC, \tag{3.9}$$

is the criterion for existence of a desired α.

Errors estimation

Lemma 3.2. *For U defined by* (3.2) *and with u_1 being the solution for* (3.1) *with $\theta = \gamma x^\alpha e^{\beta x}$,there exists real value K such that ;*

$$\|\theta\|_1 = \|U - u_1\|_1 \le \varepsilon K. \tag{3.10}$$

Proof. First we assume that β >0, and $\|\theta\|_1 = \int_0^1 \gamma x^\alpha e^{\beta x} dx \leq \frac{\gamma e^\beta}{\alpha+1}$. Then, from (3.3) we will have , $\|\theta\|_1 \leq \frac{u_1(1)}{\alpha+1}$, where α is a root of the equation (3.8) so that we have $|B| = |b(1)| + \varepsilon_1$ where ε_1 is a small number, so that

$$\frac{u_1(1)}{\alpha+1} = \frac{u_1(1)}{(\frac{b(1)+\varepsilon_1}{\varepsilon}) + 1} \leq \varepsilon K.$$

If β <0, because $e^\beta < 1$, for positive γ we have $\gamma > u_1(1)$ then we get $\|\theta\|_1 \leq \int_0^1 \gamma x^\alpha dx \leq \frac{\gamma}{\alpha+1}$. We choose K_1 so that $\gamma = u_1(1) + K_1$ remains positive

$$\frac{\gamma}{\alpha+1} = \frac{u_1(1)+K_1}{(\frac{b(1)+\varepsilon_1}{\varepsilon}) + 1} \leq \varepsilon K.$$

□

Now we stat, without proof, a lemma that gives information about u, for a proof see [1].

Lemma 3.3. *Let* $0 < b_0 \leq b(x)$.*Then, for* $i = 0, 1, 2, \ldots$, *the solution* u *of* (2.1) *satisfies*

$$|u^i(x)| \leq \varepsilon C(1 + \varepsilon^{-i} e^{-\varepsilon^{-1} b_0 (1-x)}).$$

Lemma 3.4. *Consider the equation*

$$\begin{cases} -\varepsilon y'' + b(x) y' = f \\ y(0) = 0, \quad y(1) = 0. \end{cases}$$

For $|f(x)| \leq k\varepsilon$ *,and* $m = min_x |b(x)|$, *we have that* $\|y\|_1 \leq \varepsilon C$

Proof. The equation can be rewritten as

$$y' = \frac{f + \varepsilon y''}{b(x)} \leq (\varepsilon/m)(y'' + k).$$

Integration yields

$$y \leq (\varepsilon/m)(y' - y'(0) + kx) \quad so \quad |y| \leq (\varepsilon/m)(|y'| + |y'(0)| + kx).$$

Then we get

$$\|y\|_1 \leq (\varepsilon/m)(\|y'\|_1 + |y'(0)| + k/2),$$

and from the above lemma first-order derivative of y is bounded at x=1 as ε tend to 0. So that we have $\|y\|_1 \leq \varepsilon C$ □

Lemma 3.5. *Let* u_1 *be the solution of the equation*

$$bu' = f, \quad u(0) = 0,$$

and u *be the exact solution of equation*

$$-\varepsilon u'' + b(x) u' = f, \quad u(0) = 0, \quad u(1) = 0.$$

Further, assume that $|u_1''| \leq k$ *and* $m = \min_x |b(x)|$, *then*

$$\|u - u_1\|_1 \leq \varepsilon C. \tag{3.11}$$

Proof. We consider

$$-\epsilon(u-u_1)''+b(u-u_1)'=-\varepsilon u_1'', \tag{3.12}$$

using the above lemma we have

$$\|u-u_1\|_1 \leq \varepsilon C.$$

□

Theorem 3.1. *Let U be defined by the equation (3.2) and u is the exact solution for (2.1) (when c=0), and $\|u_1''\| \leq k$. Then there exist a constant C such that*

$$\|u-U\|_1 \leq \varepsilon C. \tag{3.13}$$

Proof. From lemma (3.2) we have evidently

$$\|U-u\|_1 \leq \|u-u_1\| + \|U-u_1\| \leq \varepsilon(C_1+C_2),$$

then

$$\|u-U\|_1 \leq \varepsilon C.$$

□

The equation (3.12) shows that if m is large, or $u_1(1)$ and k are small numbers then $\|U-u\|_1 << \varepsilon$.

Theorem 3.2. *For $0 < b_0 \leq b(x)$ and $|u_1''| \leq k$ and with the residual defined by $R(U) = -\varepsilon U'' + bU' - f$, we have that there exists a constant K such that $\|R\|_1 \leq \varepsilon K$.*

Proof. For

$$-\varepsilon(u-U)''+b(x)(u-U)'=R(U), \tag{3.14}$$

we have $\|R\|_1 = \varepsilon|(U-u)'|_{x=0}^{x=1}$ (because $[U-u]_0^1 = 0$).Using mean value theorem we get for some $0 < \theta < 1$

$$\|R\|_1 = \varepsilon|(U-u)'|_{x=0}^{x=1} = \varepsilon|(U-u)''(\theta)|.$$

If θ is close to 0, from (3.2) we conclude that in a neighborhood of 0 we have $U = u_1$, $U' \simeq u_1'$, $U'' \simeq u_1''$. we replace (3.14) by $-\varepsilon(u-u_1)'' + b(u-u_1)' = R$, and from (3.13) we have that -$\varepsilon u_1'' = R$. Now from $|u_1''| \leq k$, we get $|R| \leq \varepsilon k$.
If $\theta >> 0$, from lemma (3.3)

$$\varepsilon|(U-u)''(\theta)| \leq \varepsilon C(1+\varepsilon^{-i}e^{-\varepsilon^{-1}b_0\theta}) \simeq C\varepsilon,$$

and then $\|R\|_1 \leq K\varepsilon$.

□

Iteration method to get solution for $c \neq 0$.
Here, to conclude our discussion, we introduce an iterative method, when c is not necessarily zero. Suppose $L(u) = -\varepsilon u'' + bu'$ and $L(u) = F$, where $F = f - cu$. Consider the sequence $\{u_1^j\}_0^n$, $L(u_1^j) = F(u_1^{j-1})$, and set $u_1^0 = u_1$ as initial guess for u: u_1 is solution of the equation

$bu' = f, u_0 = 0$. To get u_1^j in each iteration we should solve an equation that is introduced above with $c = 0$. We iterate with an stopping criterion $\|u_1^j - u_1^{j-1}\| \leq \delta$.

If b(0)=0 or b(1)=0 we can use the above method too, we add or subtract a small value to $b(x)$, so we consider a new equation viz,

$$-\varepsilon u'' + (b(x) + \kappa)u' = F(x), \qquad F(x) = f(x) + \kappa u_1' - cu_1,$$

and use the above iteration technique to solve it.

Numerical example. In this part we present some numerical example demonstrating the accuracy of our iteration procedure. We choose Matlab 7 as the computational environment and compares the results of the current method with the exact solution for each example. We observe a good agreement between the theory and the numerical results in our approach. In particular the correlation between the size of ε and errors in our examples justifying the accuracy of the iterative method.

Example 1. Consider the equation

$$\begin{cases} \varepsilon u''(x) + \frac{1}{xsin(x)}u' + \frac{1}{x^2}u = f(x), & 0 < x \leq 1 \\ u(0) = 0, \qquad u(1) = 0, \end{cases}$$

where

$$f(x) = \frac{\pi \cos(\pi x)\csc(x)}{x} - \varepsilon\pi^2 \sin(\pi x) + \frac{\sin(\pi x)}{x^2}.$$

The exact solution is $\sin(\pi x)$. We take $\varepsilon = 10^{-6}, \varepsilon = 10^{-10}, \ \varepsilon = 10^{-13}$, respectively. The numerical results are given in the following three tables.

Table 1. Numerical results for example 1 ($\varepsilon = 10^{-6}, \kappa = -0.04$)

x	Exact u(x)	Approx U(x)	Absolute error
.0001	3.1416e-005	3.1416e-005	3.2402e-006
0.08	0.2487	0.2487	3.0467e-006
0.2	0.5878	0.5878	2.6923e-006
0.5	1	1.0000	1.6968e-006
0.8	0.5878	0.5878	4.2393e-007
0.96	0.1253	0.1253	2.0167e-008
1	0	0.0000	2.6988e-013

Table 2. Numerical results for example 1 ($\varepsilon = 10^{-10}, \kappa = -0.04$)

x	Exact u(x)	Approx U(x)	Absolute error
.0001	3.1416e-005	3.1416e-005	3.2879e-010
0.08	0.2487	0.2487	3.0422e-010
0.2	0.5878	0.5878	2.6997e-010
0.5	1	1.0000	1.6865e-010
0.8	0.5878	0.5878	4.3071e-011
0.96	0.1253	0.1253	2.6425e-012
1	0	0.0000	2.7610e-013

Table 3. Numerical results for example 1 ($\varepsilon = 10^{-13}, \kappa = -0.04$)

x	Exact u(x)	Approx U(x)	Absolute error
.0001	3.1416e-005	3.1416e-005	9.4999e-013
0.08	0.2487	0.2487	1.4450e-013
0.2	0.5878	0.5878	1.0101e-012
0.5	1	1.0000	8.5620e-013
0.8	0.5878	0.5878	7.2131e-013
0.96	0.1253	0.1253	6.2811e-013
1	0	0.0000	2.7655e-013

Example 2. Consider equation

$$\begin{cases} \varepsilon u''(x) + \frac{1}{x}u' + \frac{1}{x^2}u = f(x), & 0 < x \leq 1 \\ u(0) = 0, \qquad u(1) = 0, \end{cases}$$

where $f(x) = \frac{2}{x} - 2\varepsilon - 3$. The exact solution in this case is $x - x^2$. We take $\varepsilon = 10^{-6}$, $\varepsilon = 10^{-10}$, $\varepsilon = 10^{-13}$, respectively and give the numerical results on results tables 4-6 below:

Table 4. Numerical results for example 2 ($\varepsilon = 10^{-6}$, $\kappa = -0.0001$)

x	Exact u(x)	Approx U(x)	Absolute error
.0001	9.9990e-005	1.9689e-004	9.6900e-005
0.08	0.0736	0.0736	1.1586e-005
0.2	0.1600	0.1600	4.0975e-007
0.5	0.2500	0.2500	4.2326e-006
0.8	0.1600	0.1600	2.5228e-007
0.96	0.0384	0.0384	1.9976e-006
1	0	0.0000	1.4042e-005

TABLE 5. Numerical results for example 2 ($\varepsilon = 10^{-10},\ \kappa = -0.0001$)

x	Exact u(x)	Approx U(x)	Absolute error
.0001	9.9990e-005	1.0001e-004	1.5385e-008
0.08	0.0736	0.0736	1.1586e-009
0.2	0.1600	0.1600	4.0975e-011
0.5	0.2500	0.2500	4.2326e-010
0.8	0.1600	0.1600	2.5227e-011
0.96	0.0384	0.0384	1.9976e-010
1	0	0.0000	1.4042e-009

TABLE 6. Numerical results for example 2($\varepsilon = 10^{-13}, \kappa = -0.0001$)

x	Exact u(x)	Approx U(x)	Absolute error
.0001	9.9990e-005	9.9990e-005	1.5351e-011
0.08	0.0736	0.0736	1.1559e-012
0.2	0.1600	0.1600	4.0773e-014
0.5	0.2500	0.2500	4.2222e-013
0.8	0.1600	0.1600	2.5091e-014
0.96	0.0384	0.0384	1.9917e-013
1	0	0.0000	1.4005e-012

Example 3 This example is about a homogeneous singularly perturbed problem, first given by Bender and Orsza(1978).

$$\begin{cases} \epsilon u''(x) + u' - u = 0, & 0 \le x \le 1 \\ u(0) = 1, \quad u(1) = 1. \end{cases}$$

The exact solution is given by

$$u(x) = \frac{(e^{m_2} - 1)e^{m_1 x} + (1 - e^{m_1})e^{m_2 x}}{e^{m_2} - e^{m_1}},$$

with the coefficients m_1 and m_2 as follows: $m_1 = \frac{-1+\sqrt{1+4\varepsilon}}{2\varepsilon}$, $m_2 = \frac{-1-\sqrt{1+4\varepsilon}}{2\varepsilon}$.

Table 7. Numerical results for example 3 ($\varepsilon = 10^{-4}$)

x	Exact u(x)	Approx U(x)	Absolute error	$\lvert R(U)\rvert$
0	1	1	0	9.0949e-013
0.0001	0.6005	0.6004	3.4861e-005	0.2326
0.2	0.4494	0.4494	2.0424e-009	-2.7311e-014
0.5	0.6066	0.6066	1.1807e-011	2.2204e-016
0.8	0.8187	0.8187	8.0713e-014	3.3307e-016
0.96	0.9608	0.9608	1.9096e-014	8.8818e-016
1	1	1	0	1.8874e-015

Table 8. Numerical results for example 3 ($\varepsilon = 10^{-8}$)

x	Exact u(x)	Approx U(x)	Absolute error	$\lvert R(U)\rvert$
0	1	1	0	0
0.0001	0.3679	0.3679	1.4430e-009	7.6350e-013
0.2	0.4493	0.4493	1.4100e-009	2.7256e-014
0.5	0.6065	0.6065	1.1896e-009	7.7716e-016
0.8	0.8187	0.8187	6.4230e-010	5.5511e-016
0.96	0.9608	0.9608	1.5075e-010	1.2212e-015
1	1	1	0	2.2204e-016

Table 9. Numerical results for example 3 ($\varepsilon = 10^{-13}$)

x	Exact u(x)	Approx U(x)	Absolute error	$\lvert R(U)\rvert$
0	1	1	0	0.0020
0.0001	0.3682	0.3679	2.9416e-004	7.6317e-013
0.2	0.4496	0.4493	2.8740e-004	2.7145e-014
0.5	0.6068	0.6065	2.4244e-004	4.4409e-016
0.8	0.8189	0.8187	1.3089e-004	1.1102e-015
0.96	0.9608	0.9608	3.0718e-005	9.9920e-016
1	1	1	0	5.5511e-016

Example 4. Consider now a semilinear version of the singularly perturbed problem given by Bender and Orszag (1978)

$$\begin{cases} \varepsilon u''(x) + 2u' + e^u = 0, \quad 0 \le x \le 1 \\ u(0) = 0, \qquad u(1) = 0. \end{cases}$$

Our results in here are compared with those given by Bender and Orszags for their, the so called , *uniformly valid approximation* given by

$$u = \ln(\frac{2}{1+x}) - e^{\frac{-2x}{\varepsilon}} \ln 2.$$

Table 10. Numerical results for example 4 ($\varepsilon = 10^{-3}$)

x	Bender and Orszag u(x)	Approx solution U(x)	$\vert R(U)\vert$
0	0	1	0
0.0001	0.12554627242589	0.12563060784551	0.27442287819082
0.2	0.51082562376599	0.51103871843734	2.0744e-010
0.5	0.28768207245178	0.28777805873158	3.4461e-011
0.8	0.10536051565783	0.10538980648510	1.2951e-010
0.96	0.02020270731752	0.02020786500263	4.1063e-011
1	0	0	8.6722e-010

Example 5. Finally we consider the following singulary perturbed problem:

$$\begin{cases} -\varepsilon u''(x) - u' = -2\varepsilon - 2x, & 0 \le x \le 1 \\ u(0) = 1, \qquad u(1) = 1, \end{cases}$$

where they use the standard Finite element method (FEM) and Streamline Diffusion Finite Element Method (SDFEM) on a Shishkin grid and a Bakhvalov grid. In the following tables we have compared our iteratively obtained solutions by the results in Chen, Xu. Here the analyric solution is given by

$$u = \frac{e^{-x/\varepsilon} - e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}} + x^2.$$

Table 11. Maximum error of FEM on a special quasi-uniform grid for example 5

N	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-8}$	$\varepsilon = 10^{-12}$
32	8.4492e-003	8.0716e-003	8.0716e-003
64	4.0535e-003	3.9732e-003	3.9731e-003
128	9.5341e-004	1.9703e-003	1.9703e-003
256	1.3007e-004	9.8127e-004	9.8127e-004
512	1.6082e-005	4.9025e-004	4.9025e-004

TABLE 12. Max error of FEM on Shishkin grid for example 5

N	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-8}$	$\varepsilon = 10^{-12}$
32	6.2475e-003	6.3344e-003	6.3344e-003
64	2.1495e-003	2.2358e-003	2.2358e-003
128	6.9143e-004	7.4446e-003	7.4448e-003
256	2.2385e-004	2.3994e-004	2.3995e-004
512	7.1234e-005	7.5245e-005	7.5255e-005

TABLE 13. Maximum error of FEM on Bakhvalov grid for example 5

N	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-8}$	$\varepsilon = 10^{-12}$
32	1.6599e-003	1.7177e-003	1.7184e-003
64	3.8046e-004	4.2546e-004	4.2556e-004
128	7.9912e-005	1.0605e-004	1.0607e-004
256	1.8511e-005	2.6464e-005	2.6471e-005
512	4.6250e-006	6.6068e-006	6.6124e-006

TABLE 14. Maximum error of our solution for example 5

itteration	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-8}$	$\varepsilon = 10^{-12}$
30	5.4130e-005	1.4654e-015	1.3240e-015

*****

4. CHAPTER 4

The two dimensional problem

Steady-state convection-diffusion problems are boundary value problems of the form

$$\begin{cases} -\varepsilon\Delta u + \mathbf{b}.\nabla u = f \qquad (x,y) \in \Omega = [0\ 1] \times [0\ 1] \\ u = 0, \qquad (x,y) \in \partial\Omega, \end{cases} \tag{4.1}$$

where Ω is some domain in R^n with boundary $\partial\Omega$. Here ε is a small positive parameter which means that the ellipticity constant is close to zero, which gives reduced stability for standard numerical methods. The term, -$\varepsilon\Delta u$ models diffusion while $\mathbf{b}.\nabla u$ corresponds to convection. The terminology convection-dominated problem is used since the

convection coefficient has much greater magnitude than the diffusion coefficient: $\frac{|\mathbf{b}|}{|\varepsilon|} >> 1$. For almost all boundary conditions this is an example of a singularly perturbed problem: the solution in the case $\varepsilon = 0$ is not identical to the pointwise limit of the solution as $\varepsilon \longrightarrow 0$. Applications of convection-diffusion problems include the linearized Navier-Stokes equations (the Oseen equations).
We divide the boundary $\partial\Omega$ into 3 parts :

inflow boundary $\partial^{-}\Omega$=$\{x \in \partial\Omega : \mathbf{b} \cdot \mathbf{n} < 0\}$

outflow boundary $\partial^{+}\Omega$=$\{x \in \partial\Omega : \mathbf{b} \cdot \mathbf{n} > 0\}$.

tangential/characteristic flow boundary $\partial^{0}\Omega = \{x \in \partial\Omega : \mathbf{b} \cdot \mathbf{n} = 0\}$,

where $\mathbf{n}$ is the outward unit normal to $\partial\Omega$. The solution u for equation (4.1) has the following asymptotic structure:

u = reduced solution + layers + negligible terms.

Here the "*reducedsolution*" is the solution u_0 of the first-order partial differential equation $\mathbf{b}\cdot\nabla u_0 = f$ on Ω, which is obtained by formally setting $\varepsilon = 0$ in (2.1), with the boundary data $u_0 = 0$ on $\partial^{-}\Omega$. The 'layers' are narrow regions where u changes rapidly, so certain derivatives of u are consequently large there. Along $\partial^{+}\Omega$ the solution usually has an exponential boundary layer.This can be written in terms of exponential functions. Finally, along $\partial^{0}\Omega$ the solution typically has parabolic or characteristic boundary layers.

Example 6. consider the following two dimension convection-diffusion problem.

$$\begin{cases} -\epsilon\Delta u + u_x + u_y = 1, & (x, y) \in \Omega = [0\ 1] \times [0\ 1] \\ u = 0, & (x, y) \in \partial\Omega. \end{cases} \tag{4.2}$$

.
Because $\mathbf{b}(x, 0) \cdot \mathbf{n} < 0$ and $\mathbf{b}(0, y) \cdot \mathbf{n} < 0$ the inflow boundaries are the sides x=0, y=0 of $\overline{\Omega}$ and because $\mathbf{b}(x, 1) \cdot \mathbf{n} > 0$ and $\mathbf{b}(1, y) \cdot \mathbf{n} > 0$, there are an exponential layers at outflow boundaries y=1 x=1, while there are not any characteristic layers because for every $(x\ y) \in \partial\Omega$, $\mathbf{b}(x\ y) \cdot \mathbf{n} \neq 0$.

Below we use reduced solution and by the same method as in the last section, we are able to give an approximation for eq(4.2). By the following lemma we have a solution for the eq(4.2) just with exponential layers. We do not give a proof of this lemma in here, a complete proof can be found in [1].

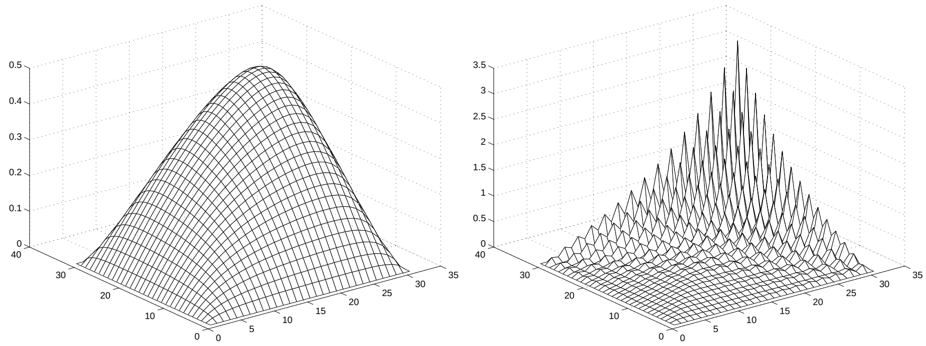

FIGURE 5. (left figure , solution of eq $-\varepsilon\nabla u + \mathbf{b}.\Delta u + cu = f \ by \ FEM \ with \ \varepsilon = 0.1 \ \mathbf{b} = (1\ 1) \ c = 1$, Right figure $\varepsilon = 0.001$ $\mathbf{b} = (1\ 1)$ $c = 1$

Lemma 4.1. *Consider the equation*

$$(4.3) \qquad \begin{cases} -\varepsilon\Delta u + \boldsymbol{b}.\nabla u + cu = f & (x,y) \in \Omega = [0\ 1] \times [0\ 1] \\ u = 0, \quad (x,y) \in \partial\Omega, \end{cases}$$

with $0 < \varepsilon << 1$. $(b_1(\boldsymbol{x}), b_2(\boldsymbol{x})) \geq (\alpha_1, \alpha_2) > (0,0)$ *a nd* $c(x) - \frac{1}{2} div \boldsymbol{b} \geq \beta > 0$. *We assume that* $\boldsymbol{b}$, *c and f are smooth. Then, the solution u of (4.3) has exponential boundary layers at sides x=1 and y=1 of* Ω.

One of the efficient methods for the numerical investigation of the boundary value problems is the finite element method. This, however, detoriates if ε becomes very small giving superious oscillations, see Fig 5. Other known techniques such as the streamline diffusion finite element method (SDFEM), although stable for small ε values, depends upon the choice of the unknown parameter and in general, no precise formula for an optimal value of SDFEM parameter δ_t is known.

In this note we introduce a meshless method with high acuracy. This method is an extension of our one dimension method of the previous sections with Dirichlet boundary condition. As a general approach we consider the equation

$$(4.4) \qquad \begin{cases} -\epsilon\Delta u + b_1 u_x + b_2 u_y = f, & (x,y) \in \Omega \\ u = 0, \quad (x,y) \in \partial\Omega \end{cases}$$

Let now u be the exact solution of (4.4). The intersection of the graph of u and the planes $y = kx + k_i, i = 1, 2, ...,\ 0 < x < 1, 0 < y < 1$ (see Fig(4)) would correspond to the shape of a one dimensional solution. Here for small k the intersections are almost paralel with the x- axis, while sufficiently larg k-values would provide intersections in the y-direction.

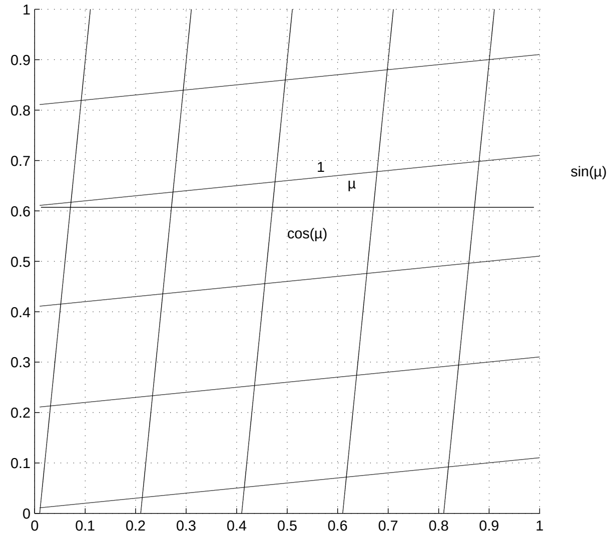

FIGURE 6

Now suppose that for a rectangular domain $\Omega = [0,1] \times [0,1]$ we have only a bounday layer on left or right side (later on we shall consider the case with layers on both sides).

We compute the one dimensional solutions u_x, u_y that lies on planes parallell to $y = kx + k_i$, so that

$$\begin{cases} u_x^d = u_x^{ind} + k u_y^{ind} \\ u_y^d = \frac{1}{k} u_x^{ind} + u_y^{ind}. \end{cases} \tag{4.5}$$

Note that x and y on the right hand side are independet variables, whereas x, and y in the left are dependet variables. More specifically u_x^{ind} and u_y^{ind} are expressed with respect to the independent variables. Likewise, u_x^d and u_y^d are expressed with respect to the dependent variables. Now we rewrite the equation (4.4) as

$$-\varepsilon \Delta u + b_1(x,y) u_x + b_2(x,y) u_y = f(x,y), \tag{4.6}$$

and use a new (different) representation, viz

$$-\varepsilon \Delta u + b_1(x,y) u_x^{ind} + b_2(x,y) u_y^{ind} = f(x,y). \tag{4.7}$$

Here, for $y = kx + k_i$ we have $u_x^d = u_x^{in} + u_y^{in} k$ or $y = kx + k_i$ we may write $u_y^d = u_x^{in} \frac{1}{k} + u_y^{in}$. To summarize we have the following case studies:

Case 1. If the exponential boudary layer lie in x=1.
Let us first modify (4.7) as

$$\begin{aligned} -\varepsilon\Delta u + b_1(x, kx+k_i)[u_x^d - ku_y^{ind}] \\ + b_2(x, kx+k_i)[\frac{1}{k}u_x^d - \frac{1}{k}u_x^{ind}] = f(x, kx+k_i). \end{aligned} \tag{4.8}$$

We split (4.8) andv write

$$-\varepsilon u_{xx}^d + (b_1 + \frac{b_2}{k})u_x^d = f + (\frac{b_2}{k})u_x^{ind} + (kb_1)u_y^{ind}, \tag{4.9}$$

where we insert approximate values for u_x^{ind} and u_x^{ind} that solve the equation

$$\begin{cases} b_1G_x + b_2G_y = f, \\ G(x,0) = 0 \end{cases} \tag{4.10}$$

i.e., $G(x, y)$ is a solution for (4.10) and the approximations for $u_y^{ind} and\, u_x^{ind}$ are G_y, G_x.

Case 2. If the exponential boudary layer lie in y=1
We modify (4.7) as

$$\begin{aligned} -\varepsilon\Delta u + b_1(\frac{1}{k}y - \frac{1}{k}k_i, y)[ku_y^d - ku_y^{ind}] \\ + b_1(\frac{1}{k}y - \frac{1}{k}k_i, y))[u_y^d - \frac{1}{k}u_x^{ind}] = f(\frac{1}{k}y - \frac{1}{k}k_i, y). \end{aligned} \tag{4.11}$$

Rewritting equation (4.11) as

$$-\varepsilon u_{yy}^d + (b_2 + kb_1)u_y^d = f + b_1ku_y^{ind} + (\frac{1}{k}b_2)u_x^{ind}, \tag{4.12}$$

the same procedure as in case 1 yields to solve

$$\begin{cases} b_1M_x + b_2M_y = f, \\ M(0,y) = 0. \end{cases} \tag{4.13}$$

And for $M(x, y)$ beiny a solution for (4.13) we insert M_x for u_x^{ind}, and M_y for u_y^{ind}.

Numerical results

Example 7. We consider Eq (4.3) with two exponential l ayers at $x = 1$ and $y = 1$. To obtain reduced solution $u_1(x, y)$ we need to solve

$$\begin{cases} (u_1)_x + (u_1)_y = 1, \\ u_1(x,y) = 0, \ on\ \partial^-\Omega. \end{cases} \tag{4.14}$$

The general solution for (4.14) is of the form

$$u_1(x,y) = \frac{x+y}{2} + \varphi(x-y).$$

For a solution $u_1(x,y)$ satisfying $u_1 = 0$ on $\partial^- \Omega$, we consider Fourier series expansion for

$$\varphi(x-y) = |x-y| \quad \text{on} - \pi < x-y < \pi,$$

and find that

$$u_1(x,y) = \frac{x+y}{2} - \frac{1}{2}\Big(\frac{\pi}{2} - \frac{4}{\pi}\sum_{1}^{\infty}\frac{cos(2n-1)(x-y)}{(2n-1)^2}\Big).$$

Here, we solve two types of one dimensional convection-diffusion equations

Type I.

$$-\varepsilon u_{xx} + u_x - k u_{1_y} + \frac{1}{k}u_x - \frac{1}{k}u_{1_x} = f(x, kx + k_i),$$

or

$$-\varepsilon u_{xx} + (1+\frac{1}{k})u_x = f(x, kx+k_i) + k u_{1_y} + \frac{1}{k}u_{1_x} \ i = 1,2,...,l.$$

Type II.

$$-\varepsilon u_{yy} + k u_y - k u_{1_y} - \frac{1}{k}u_{1_x} + u_y = f(\frac{y}{k} - \frac{k_i}{k}, y)$$

or

$$-\varepsilon u_{yy} + (k+1)u_y = f(\frac{y}{k} - \frac{k_i}{k}, y) + k u_{1_y} + \frac{1}{k}u_{1_x} \ i = 1,2,...,l.$$

The first type will generate solutions with exponential layer at x=1 and the seconde type will generate solutions with exponential layer at y=1. (see fig 7).

Example 8. In this example we compare the approximate solution by the exact solution, we consider the following two dimension convection-diffusion problem.

$$\begin{cases} -\varepsilon\Delta u + (1+x)u_x = f, (x,y) \in \Omega = [0\ 1] \times [0\ 1], \\ u = 0\ (x,y) \in \partial\Omega. \end{cases}$$

The exact solution is given by (see Fig 8, 9)

$$u(x,y) = xy(1-y)\Big(1 - e^{(-(1-x)/\varepsilon)}\Big).$$

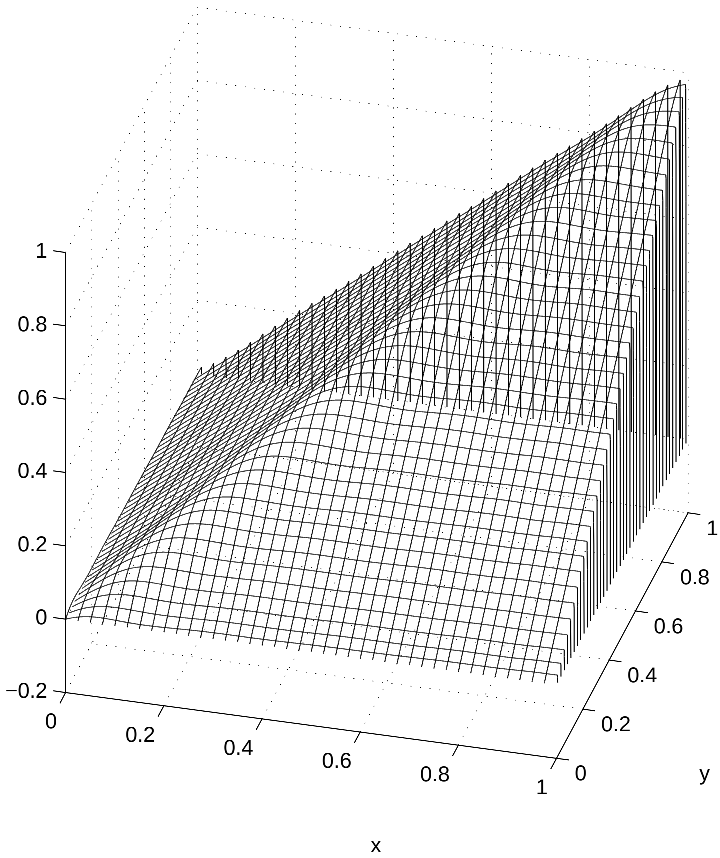

FIGURE 7. solution of example 6 , $\varepsilon = 10^{-2}$

Example 9. In this example, we use our meshless method to solve the following two dimension convection-diffusion problem and compare the result with the exact solution

$$\begin{cases} -\varepsilon \Delta u + u_x = f, (x, y) \in \Omega = [0\ 1] \times [0\ 1] u = 0\ (x, y) \in \partial\Omega. \end{cases}$$

The exact solution is given by (see Fig 10 ,11,12)

$$u(x, y) = xy(1 - e^{(-(1-x)/\varepsilon)})(1 - e^{(-(1-y)/\varepsilon)}).$$

Example 10. Here, we apply the meshless method to the following two dimension convection-diffusion problem.

$$\begin{cases} -\varepsilon \Delta u + u_y = f, (x, y) \in \Omega = [0\ 1] \times [0\ 1] \\ u = 0\ (x, y) \in \partial\Omega \end{cases}$$

The exact solution (given below) is ploted in Figs 13 and 14:

$$u(x, y) = \left(1 - e^{\frac{-1}{\epsilon} + \frac{x}{\epsilon})} - cos(\frac{\pi x}{2})\right) \times sin(2\pi y).$$

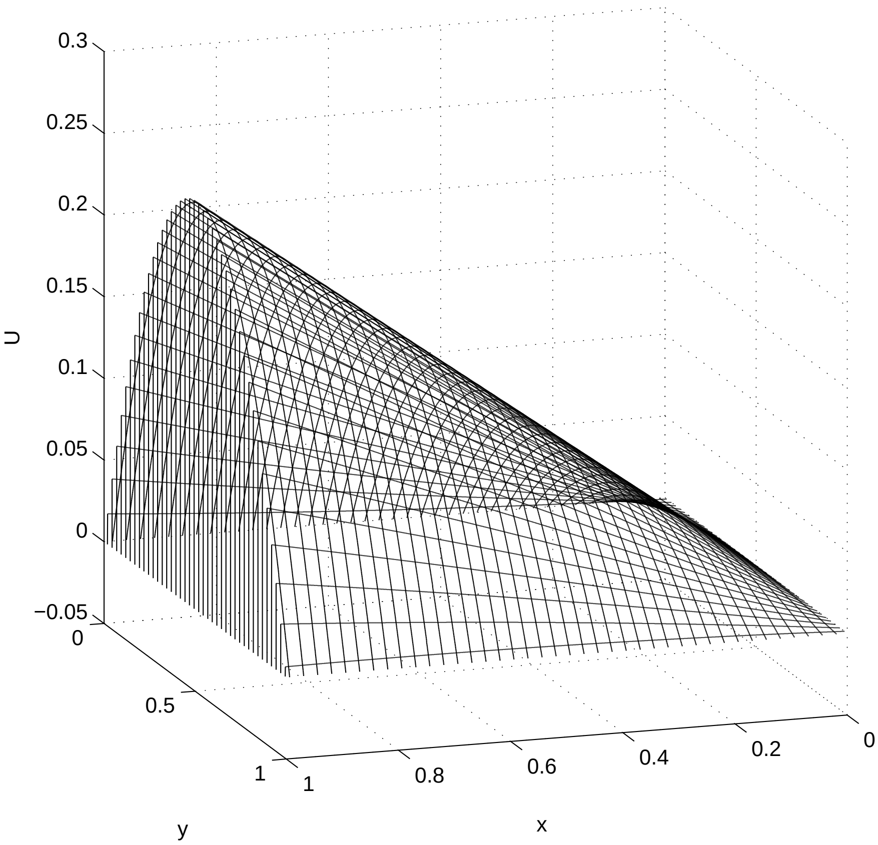

FIGURE 8. solution of example 7 , $\varepsilon = 10^{-6}$

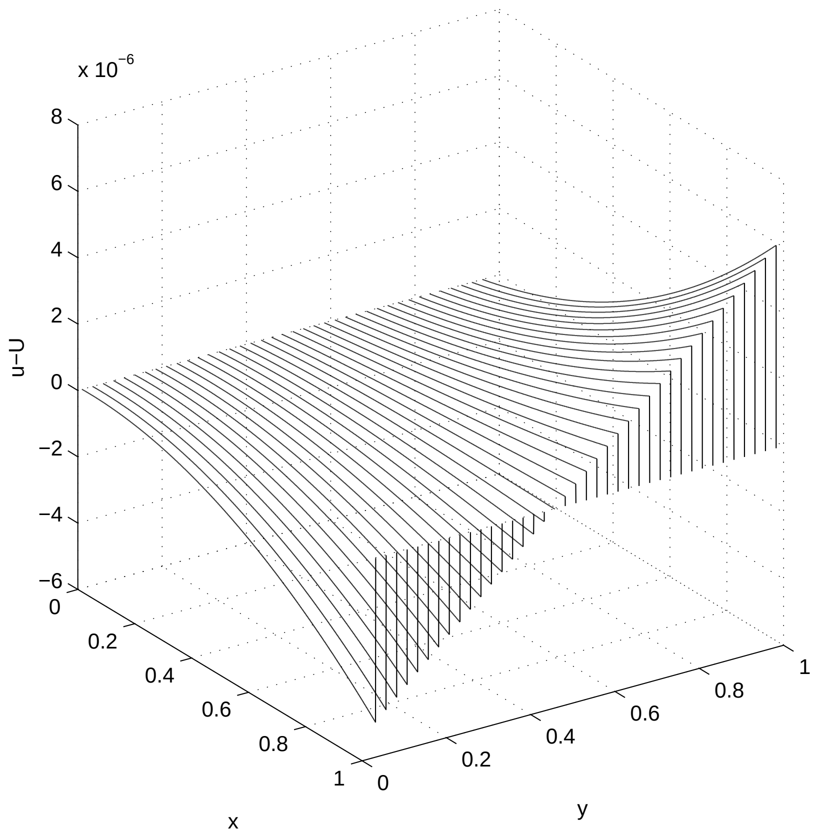

FIGURE 9. Error of example 7, $\varepsilon = 10^{-6}$

Example 11. We continue and consider the following two dimension convection-diffusion problem (see Fig 15):

$$\begin{cases} -\varepsilon\Delta u - u_x = exp(-x-y+2), \quad (x,y) \in \Omega = [0\ 1] \times [0\ 1] \\ u = 0\ (x,y) \in \partial\Omega \end{cases}$$

Example 12. In this example, we consider the following two dimension convection-diffusion problem (see Fig 16).

$$\begin{cases} -\varepsilon\Delta u + u_x = 2\pi sin(\frac{\pi x}{2})^3 cos(\frac{\pi x}{2}), \quad (x,y) \in \Omega = [0\ 1] \times [0\ 1] \\ u = 0, \quad (x,y) \in \partial\Omega. \end{cases}$$

The time dependent problems

In the remaining part of the paper we consider a two dimensional time dependent problem formulated as the initial boundary value problem below

$$(4.15) \quad \begin{cases} u_t - \varepsilon\Delta u + \mathbf{b}\cdot\nabla u + cu = f \quad (\mathbf{x},t) \in \Omega = [0\ 1]^2 \times (0\ \infty) \\ u = 0, \qquad (x,y) \in \partial\Omega, \\ u(x,y,0) = g(x,y), \end{cases}$$

whwere $\mathbf{x} := (x,y)$ andc $0 < \varepsilon << 1$. We start by solving the following equation

$$(4.16) \qquad \begin{cases} u_t + \mathbf{b}\cdot\nabla u + cu = f \quad (\mathbf{x},t) \in \Omega = [0\ 1]^2 \times (0\ \infty) \\ u(x,y,t) = 0,\ x = 0\ or\ x = 1\ or\ y = 0\ or\ y = 1 \\ u(x,y,0) = g(x,y), \end{cases}$$

(without ε part, atleast on one side of the rectangle $[0\ \ 1]\times\ [0\ \ 1]$ the new solution is zero.) We denote the new solution by u_{in}, and find $\frac{\partial u_{in}}{\partial t}$ from (4.16) and insert u_t in the orginal form and continue to solve the problem by using the method that was introduced in Section 3 for two dimensional equation.

Example 13. Now, we consider the two dimension convection-diffusion problem with the special data function $f = tcos(x) + sin(x)$ on the right hand side (see Fig 17).

$$(4.17) \quad \begin{cases} u_t - \varepsilon\Delta u + u_x = f(x,t) \quad ((x,y) \in \Omega = [0\ 1]^2) \times (0\ \infty) \\ u = 0, \qquad (x,y) \in \partial\Omega, \\ u(x,y,0) = 0. \end{cases}$$

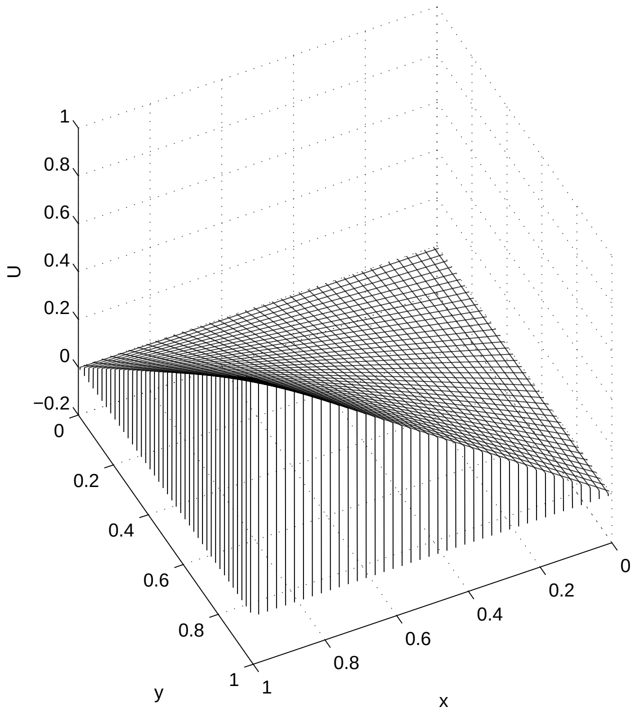

FIGURE 10. Solution of example 8, $\epsilon = 10^{-15}$

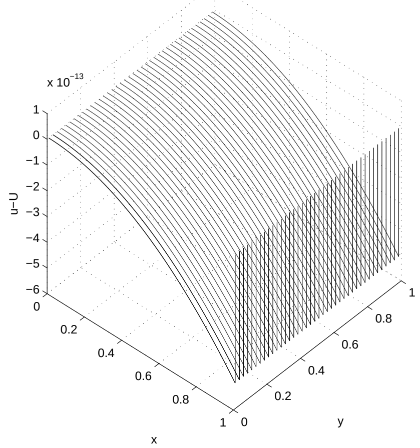

FIGURE 11. Error of example 8, $\epsilon = 10^{-15}$

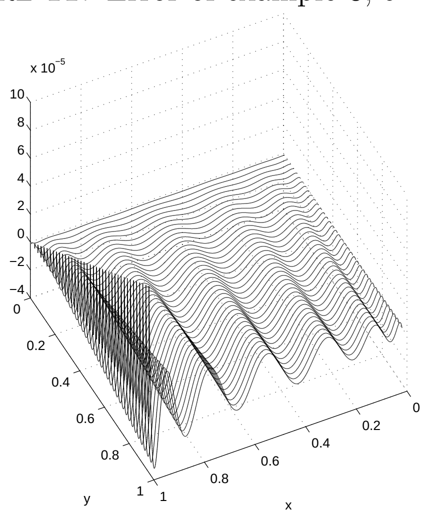

FIGURE 12. Error of example 8, $\epsilon = 10^{-8}$

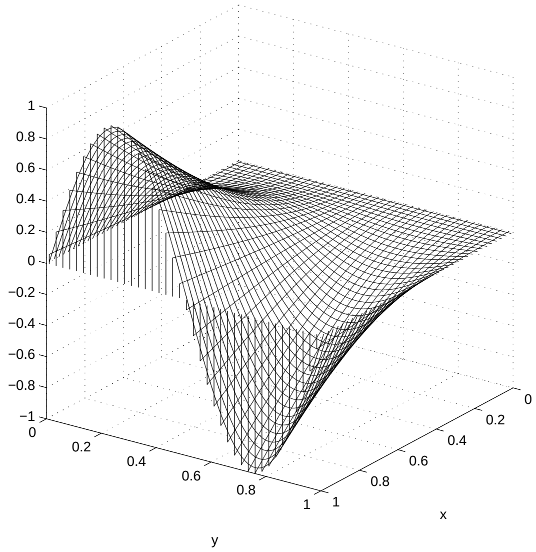

FIGURE 13. Solution of example 9, $\epsilon = 10^{-8}$

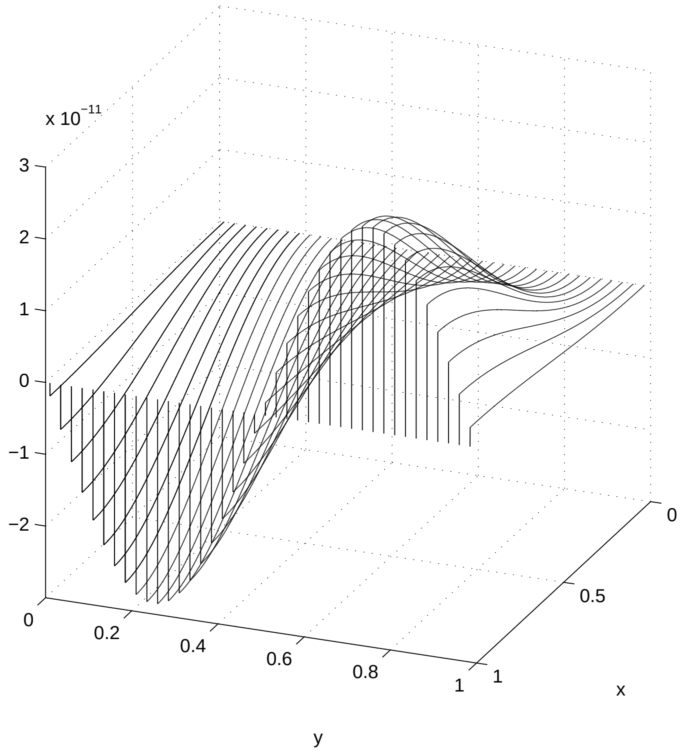

FIGURE 14. Error of example 9, $\epsilon = 10^{-8}$

Example 14 Finally, we study the following two dimensional convection-diffusion problem,

$$\begin{cases} u_t - \varepsilon \Delta u + u_y = f & ((x,y) \in \Omega = [0\ 1] \times [0\ 1]) \times (0\ \infty) \\ u = 0, & (x,y) \in \partial\Omega, \\ u(x,y,0) = 0, & \end{cases} \tag{4.18}$$

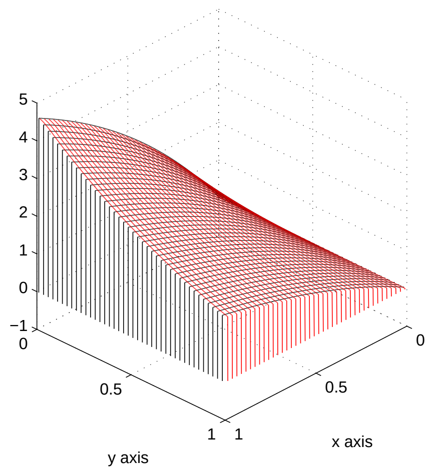

FIGURE 15. Solution of example 10, $\epsilon = 10^{-5}$

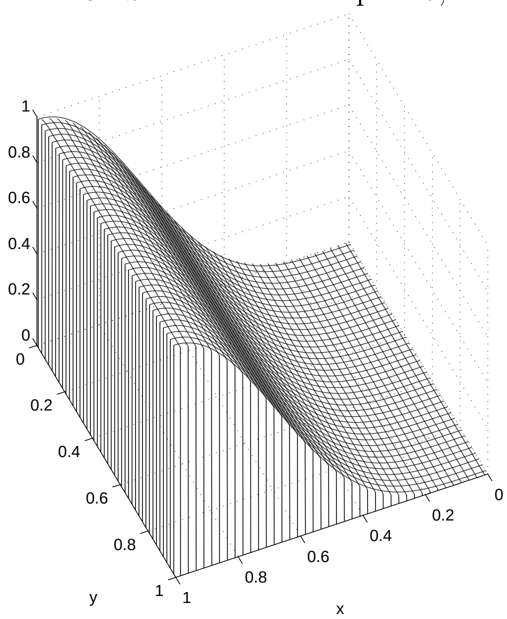

FIGURE 16. Solution of example 11, $\epsilon = 10^{-4}$

with the exact solution given by

$$f = xy \sin(5/6\pi x) \ln(10^5\pi(y-2))e^t + x \sin(5/6\pi x) \ln(10^5\pi(y-2))e^t + xy \sin(5/6\pi x)/(y-2)e^t.$$

(see Fig 18)

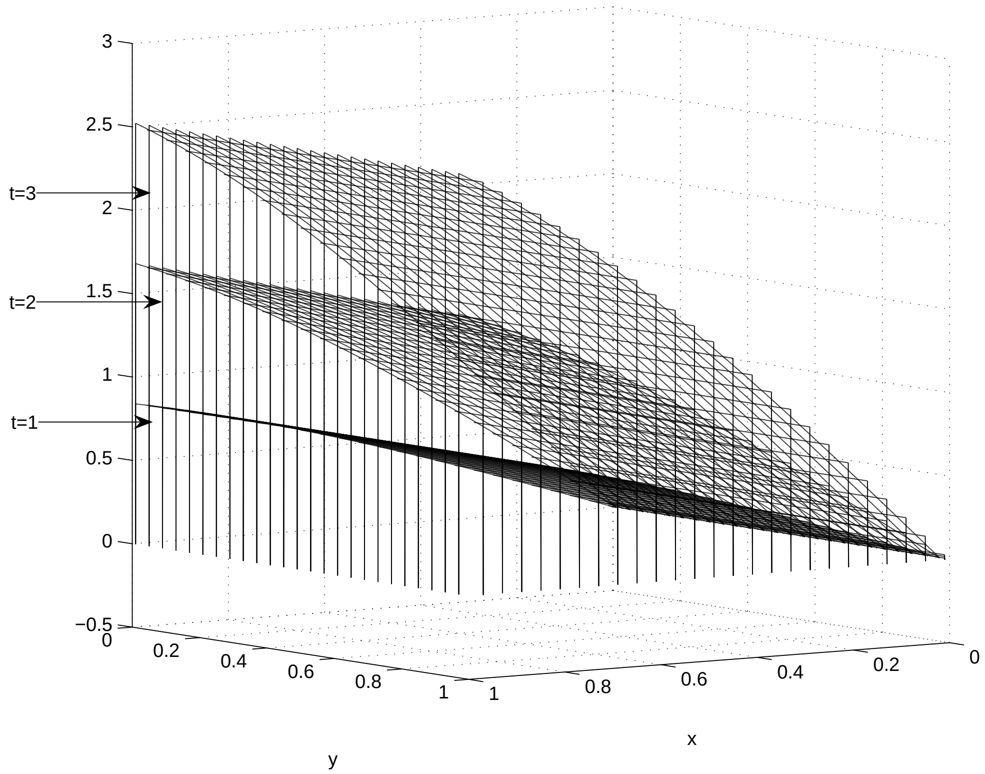

FIGURE 17. Numerical solution of example 13 ,ε=10^{-8}

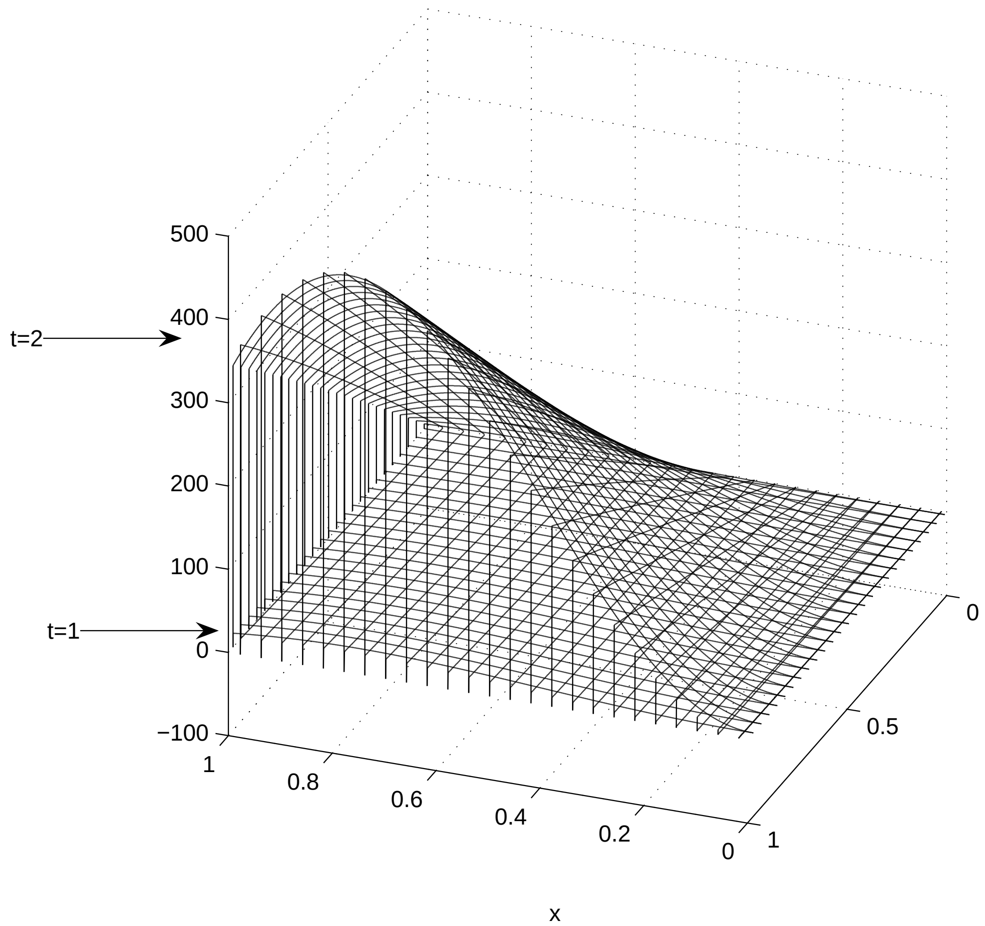

FIGURE 18. Numerical solution of example 14 ,ε=10^{-5}

5. CHAPTER 5

Solution of linear ODE/BVP by using an iteration approach

We consider linear ODE/BVP,

$$\begin{cases} -au'' + b(x)u' + c(x)u = f(x) \\ u(0) = 0, \quad u(1) = 0; \end{cases} \tag{5.1}$$

where a=constant and b,c and f are infinitly differntiable in a neighbourhood of zero. There are many methods to solve ODE/BVP problems for example shooting method, Finit difference method, Galerkin methods, Rayleigh-Ritz, One of the well - known methods to solve BVP problem is shooting method, where we want to get a good approximation for $u'(0)$ by guessing, initial values. In Shooting method, we use several times the solution of IVP problem untill we get a good approximate for $u'(0)$. We try to get $u'(0)$ directly with hight accuracy. We start with simple case and use that result to solve more involved problemes by indution to get $u'(0)$ at following three steps.

1. Approximation of $u'(0)$ if $b(x) = b$ is constant and $c(x) = 0$.

Let us consider the linear boundary value problem

$$\begin{cases} -au'' + b(x)u' = f(x) \\ u(0) = 0, \quad u(1) = 0. \end{cases} \tag{5.2}$$

We assume that U is infinitly differntiable in a nighborhood of zero. We want to compute $u^{(n)}(0)$, $(n = 1, 2, 3, ...)$. From (5.2), we know that

$$[u''(x) = \frac{b}{a}u'(x) - a^{-1}f(x)]_{x=0},$$

differentialy (5.2) we get $[-au''' + bu'' = f']_{x=0}$. Thus we have

$$[u^3(x) = \frac{b^2}{a^2}u'(x) - \frac{b}{a^2}f(x) - a^{-1}f'(x)]_{x=0}$$

by the same way

$$[u^4(x) = \frac{b^3}{a^3}u'(x) - \frac{b^2}{a^3}f(x) - \frac{b}{a^2}f'(x) - a^{-1}f''(x)]_{x=0},$$

and for all $n > 1$:

(5.3)

$$[u^{(n)}(x) = \frac{b^{n-1}}{a^{n-1}}u'(x) - \frac{b^{n-2}}{a^{n-1}}f(x) - \frac{b^{n-3}}{a^{n-2}}f'(x) - \frac{b^{n-4}}{a^{n-3}}f''(x) + ...]_{x=0},$$

By Taylor expasion of u(x) at a neighbourhood of zero and letting x=1 we get,

$$u(1) = [u + (1-x)u' + \frac{(1-x)^2}{2}u'' + ... + \frac{1}{n!}u^{(n)} + ...]_{x=0} \tag{5.4}$$

Equations (5.3) in (5.4) and because

$$e^{\frac{b}{a}} = 1 + \frac{b}{a} + \frac{1}{2}(\frac{b}{a})^2 + ... \tag{5.5}$$

yields

$$\frac{a}{b}(e^{\frac{b}{a}} - 1)[u'(x)]_{x=0} - \frac{a}{b^2}(e^{\frac{b}{a}} - 1 - \frac{b}{a})[f(x)]_{x=0} -,$$

$$\frac{a^2}{b^3}(e^{\frac{b}{a}} - 1 - \frac{b}{a} - \frac{b^2}{a^2}\frac{1}{2!})[f'(x)]_{x=0} - \frac{a^3}{b^4}(e^{\frac{b}{a}} - 1 - \frac{b}{a} - \frac{b^2}{a^2}\frac{1}{2!} - \frac{b^3}{a^3}\frac{1}{3!})[f''(x)]_{x=0} - ... = 0.$$

Hence, we may write

$$[u'(x)]_{x=0} = \frac{\Sigma_{i=1}^{+\infty}((\frac{a}{b})^i(e^{\frac{b}{a}} - S_i)[f^{i-1}(x)]_{x=0})}{a(e^{\frac{b}{a}} - 1)}. \tag{5.6}$$

such that , $S_i = \Sigma_{k=1}^{i}(\frac{(\frac{b}{a})^k}{k!})$.

Note. At (5.6) because $\lim(e^{\frac{b}{a}} - S_i) = 0$ as $i \to \infty$, if $\frac{b}{a} << 2$. We see a fast convergence of S_i to $e^{\frac{b}{a}}$. Later on we return to this phonomenon. Suppose $u' = U$ in (5.2) then $-aU' + bU = f$ and

$$U = e^{-g(x)}\Big(\int_0^x \frac{-f}{a}e^{g(t)}dt + u'(0)\Big), \qquad g(x) = \int_0^x \frac{-b}{a}dt.$$

2. Approximation of $u'(0)$ when b(x) is a function and c(x)=0.

Let us consider the equation

$$-au'' + b(x)u' = f(x), \quad u(0) = u(1) = 0. \tag{5.7}$$

If B is a fixed number, then

$$-au'' + b(x)u' = f(x) \Leftrightarrow -au'' + Bu' = (B-b)u' + f. \tag{5.8}$$

Substituting (5.7) we end up with the following three equations:

$$\begin{cases} 1. -a\hat{u}_1'' + B\hat{u}_1' = f \\ 2. -a\hat{u}_2'' + B\hat{u}_2' = (B-b)(e^{-g(x)}\int_0^x \frac{-f}{a}e^{g(x)}dt) \\ \begin{cases} 3. -a\hat{u}_3'' + B\hat{u}_3' = (B-b)u'(0)e^{-g(x)} \\ \hat{3}. -a\hat{u}_4'' + B\hat{u}_4' = (B-b)e^{-g(x)}, \end{cases} \end{cases} \tag{5.9}$$

$\hat{u}_j(0) = \hat{u}_j(1) = 0,\ j = 1,2,3,4\ \textit{and}\ \hat{u}_3 = u'(0)\hat{u}_4.$

It is clear that$u' = \hat{u}'_1 + \hat{u}'_2 + \hat{u}'_3$. Using each of the above three equations in (5.9) we may get $\hat{u}_j(0)$ using also (5.6). Then

$$[u'(x)]_{x=0} = [\frac{\hat{u}'_1(x) + \hat{u}'_2(x)}{1 - \hat{u}'_4(x)}]_{x=0} \quad \text{if} \quad \hat{u}'_4(0) \neq 1.$$

3. Approximation of $u'(0)$ if b(x) and c(x) are functions

In this part we are applying the same iteration procedure as introdued in chapter 3, let us consider general form

$$-au'' + b(x)u' + c(x) = f(x),\ u(0) = u(1) = 0.$$

As initial guess we choose $u = 0$, at first we solve

$$-au'' + b(x)u' = f(x),\ u(0) = u(1) = 0,$$

by the method of section 2 for $(c = 0)$. Suppose $u_1(x)$ is a solution, then we consider

$$-au'' + b(x)u' = f(x) - c(x)u_1(x).$$

Now, assume $u_2(x)$ is a solution and continue in this way to get a sequence $\{u'_i(x)\}_{x=0},\ i = 1, 2, ...$ End the procedure when

$$|u'_i(x) - u'_{i-1}(x)|_{x=0} \leq \varepsilon.$$

Example 15. Consider the problem

$$-6u'' + \frac{e^x}{1+x}u' + 2^x u = f, \qquad u(0) = u(1) = 0.$$

The exact solution is $u = \frac{\sin(\pi x)}{20-x}$. We will compare two solutions that obtained by using the above technique and Matlab codes(bvp4c).see fig(19)

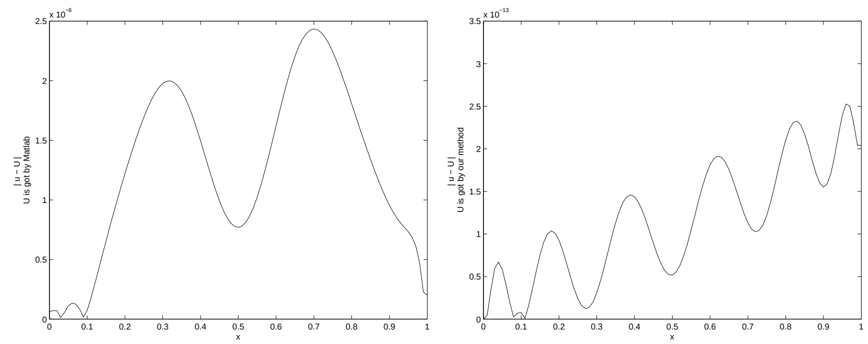

FIGURE 19. At the left Error when we use Matlabs code(bvp4c) and in the right error of our solution for problem(6.1)

Note: in almost all methods, as the basis for solving ODE and BVP problem we have to guess the initial value or functionality to start the method and the result is dependent on how to choose initial guess, but in the above method the solution is not sensitive to the choice of the initial guess.

Error analysis

In this section we analyse the error when the coefficients are fixed and $c = 0$. Since the structure of this method is based on solving problem (5.9), let B=2a (B in (5.9) is an arbitary and constant.) From (5.6) we have

$$[u'(x)]_{x=0} = \frac{\sum_{i=1}^{+\infty}((\frac{1}{2})^i(e^2 - S_i)[f^{i-1}(x)]_{x=0})}{a(e^2-1)} \tag{5.10}$$

where $S_i = \sum_{k=1}^{i} \frac{2^k}{k!}$. Applying Taylor expansion for e^x, then $\exists \lambda_i, 0 < \lambda_i < 2$,
such that

$$e^2 - S_i = 2^{i+1}\frac{e^{\lambda_i}}{(i+1)!}. \tag{5.11}$$

Inserting (5.11) in (5.10) we get

$$[u'(x)]_{x=0} = \Sigma_{i=1}^{+\infty}\frac{2e^{\lambda_i}}{a(e^2-1)}\frac{[G^{i+1}(x)]_{x=0}}{(i+1)!}. \tag{5.12}$$

In (5.12) , $G(x)$ is defined by $G''(x) = f(x)$

Theorem 5.1. *Assume that $G(x)$ has continuous derivatives of order (i+1) in an interval (0 1). Then the series* (5.12) *convergences to* $[u'(x)]_{x=0}$.

Proof. From (5.12),

$$|[u'(x)]_{x=0} - \Sigma_{i=1}^{+\infty}\frac{2e^{\lambda_i}}{a(e^2-1)}\frac{[G^{i+1}(x)]_{x=0}}{(i+1)!}| = |\Sigma_{i=k+1}^{+\infty}\frac{2e^{\lambda_i}}{a(e^2-1)}\frac{[G^{i+1}(x)]_{x=0}}{(i+1)!}|$$

Since $0 < \lambda_i < 2$ we have

(5.13)

$$\begin{aligned} |\Sigma_{i=k+1}^{+\infty}\frac{2}{a(e^2-1)}\frac{[G^{i+1}(x)]_{x=0}}{(i+1)!}| &\leq |[u'(x)]_{x=0} - \Sigma_{i=1}^{k}\frac{2e^{\lambda_i}}{a(e^2-1)}\frac{[G^{i+1}(x)]_{x=0}}{(i+1)!}| \\ &\leq |\Sigma_{i=k+1}^{+\infty}\frac{2e^2}{a(e^2-1)}\frac{[G^{i+1}(x)]_{x=0}}{(i+1)!}| \end{aligned}$$

but,

$$|G(1) - \Sigma_{i=1}^{k}\frac{[G^{i+1}(x)]_{x=0}}{(i+1)!}| = |\Sigma_{i=k+1}^{+\infty}\frac{[G^{i+1}(x)]_{x=0}}{(i+1)!}|, \tag{5.14}$$

when k $\to +\infty$ (Taylor remainder for function G). The right hand side in (5.14) tends to 0, so both right and left side of (5.13) tend to 0 and this completes the proof. □

we get an approximation of $[u'(x)]_{x=0}$ with a convergence of order $\mathcal{O}(\frac{1}{(k+1)!})$

$$[u'(x)]_{x=0} = [\hat{u}'(x)]_{x=0} + \mathcal{O}\Big(\frac{1}{(k+1)!}\Big).$$

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