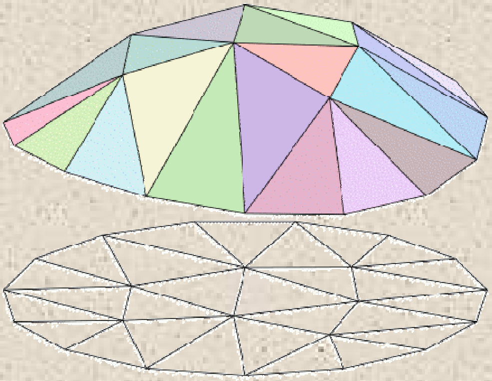
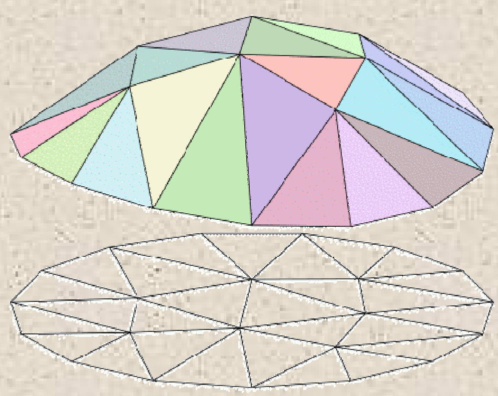


Streamline Diffusion Method for Coupling of two Hyperbolic Conservation Laws



M. Izadi

August 20, 2005

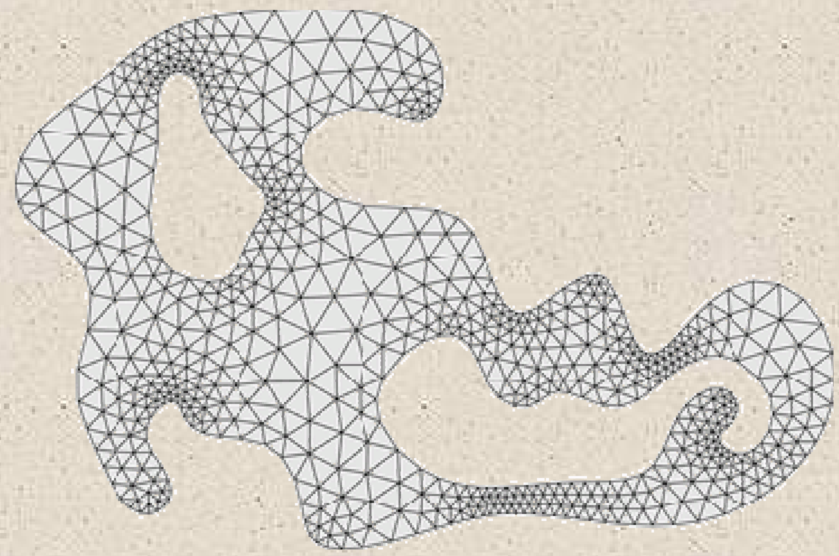
Institute for Advance studies in Basic Sciences

The Plane

- Review of Finite Element Method (FEM)
- The Coupled Problem
- Streamline-Diffusion Formulation
- A priori Error Estimates for Sd-Method
- Numerical Examples

✓ Why the Finite Element Method?

- Finite element method provides a greater flexibility to model complex geometries than finite difference and finite volume methods do.



- The construction of higher order approximation

Basic Principles of FEM

- Finding a variational formulation of the problem:
- Integrating by parts in order to decease the number of differentiations involved, thereby decreasing the smoothness demands on u.
- Retaining only the essential (Dirichlet) boundary conditions.
- Approximating the solution by a finite number of degrees of freedom, i.e. within a finite dimensional space V.
- Choosing basis functions, e.g. in V, that are locally supported (vanish on most of the domain).

One Dimensional Example

We consider

$$-u''(x) = f(x), \quad 0 < x < 1$$

$$u(0) = u(1) = 0$$

THE STRONG OR DIFFERENTIAL PROBLEM (D)

- $u' = \frac{du}{dx}$ and $f \in L_2(0,1)$,
- By integrating twice, we can see that this problem has a unique solution

The Sobolev Space

- Define

$$\mathcal{V} = H_0^1(0,1) = \{v \in H^1(0,1) : v(0) = v(1) = 0]$$

where

- We define the space $H^1(I)$ as follows:

$$H^1(I) = \{v : v \text{ and } v' \in L_2(I) \}$$

- We associate $H^1(I)$ with the scalar product:

$$(v,w)_{H^1(I)} = \int_I [vw + v'w']dx$$

- The corresponding norm is:

$$\|v\|_{H^1(I)} = \left(\int_I [v^2 + (v')^2]dx\right)^{1/2}$$

Variational Formulation

- Multiplying both sides of (D) by any function $v \in \mathcal{V}$ yields

$$\int_0^1 f v\,dx = -\int_0^1 u'' v\,dx$$

Integrating by part + B.C.

- Find u such that

$$\int_0^1 f v\,dx = \int_0^1 u' v'\,dx, \qquad \forall\, v \in \mathcal{V}. \qquad \text{(VF)}$$

Variational Formulation

- Note that (D) is equivalent to (VF). (*)

- With the notations

$$a(u,v) = \int_0^1 u'v'dx,$$

and

$$L(v) = \int_0^1 fv dx,$$

(*) can be written as:

Find u such that a(u,v)=L(v) for all admissible v

Uniqueness & Existence Theorem

Thm.(Lax-Milgram) Let a(.,.) be a bilinear form on a Hilbert space $\mathcal{H}$ equipped with $\|.\|_{\mathcal{H}}$ and the following properties:

- *a(.,.) is continuous, that is*

$$\exists \gamma_1 > 0 \quad \textit{such that} \quad |a(w,v)| \le \gamma_1 \|w\|_{\mathcal{H}} \|v\|_{\mathcal{H}} \quad \forall w, v \in \mathcal{H},$$

- *a(.,.) is coercive* (or $\mathcal{H}$-elliptic), *that is*

$$\exists \alpha > 0 \quad \textit{such that} \quad a(v,v) \ge \alpha \|v\|^2_{\mathcal{H}}, \quad \forall v \in \mathcal{H}.$$

Further

- *L(.) is a linear mapping on* $\mathcal{H}$ *,that is*

$$\exists \gamma_2 > 0 \quad \textit{such that} \quad |L(w)| \le \gamma_2 \|w\|_{\mathcal{H}}, \quad \forall w \in \mathcal{H}.$$

Then there exist a unique $u \in \mathcal{H}$ *such that*

$$\boxed{a(w,u) = L(w), \quad \forall w \in \mathcal{H}.}$$

Example

- $a(.,.)$ is obviously symmetric and bilinear and L is linear.

- The continuity of L is shown using the Cauchy inequality in L_2:

$$|L(v)| \leq |\int_\Omega f v dx| \leq \|f\|_{L_2(\Omega)} \|v\|_{L_2(\Omega)} \leq \|f\|_{L_2(\Omega)} \|v\|_{H^1(\Omega)}$$

- The continuity of $a(.,.)$ is shown as follows:

$$|a(v,w)| \leq \|v'\|_{L_2(I)} \|w'\|_{L_2(I)} \leq \|v\|_{H_0^1(I)} \|w\|_{H_0^1(I)}$$

- The $V-$elliptic condition for $a(.,.)$ can be shown using the fact that

$$\int_I v^2 dx \leq \int_I (v')^2 dx \qquad \forall\, v \in H_0^1(I)$$

- $a(v,v) = \int_I (v')^2 dx \geq \frac{1}{2} \left(\int_I v^2 dx + \int_I (v')^2 dx \right) = \frac{1}{2} \|v\|_{H_0^1(I)},\ \forall\, v \in H_0^1(I).$

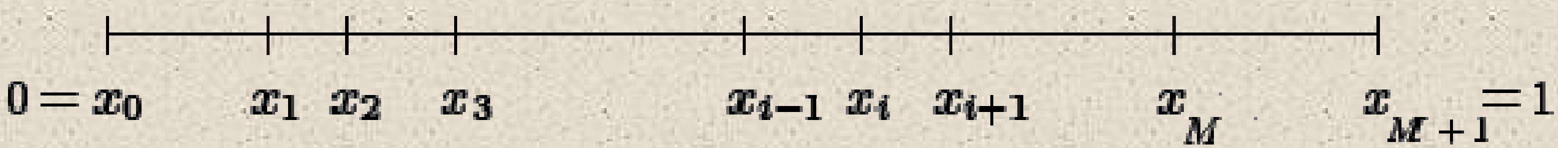
Interval Partition (FEM)

➢ Construct a finite-dimensional subspace $V_h \subset V$ as follows:

- For a given interval $I = [0, 1]$ let

$$\mathcal{T}_h : 0 = x_0 < x_1 < x_2 < \ldots < x_{M+1} = 1,$$

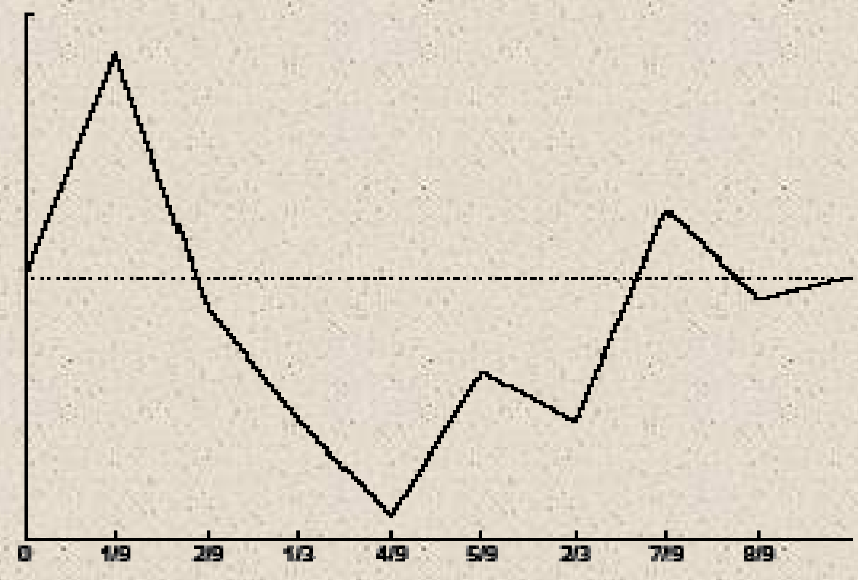
be a partition of I into intervals $I_i = (x_{i-1}, x_i)$ of length $h_i = x_i - x_{i-1}$.



- The quantity $h = \max_j h_j$ is a measure of how fine the partition is.

Finite Element Space

- Let $\mathcal{V}_h$ be a set of functions v such that:
  - v is linear on each subinterval I_j
  - v is continuous on $[0,1]$ and
  - $v(0) = v(1) = 0.$



Continuous piecewise linear basis function

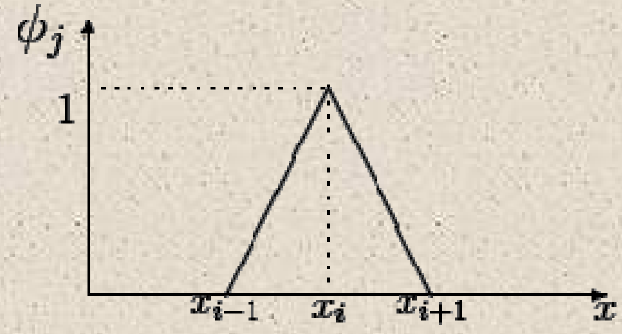
- A function $v \in \mathcal{V}_h$ has the representation:

  $v(x) = \Sigma_{i=1}^{M} \eta_i \, \phi_i(x)$, $x \in [0,1]$, where:

- $\eta_i = v(x_i)$ and

- $\phi_j(x_i) = \delta_{ij} \equiv \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \qquad i, j = 1, \ldots, M.$

- The space $\mathcal{V}_h$ is a linear space of dimension M with basis $\{\phi_i\}_{i=1}^{M}$.



Finite Element Approximation

- The problem (VF) is reduced to

$$\begin{cases} \text{Find} \quad u_h \in \mathcal{V}_h, \quad \text{such that} \\ a(u_h, v) = L(v), \qquad \forall v \in \mathcal{V}_h, \end{cases} \qquad (V_h)$$

- where
  - $u_h(x) = \Sigma_{j=1}^{M} \xi_j \phi_j(x)$, with
  - $\xi_j = u_h(x_j)$ (nodal values of $u_h(x)$).

Linear system of Equations

(V_h): Find $u_h \in \mathcal{V}_h$ such that $(u_h', \phi_i') = (f, \phi_i)$, $i = 1, 2, \ldots, M$.

We finally obtain the following linear system of equations:

$$\Sigma_{j=1}^{M} \xi_j \, (\phi_j', \phi_i') = (f, \phi_i), \qquad i = 1, 2, \ldots, M$$

- This is equivalent to the system A =b, where
  - $A = (a_{ij})$ is the $M \times M$ <u>stiffness matrix</u> with $a_{ij} = (\phi_j', \phi_i') = \int_0^1 \phi_i'(x)\phi_j'(x)dx$
  - $b = (b_i)$ is the <u>force vector</u> with: $b_i = (f, \phi_i) = \int_0^1 f\phi_i(x)dx$ and
  - $\xi = (\xi_i)$ is the <u>solution vector</u> with: $\xi_i = u_h(x_i)$, $i = 1, 2, \ldots, M$.

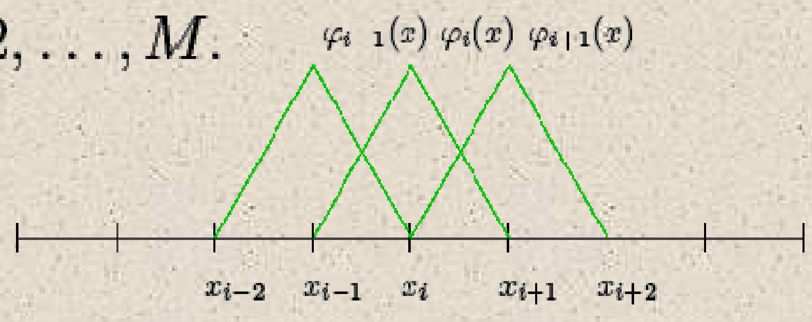
Properties of the Stiffness matrix A

- A is symmetric, $A_{ij} = A_{ji}$, $(\phi'_j, \phi'_i) = (\phi'_i, \phi'_j)$, $i, j = 1, 2, \ldots, M$.
- A is sparse (i.e. only a few elements of A are nonzero)

$$(\phi'_j, \phi'_j) = \frac{1}{h_j} + \frac{1}{h_{j+1}}, \quad j = 1, 2, \ldots, M.$$

$$(\phi'_j, \phi'_{j-1}) = (\phi'_{j-1}, \phi'_j) = -\frac{1}{h_j}, \; j = 2, \ldots, M.$$

$$(\phi'_i, \phi'_j) = 0 \quad \text{if} \quad |i - j| > 1.$$



- A is positive definite. Indeed for $\forall\ \eta \in \Re^M$ we obtain:

$$\begin{aligned} \eta^t A \eta &= \Sigma_{i=1}^M \Sigma_{j=1}^M \eta_i A_{ij} \eta_j = \Sigma_{i=1}^M \Sigma_{j=1}^M \eta_i (\phi'_i, \phi'_j) \eta_j \\ &= (\Sigma_{i=1}^M \eta_i \phi'_i, \Sigma_{j=1}^M \eta_j \phi'_j) = (v', v') \geq 0, \quad v = \Sigma_{i=1}^M \eta_i \phi_i(x). \end{aligned}$$

Also, $\quad \eta^t A \eta = 0$ only if $\eta_j = 0$, $j = 1, \ldots, M$.

Properties of the Stiffness matrix A

- Since A is a positive definite matrix, we conclude that A is non-singular.
- It follows that the system $A\xi = b$ has a unique solution.
- For the particular case of $h_j = h = \frac{1}{M+1}$, the system $A\xi = b$ becomes:

$$\frac{1}{h}\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & \cdot & \cdot & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & \cdot & \cdot \\ 0 & -1 & 2 & -1 & 0 & 0 & \cdot & \cdot \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & 0 & -1 & 2 & -1 \\ 0 & \cdot & \cdot & 0 & 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \\ \cdot \\ \cdot \\ \cdot \\ \\ \\ \xi_M \end{bmatrix} = \begin{bmatrix} b_1 \\ \\ \cdot \\ \cdot \\ \cdot \\ \\ \\ b_M \end{bmatrix}$$

✓ The Coupled Problem

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f_R(u) = 0, \quad x > 0, \quad t > 0,$$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f_L(u) = 0, \quad x < 0, \quad t > 0,$$

- Initial Condition

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R},$$

- Coupling Condition

$$\boxed{u(0, t) = u^b(t), \quad t \geq 0,}$$

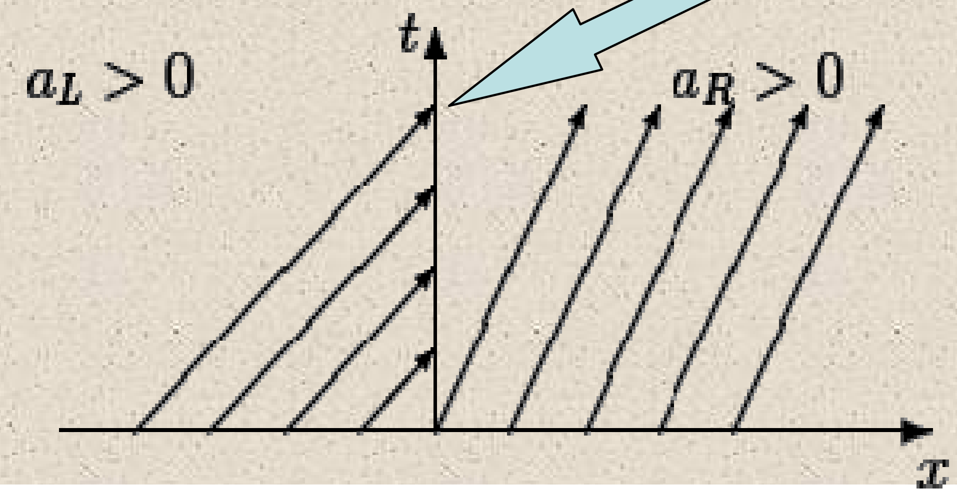
Coupled problem

- One dimensional example

$$f_L = a_L u, \quad f_R = a_R u,$$

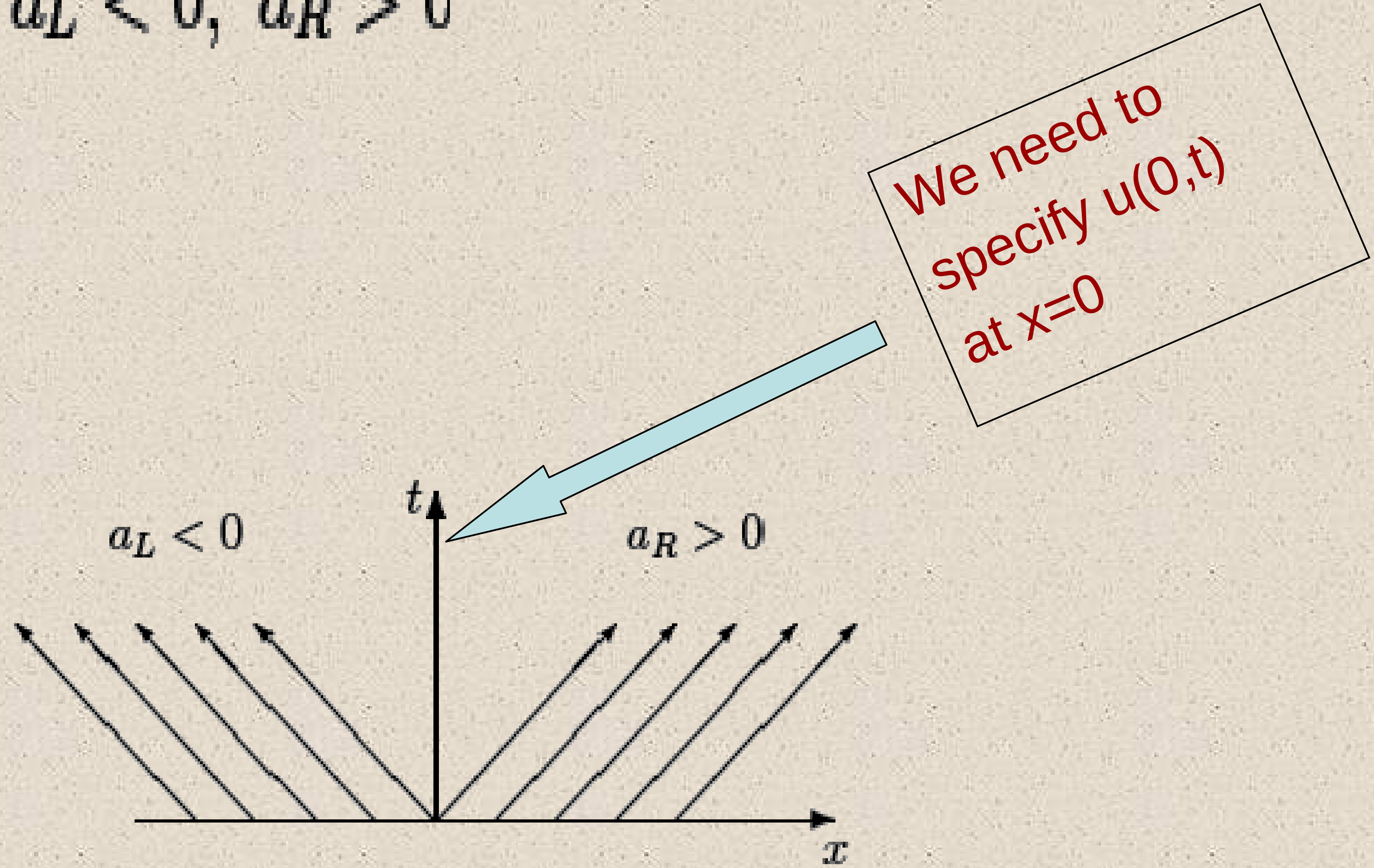
- $a_L > 0, a_R > 0$ or $a_L < 0, a_R < 0$

We can impose coupling condition at x=0



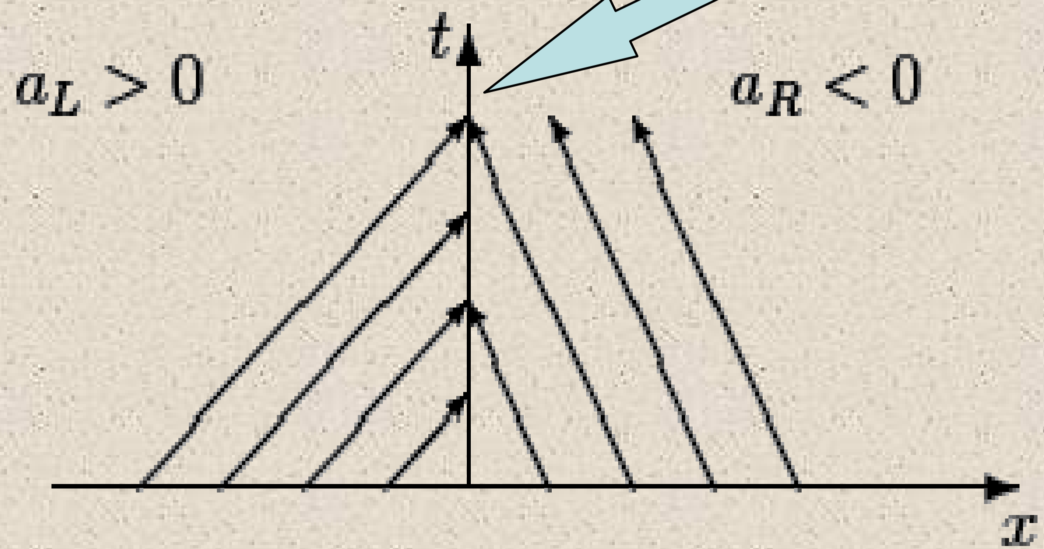
Coupled problem

- $a_L < 0, \ a_R > 0$



Coupled problem

- $a_L > 0, \; a_R < 0$



No coupling condition need at x=0

In general

- $f'_R(u) \leq 0$ & $\frac{\partial}{\partial x}(f'_R(u)) \leq 0$
- $f'_L(u) \geq 0$ & $\frac{\partial}{\partial x}(f'_L(u)) \geq 0$

✓ Sd-Formulation

- We consider

$$\begin{cases} \dfrac{\partial u}{\partial t} + f'_R(u)\dfrac{\partial u}{\partial x} = 0, & (x,t) \in \Omega := \mathbb{R}_+ \times (0,T), \\ u(x,0) = u_0, & (x,t) \in \Omega_0 := \mathbb{R}_+ \times \{0\}, \\ u(0,t) = u^b, & (x,t) \in \Gamma := \{0\} \times (0,T), \end{cases} \quad (1)$$

with T is a given final time value and

$$f'_R(u) \leq 0 \quad \& \quad \frac{\partial}{\partial x}(f'_R(u)) \leq 0$$

Space-time discretization

Let $\{0 = t_0 < t_1 < ... < t_N = T\}$ be a partition of $I = (0, T)$

into $I_n = (t_n, t_{n+1})$, with time steps $k_n = t_{n+1} - t_n$,

and introduce the corresponding space-time "slabs", i.e.,

$$\mathbf{S}_n = \{(x,t) : x > 0, \quad t_n < t < t_{n+1}\}, \qquad n = 0, 1, \ldots, N-1.$$

Figure: *Space-time* discretization.

Space-time discretization

For each slab S_n, let x_i^n be a mesh on $\mathbb{R}_+$,

portioned in intervals $J_i^n = (x_{i-1}^n, x_i^n)$, with $h_i^n = x_i^n - x_{i-1}^n$.

For $h > 0$, let T_h^n be a triangulation of the slab $\mathbf{S}_n$ into triangles K,

satisfying quasi-uniformity conditions for finite element meshes

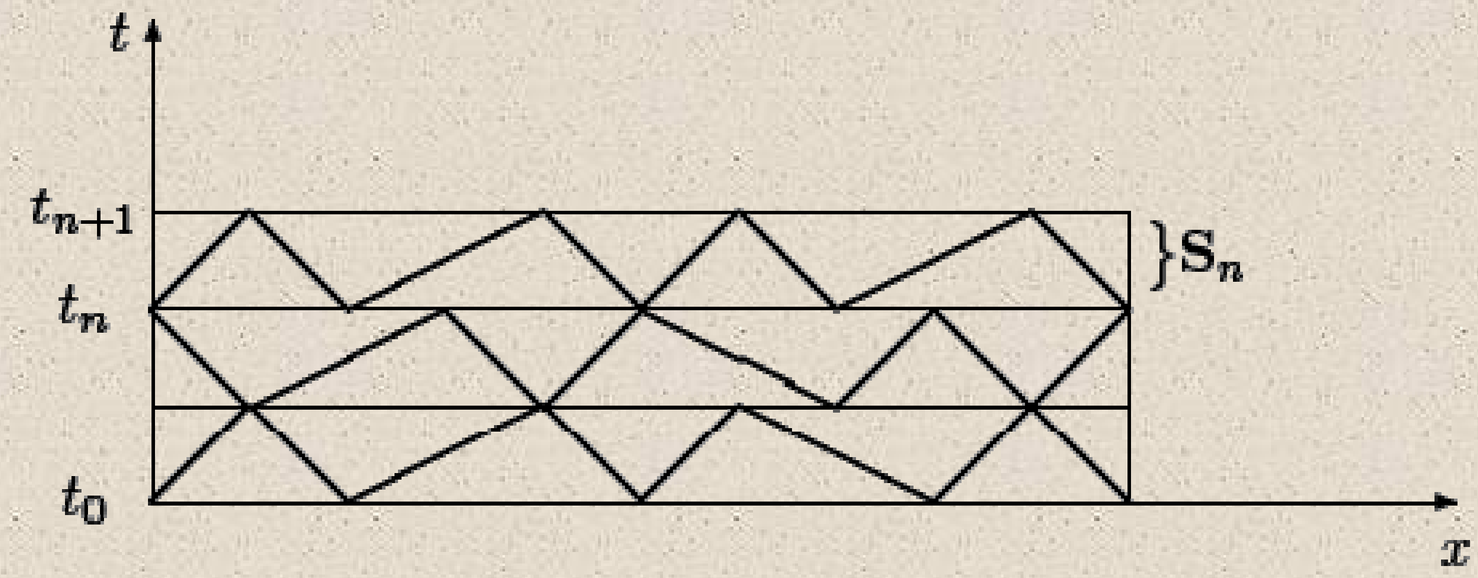


Figure 3.1: The space-time triangulation.

Finite element spaces

- Let *k* be a positive integer, introduce

$$U_h^n = \{u \in H^1(\mathbb{S}_n) : u\big|_K \in P_k(K), \quad K \in T_h^n\},$$

- Define the trial & test function spaces

$$V_h^n = \{v \in U_h^n : v\big|_\Gamma = u_h^b\},$$

$$W_h^n = \{w \in U_h^n : w\big|_\Gamma = 0\},$$

Some notations

$$(u,v)_n = \int_{\mathbf{S}_n} uv\,dx\,dt, \qquad ||v||_n = (v,v)_n^{1/2},$$

$$< u, v >_n = \int_{\mathbb{R}_+} u(x,t_n)v(x,t_n)dx, \qquad |v|_n = < v, v >_n^{1/2},$$

$$v_+ = \lim_{s \longrightarrow 0+} v(x, t+s), \qquad v_- = \lim_{s \longrightarrow 0-} v(x, t+s).$$

$$||\cdot|| = \|\cdot\|_{L_2(Q)} \qquad \|\cdot\|_{\infty,Q} = \|\cdot\|_{L_\infty(Q)}$$

$$||\cdot||_{s,Q} = ||\cdot||_{H^s(Q)}$$

Space-time Sd Formulation

- Find $u \in H^1(\Omega)$ with $u|_\Gamma = u^b$, such that

$$\left(u_t + f'_R(u)u_x,\ v + \delta\left(v_t + f'_R(u)v_x\right)\right)_\Omega + \int_\Gamma uv\, d\sigma dt = \int_\Gamma u^b v\, d\sigma dt, \qquad \forall v \in H_0^1(\Omega), \tag{2}$$

- For $n = 0, 1, \ldots, N-1$ find $u \in H^1(\mathbf{S}_n)$, such that

$$\left(u_t + f'_R(u)u_x, v + \delta\left(v_t + f'_R(u)v_x\right)\right)_n + < u_+, v_+ >_n + \int_\Gamma uv\, d\sigma dt = < u_-, v_+ >_n + \int_\Gamma u^b v\, d\sigma dt, \qquad \forall v \in H_0^1(\mathbf{S}_n). \tag{3}$$

Continue…

- After summing over n, we rewrite (3) as follows

- Find $u \in \prod_{n=0}^{N-1} H^1(\mathbf{S}_n)$, such that

$$B(u,v) = L(v), \qquad \forall v \in \prod_{n=0}^{N-1} H_0^1(\mathbf{S}_n), \tag{4}$$

where

$$B(u,v) = \sum_{n=0}^{N-1} \left\{ \Big(u_t + f_R'(u)u_x, v + \delta(v_t + f_R'(u)v_x) \Big)_n + < u_+ - u_-, v_+ >_n + \int_{\Gamma_n} u_+ v_+ dt \right\},$$

$$L(v) = < u_0, v_+ >_0 + \int_{\Gamma} u^b v_+ dt.$$

Continue ...

and finally

- Find $u_h^n \in V_h^n$, such that for $n = 0, 1, \ldots, N-1$

$$\left(u_{h,t}^n + f_R'(u_h^n)u_{h,x}^n, v_h^n + \delta(v_{h,t}^n + f_R'(u_h^n)v_{h,x}^n)\right)_n + < u_{h,+}^n, v_{h,+}^n >_n + \int_{\Gamma_n} u_{h,+}^n v_{h,+}^n dt =< u_{h,-}^n, v_{h,+}^n >_n + \int_{\Gamma_n} u^b v_{h,+}^n dt, \qquad \forall v_h^n \in W_h^n, \tag{5}$$

where

$$\delta = \bar{C}h \qquad \Gamma_n = \{0\} \times I_n$$

and $u_{h,-}^0 = u_0$ is the initial data

Continue …

- After summing over n, we have

$$\mathcal{V}_h = \prod_{n=0}^{N-1} V_h^n, \qquad \mathcal{W}_h = \prod_{n=0}^{N-1} \mathcal{W}_h^n,$$

We shall seek an approximate solution $u_h \in \mathcal{V}_h$ such that for $n = 0, 1, ..., N$ we will have that $u_h|_{S_n} = u_h^n$.

Functions in $\mathcal{V}_h$ are continuous in x & discontinuous in t

- Define

$$[v](x, t_n) = \begin{cases} v_+, & \text{if } n = 0 \\ v_+ - v_-, & \text{if } n \neq 0. \end{cases}$$

Continue …

Summing (5) over n=0,1,…,N-1, we get the following analogue to (4)

- Find $u_h \in \mathcal{V}_h$ such that

$$B(u_h, v) = L(v), \qquad \forall v \in \mathcal{W}_h. \qquad (6)$$

Basic Stability Estimates for the Sd-method

- Thm. *For* $u \in \prod_{n=0}^{N-1} H^1(S_n)$, *and with the assumptions* $f_R'(u^b) \leq 0$, *and*

  $\frac{\partial}{\partial x}\left(f_R'(u)\right) \leq 0$, *we have that*

$$B(u,u) \geq |||u|||^2,$$

where

$$|||u|||^2 := \frac{1}{2}\left[|u_-|_N^2 + |u_+|_0^2 + \sum_{n=1}^{N-1} |[u]|_n^2 + 2\delta\|u_t + f_R'(u)u_x\|_\Omega^2\right] + \|u_+\|_\Gamma^2.$$

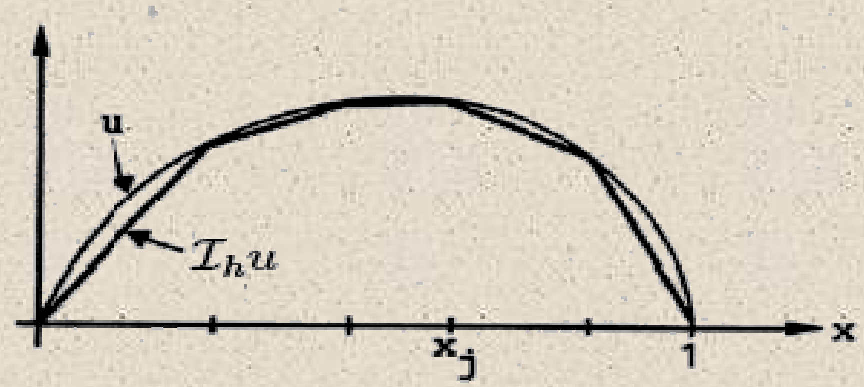
A priori Error Estimate for the Sd-method

To do this introduce interpolant $\mathcal{I}_h u \in \mathcal{V}_h$ of exact solution u and set

$$\eta = u - \mathcal{I}_h u \qquad \xi = u_h - \mathcal{I}_h u.$$

Then we have

$$e := u - u_h = (u - \mathcal{I}_h u) - (u_h - \mathcal{I}_h u) = \eta - \xi.$$



Continue ...

Theorem *If* $u_h \in V_h$ *satisfies* (6) *and the exact solution* u *satisfies* (1), *and further*

$$\|f_R'\|_{\infty,\Omega} \leq C,$$

then there is a constant C *such that*

$$|||u - u_h||| \leq C h^{k+\frac{1}{2}} \|u\|_{k+1,\Omega}.$$

Numerical Example

$$f_\alpha(u) = a_\alpha u \ (\alpha = L, R)$$

$$\begin{cases} u_t + a_R u_x &= 0, & x > 0, & t > 0, \\ u_t + a_L u_x &= 0, & x < 0, & t > 0, \\ u(x,0) &= u_0(x), & x \in [-a, a], & \\ u(-a,t) &= g(t), & & t > 0, \\ u(a,t) &= h(t), & & t > 0, \end{cases}$$

where a>0. This problem has the explicit solution

$$u(x,t) = \begin{cases} u_0(x - a_R t), & x \in (0, a] \\ u_0(x - a_L t), & x \in [-a, 0) \end{cases}$$

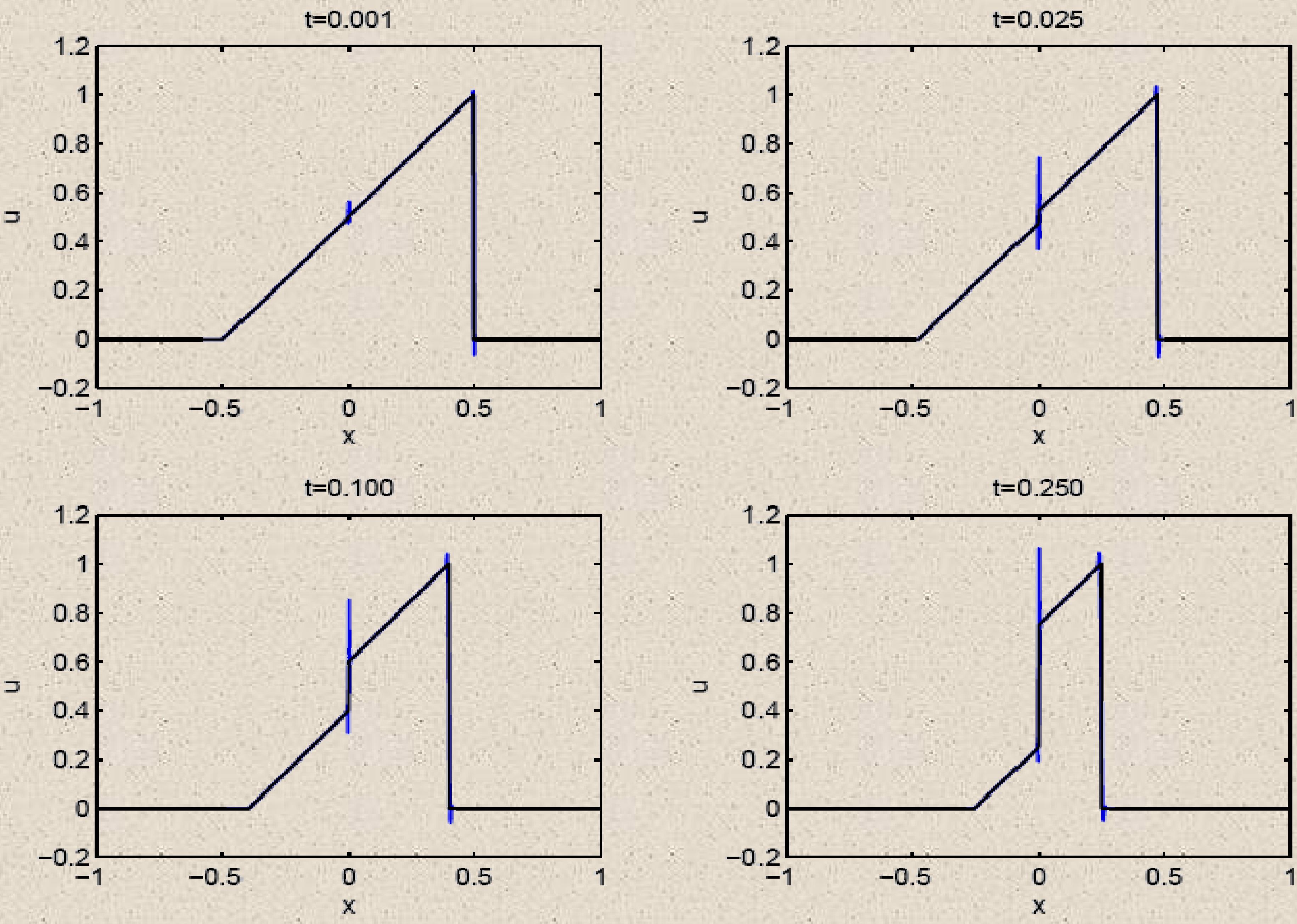
Test problem 1

$$\begin{cases} u_t + u_x &= 0, \qquad -1 < x \leq 0, \quad t > 0, \\ u_t - u_x &= 0, \qquad 0 \leq x < 1, \quad t > 0, \\ u(x,0) &= \begin{cases} 0.5 + x & \text{if} \quad -0.5 \leq x < 0.5, \\ 0 & \text{if} \quad o.w \end{cases} \end{cases}$$

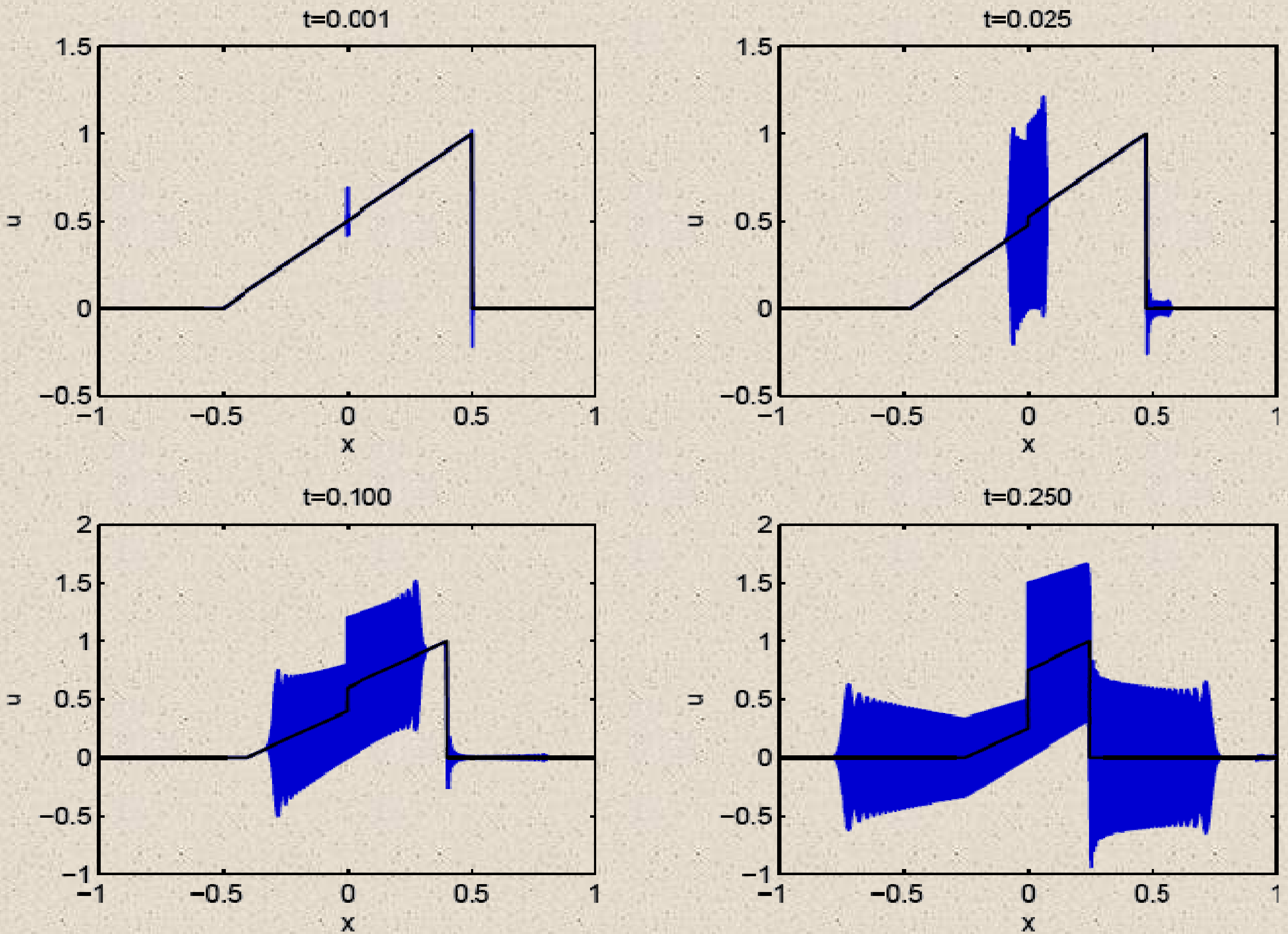
with the boundary conditions

$$u(-1,t) = u(1,t) = 0.$$

$$\delta = h$$



$\delta = 0$



Test problem 2

$$\begin{cases} u_t + 3u_x = 0, & -1 < x \leq 0, \quad t > 0, \\ u_t - 2u_x = 0, & 0 \leq x < 1, \quad t > 0, \end{cases}$$

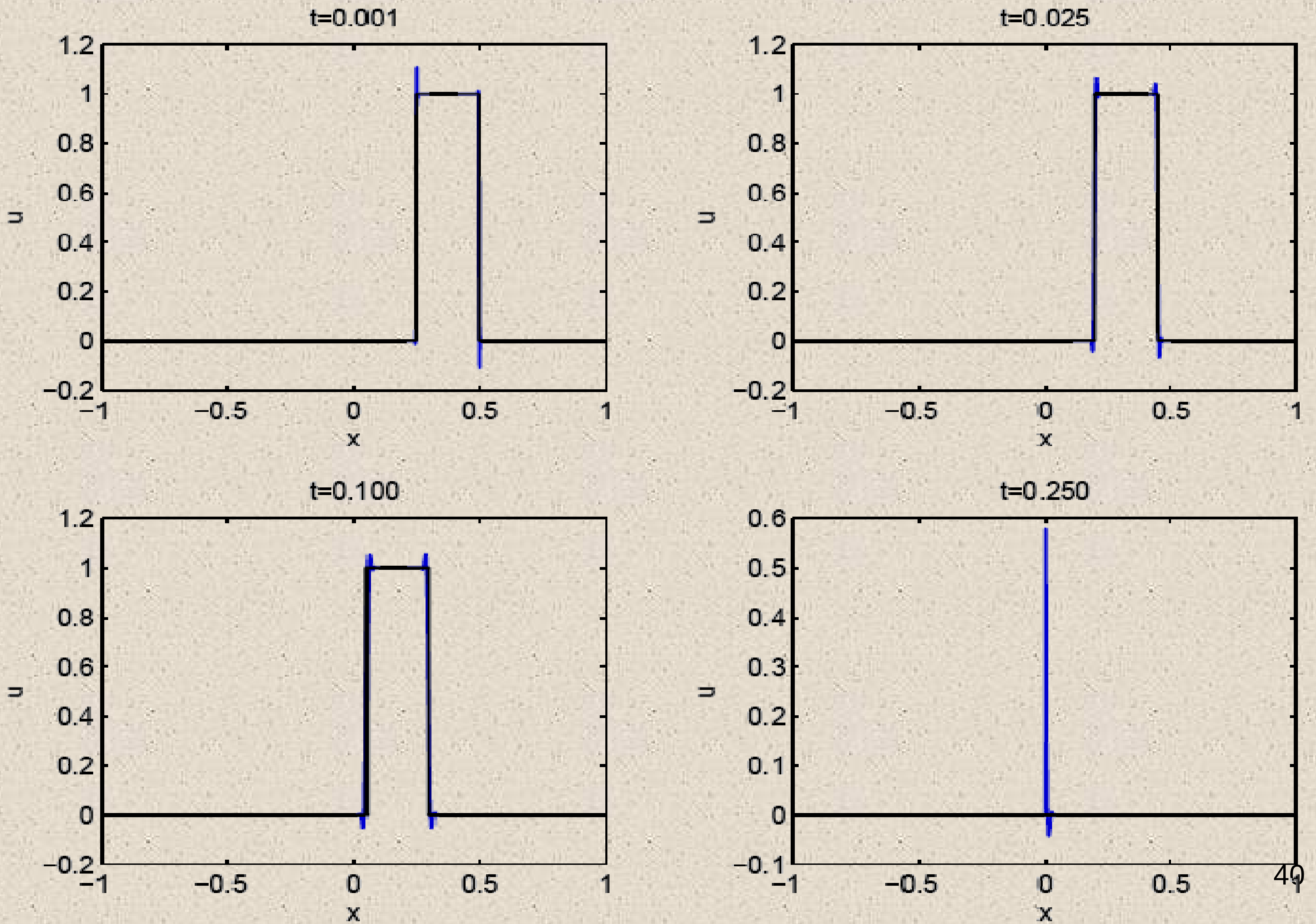
with the following initial condition

$$u_0(x) = \begin{cases} 0, & x \leq 0.25, \\ 1, & 0.25 < x \leq 0.5, \\ 0, & x > 0.5. \end{cases}$$

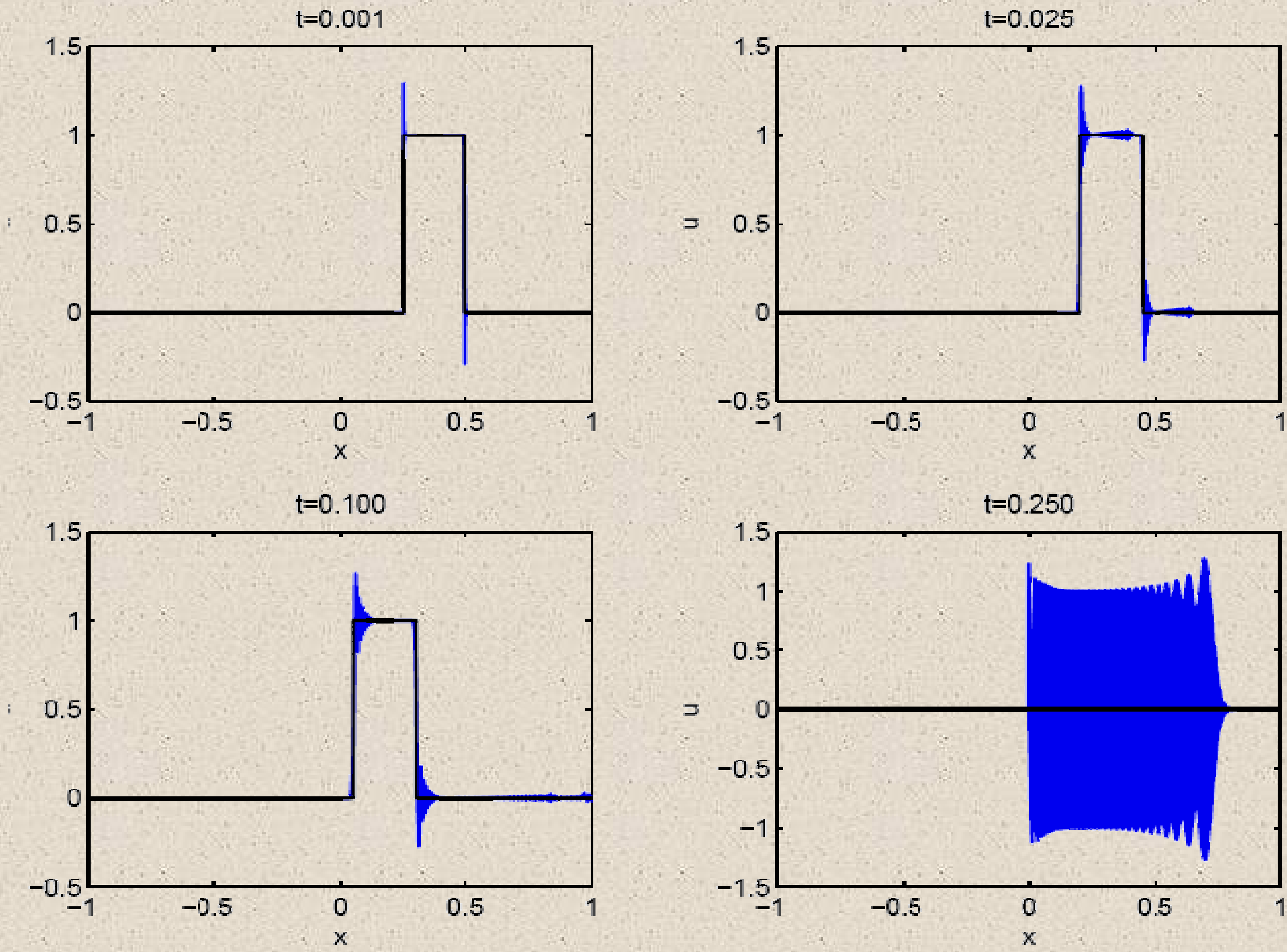
and boundary condition

$$u(-1,t) = u(1,t) = 0$$

$\delta = h$


t=0.001
t=0.025
t=0.100
t=0.250
u
x
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−1
−0.5
0
0.5
1
0.6
0.5
0.4
0.3
0.2
0.1
−0.1

$$\delta = 0$$



Thank You!