# Comparing the Streamline Diffusion Method and Bipartition Model for Electron Transport

Jiping Xin

Department of Mathematical Science,
Chalmers University of Technology

# Content

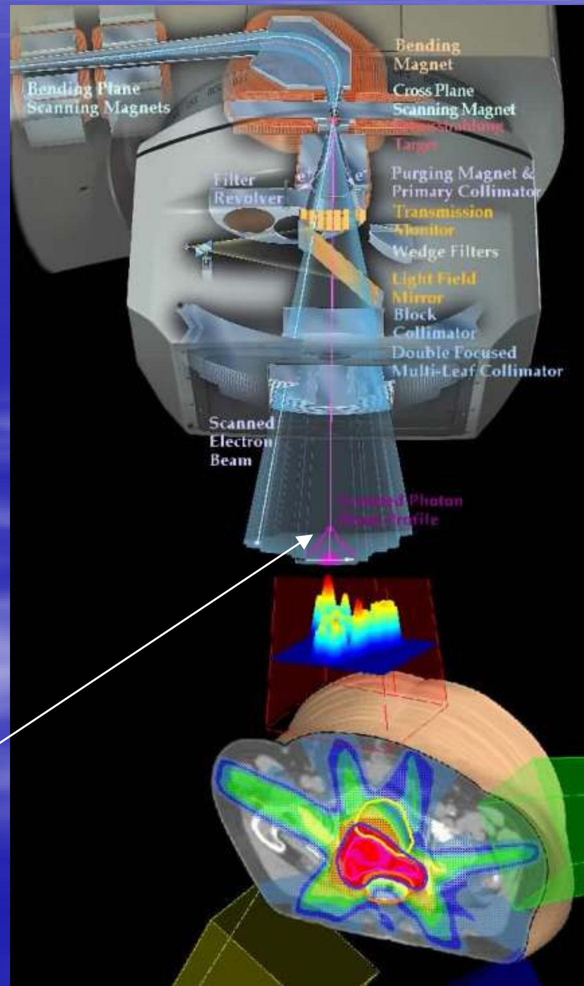
- A brief introduction of the background
- Comparing Sd-method and bipartition model for electron transport
- Future work

# Background

- Radiation therapy and TPS
- Two mathematical problems in TPS
- Transport theory and conservative law
- Equations
- Overview of different research ways

# Radiation therapy and TPS

**Radiation therapy** is the medical use of ionizing radiation as part of cancer treatment to control malignant cells.

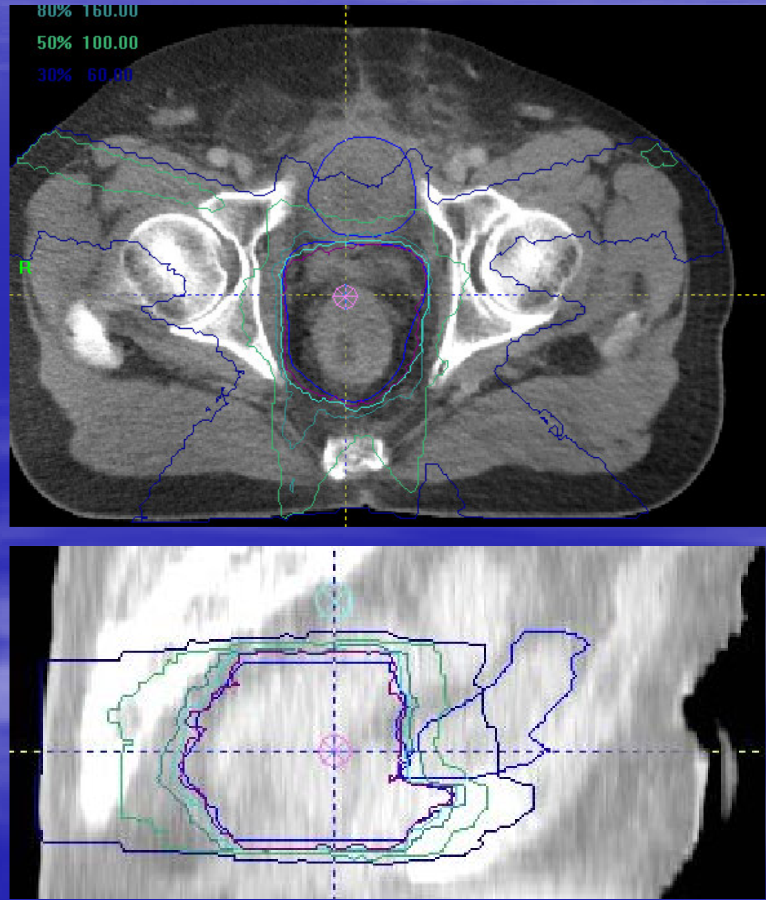



From the website of Medical Radiation Physics, KI

Pencil beam

# Radiation therapy and TPS

In radiotherapy, **Treatment Planning** is the process in which a team consisting of radiation oncologists, medical radiation physicists and dosimetrists plan the appropriate external beam radiotherapy treatment technique for a patient with cancer.




From Phoenix TPS

# Two mathematical problems in TPS

- **Radiation transport simulations:** This process involves selecting the appropriate beam type (electron or photon), energy (e.g. 6MV, 12MeV) and arrangements.

  Our problem! General, flexible, accurate, efficient algorithm!

- **Optimization:** The more formal optimization process is typically referred to as forward planning and inverse planning inference to intensity modulated radiation therapy (IMRT).

# Two mathematical problems in TPS

**GMMC, Dep. of Math. Sci., CTH:**

Research project: Cancer treatment through the IMRT technique: modeling and biological optimization

- Models and methods for light ion-beam transport
- Biological models and optimization for IMRT planning

# Questions

How to describe this kind of physical phenomena?

# Transport theory and conservative law

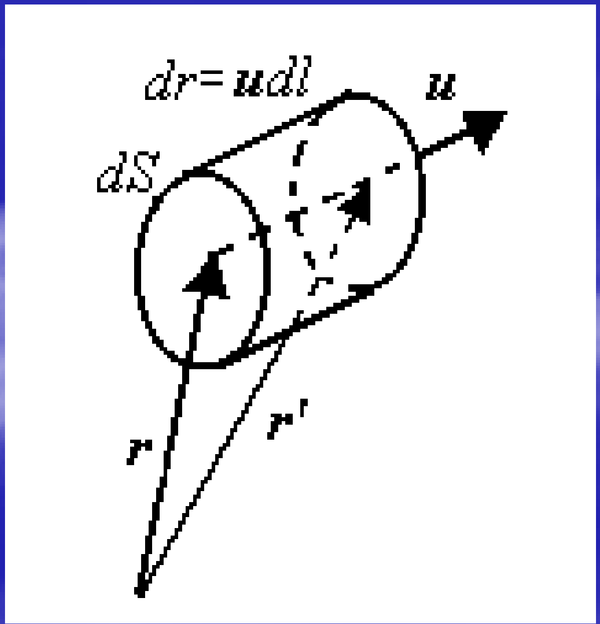
Transport theory:

1. Transport theory is based on nuclear physics, quantum physics, statistical physics, etc.
2. Gas dynamic, neutron transport (nuclear weapon after WWII), astrophysics, plasma, medical physics
3. Transport theory (discontinuous field) is more microscopic compared with fluid dynamic or heat transfer (continuous field) and more macroscopic compared with quantum mechanics or nuclear physics, it describes a group of particles. Very important!!!
4. We study charged particle transport theory!

# Transport theory and conservative law

The particle phase space density: Boltzmann

$$f(r,u,E) \qquad r=(x,y,z) \qquad u=(\theta,\phi)$$




- **Cross sections!**
- **Particle phase space density!**
- **6 variables!**

Photon: Compton scattering, pair production, photon-electron

Electron: elastic scattering, inelastic scattering, bremsstrahlung

# Equations (Models)

- Transport equation (Boltzmann equation)

$$u\cdot\nabla f=\int_0^\infty\int_{4\pi} f(r,u',E')\sigma_s(E',E,u'\cdot u)\,du'\,dE'-f(r,u,E)\int_0^\infty\int_{4\pi}\sigma_s(E,E',u'\cdot u)\,du'\,dE'+S$$

- Fokker-Planck equation (approximation)

$$u\cdot\nabla f=T(E)\left[\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial f}{\partial\mu}+\frac{1}{1-\mu^2}\frac{\partial^2 f}{\partial\phi^2}\right]+\frac{\partial fS}{\partial E}+\frac{\partial^2 fR}{\partial E^2}+S$$

- Fermi equation (approximation)

$$\frac{\partial f}{\partial z}+\theta_x\frac{\partial f}{\partial x}+\theta_y\frac{\partial f}{\partial y}=\frac{\partial fS}{\partial E}+\frac{1}{2}\frac{\partial^2 fR}{\partial E^2}+T(E)\left[\frac{\partial^2 f}{\partial\theta_x^2}+\frac{\partial^2 f}{\partial\theta_y^2}\right]+S$$

- Bipartition model

Our final goal is to solve the 6 dimensional pencil beam equation!!!

# Questions

- Which equation we should begin with? (M)
- The approximation (Fermi and Fokker-Planck) is accurate? In which cases they are accurate? (M)
- Which particle we should begin with? (photon - popular, electron - complex, ion - hot topic, proton) (M)
- How to solve the equation? Analytical method (FT), numerical method (FEM), stochastic method (MC). (S)
- Obviously it will be difficult to solve the 6 dimensional equations directly, then how to simplify the equations? (S) broad beam model and 2D pencil beam model
- How to go back to 6D model from simplified equations?

# Overview of different research ways

Sichuan University
Zhengming Luo's group

- Proton, Ion: Fermi, FT, 6
- Photon: TE, CL, 6
- Electron: BiM, FT, FDM, 3 Hybrid PBM, 6

Chalmers
Mohammad Asadzadeh

- Fermi and FP, 3, FEM

Karolinska Institute
Anders Brahme's group

- Ion: Fermi, FT, 6

University of Kuopio
Jouko Tervo's group

- TP 4 FEM
  TP 6 FEM (failed)

EGSnrc
Monte Carlo

# Overview of different research ways

- Electron transport (more complex than ion)
- 1-50 MeV (Fokker-Planck approximation is accurate)
- Fokker-Planck equation (PDE)
- Broad beam model (3 dimensions)
- Finite element method (Sd-method)

# Comparing Sd-method and bipartition model for electron transport

- Fokker-Planck equation for 1-50 MeV electron transport
- Broad beam and 2D pencil beam models
- Bipartition model
- Results by Sd-method

# Fokker-Planck equation for 1-50 MeV electron transport

## Transport equation

$$
\begin{aligned}
u\cdot\nabla f(r,u,E) = {} & \frac{N_A D}{A}\int_{4\pi}\left[f(r,u',E)-f(r,u,E)\right]\sigma_N(E,u'\cdot u)\,du' \\
& +\frac{N_A D}{A}\int_E^{E_0} f(r,u,E')\sigma_r(E',E'-E)\,dE' - \frac{N_A D}{A}\int_0^E f(r,u,E)\sigma_r(E,E-E')\,dE' \\
& +\frac{N_A D}{A}Z\int_E^{E_0}\int_{4\pi} f(r,u',E')\sigma_M(E',E'-E)\,\delta\left[u'\cdot u-\varphi(E',E'-E)\right]du'\,dE' \\
& -\frac{N_A D}{A}Z\int_{E/2}^{E}\int_{4\pi} f(r,u,E)\sigma_M(E,E-E')\,\delta\left[u'\cdot u-\varphi(E,E-E')\right]du'\,dE'
\end{aligned}
$$

# Fokker-Planck equation for 1-50 MeV electron transport

The screened Rutherford cross section:

$$\sigma_N(E,\xi) = \frac{r_0^2 Z(Z+1)(m_0c^2)^2(E+m_0c^2)^2}{E^2(E+2m_0c^2)^2}\left[\frac{1}{(1-\xi+2\eta)^2} + \frac{\pi\beta}{\sqrt{2}}\frac{Z}{137}\frac{1}{(1-\xi)^{3/2}} - \frac{1}{2}\left(\beta^2 + \frac{\pi\beta Z}{137}\right)\frac{1}{1-\xi}\right]$$

$$\eta = \frac{1}{4}\left[\frac{Z^{1/3}}{121.25}\right]^2\left[1.13 + 3.76\left(\frac{Z}{137}\right)^2\frac{(E+m_0c^2)^2}{E(E+2m_0c^2)}\right]\frac{(m_0c^2)^2}{E(E+2m_0c^2)}$$

$$\beta = \frac{\sqrt{E(E+2m_0c^2)}}{E+m_0c^2}$$

# Fokker-Planck equation for 1-50 MeV electron transport

## Bremsstrahlung cross section

$$\sigma_r(E,T) = \frac{4r_0^2 Z^2}{137T}\left[\left(1+\frac{(E-T)^2}{E^2}-\frac{2}{3}\frac{E-T}{E}\right)\left(\frac{\phi_1(\gamma)}{4}-\frac{1}{3}\ln Z\right)+\frac{E-T}{6E}\Delta(\gamma)\right]$$

$$\gamma = 100\frac{m_0c^2T}{E(E-T)Z^{1/3}} \qquad \phi_1(\gamma) = a_1e^{-a_2\gamma}+b_1e^{-b_2\gamma} \qquad \Delta(\gamma) = c_1e^{-c_2\gamma}+d_1e^{-d_2\gamma}$$

## Moller cross section

$$\sigma_M(E,T) = \frac{r_0^2 m_0c^2\left(E+m_0c^2\right)^2}{E\left(E+2m_0c^2\right)}\frac{1}{T^2}\left[1+\frac{T^2}{(E-T)^2}+\frac{E^2}{\left(E+m_0c^2\right)^2}\left(\frac{T}{E}\right)^2-\frac{\left(2E+m_0c^2\right)m_0c^2}{\left(E+m_0c^2\right)^2}\frac{T}{E-T}\right]$$

# Questions

How to derive Fokker-Planck equation from transport equation and check the accuracy?

# Fokker-Planck equation for 1-50 MeV electron transport

The classical contributions about the Fokker-Planck approximation are summarized by Chandrasekhar in [28] and Rosenbluth in [32]. [29,30,31] are studying the case of the linear particle transport. These works give a heuristic derivation of the Fokker-Planck operator. In [33] Pomraning gave a formalized derivation of the Fokker-Planck operator as an asymptotic limit of the integral scattering operator where a peaked scattering kernel is a necessary but not sufficient condition in the asymptotic treatment.

# Fokker-Planck equation for 1-50 MeV electron transport

## Laplace's method for Laplace integrals

$$I(x) = \int_a^b f(t) e^{x\phi(t)} dt$$

If $\phi(t)$ has a global maximum at $t = c$ with $a \leq c \leq b$ and if $f(c) \neq 0$, then it is only the neighbourhood of $t = c$ that contributes to the full asymptotic expansion of $I(x)$ as $x \to +\infty$

# Fokker-Planck equation for 1-50 MeV electron transport

Step 1. we may approximate $I(x)$ by $I(x;\varepsilon)$ where

$$I(x;\varepsilon)=\begin{cases}\displaystyle\int_{c-\varepsilon}^{c+\varepsilon} f(t)e^{x\phi(t)}dt, \text{ if } a<c<b,\\ \displaystyle\int_{a}^{a+\varepsilon} f(t)e^{x\phi(t)}dt, \text{ if } c=a,\\ \displaystyle\int_{b-\varepsilon}^{b} f(t)e^{x\phi(t)}dt, \text{ if } c=b.\end{cases}$$

This is a justified approximation since the changing limits of integration only introduces exponentially small errors.

# Fokker-Planck equation for 1-50 MeV electron transport

Step 2. Now $\varepsilon > 0$ can be chosen small enough so that it is valid to replace $\phi(t)$ by the first few terms in its Taylor or asymptotic series expansion.

Step 3. Having substituted the approximations for $\phi$ and with $f$ being defined above (cont. $f(c) \neq 0$), we now extend the endpoints of integration to infinity, in order to evaluate the resulting integrals (again this only introduces exponentially small errors).

# Fokker-Planck equation for 1-50 MeV electron transport

Surface harmonic expansion and Legendre polynomial expansion (important part no used in the heuristic derivation for the angular integration)

$$f(r,u,E)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left(\frac{2n+1}{4\pi}\right)a_{nm}f_{nm}(r,E)Y_{nm}(u)$$

$$\sigma_s(E',E,u'\cdot u)=\sum_{k=0}^{\infty}\left(\frac{2k+1}{4\pi}\right)\sigma_{sk}(E',E)P_k(u'\cdot u)$$

$$\int_0^{\infty}\int_{4\pi}\sigma_s(E',E,u'\cdot u)f(r,u',E')du'dE'=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left(\frac{2n+1}{2}\right)a_{nm}Y_{nm}(u)\cdot$$

$$\int_0^{\infty}f_{nm}(E')\int_{-1}^{1}P_n(\xi)\sigma_s(E',E,\xi)d\xi dE'$$

# Fokker-Planck equation for 1-50 MeV electron transport

Example: the screened Rutherford cross section for elastic scattering

$$\frac{N_A D}{A}\int_{4\pi}\left[f(r,u',E)-f(r,u,E)\right]\sigma_N(E,u'\cdot u)\,du'$$

$$\frac{N_A D}{A}\int_{4\pi} f(r,u',E)\sigma_N(E,u'\cdot u)\,du' = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left(\frac{2n+1}{2}\right)a_{nm}f_{nm}(r,E)Y_{nm}(u)\int_{-1}^{1}P_n(\xi)\sigma_N(E,\xi)\,d\xi$$

$$\int_{-1}^{1}P_n(\xi)\sigma_N(E,\xi)\,d\xi$$

# Fokker-Planck equation for 1-50 MeV electron transport

Step 1:

$$I = \int_{-1}^{1} P_n(\xi)\sigma_N(E,\xi)d\xi$$

$$= \int_{-1}^{\varepsilon} P_n(\xi)\sigma_N(E,\xi)d\xi + \int_{\varepsilon}^{1} P_n(\xi)\sigma_N(E,\xi)d\xi := I_1 + I_2$$

Choose $\varepsilon \in (\xi_n, 1)$. Here $\xi_n$ is the largest positive root of the Legendre polynomial $P_n(\xi)$.

$$\frac{|I_1|}{I_2} \le \lim_{a\to 0+} \frac{\int_{-1}^{\varepsilon} \sigma_N(E,\xi)d\xi}{P_n(\varepsilon)\int_{\varepsilon}^{1-a} \sigma_N(E,\xi)d\xi} = 0 \quad \Rightarrow \quad I \approx I_2$$

# Fokker-Planck equation for 1-50 MeV electron transport

Step 2: $$I_2 \approx \int_{\varepsilon}^{1} \left(P_n(1) + P_n^{(1)}(1)(\xi - 1)\right)\sigma_N(E,\xi)\,d\xi := I_2'$$

Step 3: $$I_1' := \int_{-1}^{\varepsilon} \left(P_n(1) + P_n^{(1)}(1)(\xi - 1)\right)\sigma_N(E,\xi)\,d\xi$$

$$\frac{I_1'}{I_2'} = \lim_{a \to 0+} \frac{\int_{-1}^{\varepsilon} \left(P_n(1) + P_n^{(1)}(1)(\xi - 1)\right)\sigma_N(E,\xi)\,d\xi}{\int_{\varepsilon}^{1-a} \left(P_n(1) + P_n^{(1)}(1)(\xi - 1)\right)\sigma_N(E,\xi)\,d\xi} = 0 \quad \Rightarrow \quad I \approx I_2 \approx I_2' \approx I_1' + I_2'$$

$$I \approx \int_{-1}^{1} \left(P_n(1) + P_n^{(1)}(1)(\xi - 1)\right)\sigma_N(E,\xi)\,d\xi$$

# Fokker-Planck equation for 1-50 MeV electron transport

$$\frac{N_A D}{A}\int_{4\pi}[f(r,u',E)-f(r,u,E)]\sigma_N(E,u'\cdot u)du'$$

$$=\frac{N_A D}{A}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left(\frac{2n+1}{2}\right)a_{nm}f_{nm}(r,E)Y_{nm}(u)\int_{-1}^{1}\left(P_n(1)+P_n^{(1)}(1)(\xi-1)\right)\sigma_N(E,\xi)d\xi$$

$$-\frac{N_A D}{A}\int_{4\pi}f(r,u,E)\sigma_N(E,u'\cdot u)du'$$

$$=\frac{N_A D}{A}\int_{4\pi}f(r,u,E)\sigma_N(E,u'\cdot u)du'+T_1(E)\left(\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial}{\partial\mu}+\frac{1}{(1-\mu^2)}\frac{\partial^2}{\partial\phi^2}\right)f$$

$$-\frac{N_A D}{A}\int_{4\pi}f(r,u,E)\sigma_N(E,u'\cdot u)du'$$

$$=T_1(E)\left(\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial}{\partial\mu}+\frac{1}{(1-\mu^2)}\frac{\partial^2}{\partial\phi^2}\right)f$$

$$T_1(E)=\frac{N_A D}{A}\pi\int_{-1}^{1}(\xi-1)\sigma_N(E,\xi)d\xi$$

# Fokker-Planck equation for 1-50 MeV electron transport

The screened Rutherford cross section without corrections for low energy electron transport

$$\sigma_N(E,\xi) = \frac{r_0^2 Z^2 (m_0 c^2)^2 (E + m_0 c^2)^2}{E^2 (E + 2m_0 c^2)^2} \frac{1}{(1 - \xi + 2\eta)^2}$$

This is the advantage of bipartiton model for large angle!

# Fokker-Planck equation for 1-50 MeV electron transport

Bremsstrahlung and inelastic scattering are similar with elastic scattering, we won't repeat them again. Then we have:

$$\frac{N_A D}{A}\int_E^{E_0} f(r,u,E')\sigma_r(E',E'-E)dE' - \frac{N_A D}{A}\int_0^E f(r,u,E)\sigma_r(E,E-E')dE' = \frac{\partial S_1(E)f}{\partial E} + \frac{\partial^2 R_1(E)f}{\partial E^2}$$

$$S_1(E) = \frac{N_A D}{A}\int_0^E \sigma_r(E,T)TdE' \qquad R_1(E) = \frac{N_A D}{2A}\int_0^E \sigma_r(E,T)T^2 dE'$$

$$\frac{N_A D}{A} Z\int_E^{E_0}\int_{4\pi} f(r,u',E')\sigma_M(E',E'-E)\delta\left[u'\cdot u-\varphi(E',E'-E)\right]du'dE'$$

$$-\frac{N_A D}{A} Z\int_{E/2}^{E}\int_{4\pi} f(r,u,E)\sigma_M(E,E-E')\delta\left[u'\cdot u-\varphi(E,E-E')\right]du'dE'$$

$$= T_2(E)\left(\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial}{\partial\mu} + \frac{1}{(1-\mu^2)}\frac{\partial^2}{\partial\phi^2}\right)f + \frac{\partial S_2(E)f}{\partial E} + \frac{\partial^2 R_2(E)f}{\partial E^2}$$

# Fokker-Planck equation for 1-50 MeV electron transport

## Fokker-Planck equation

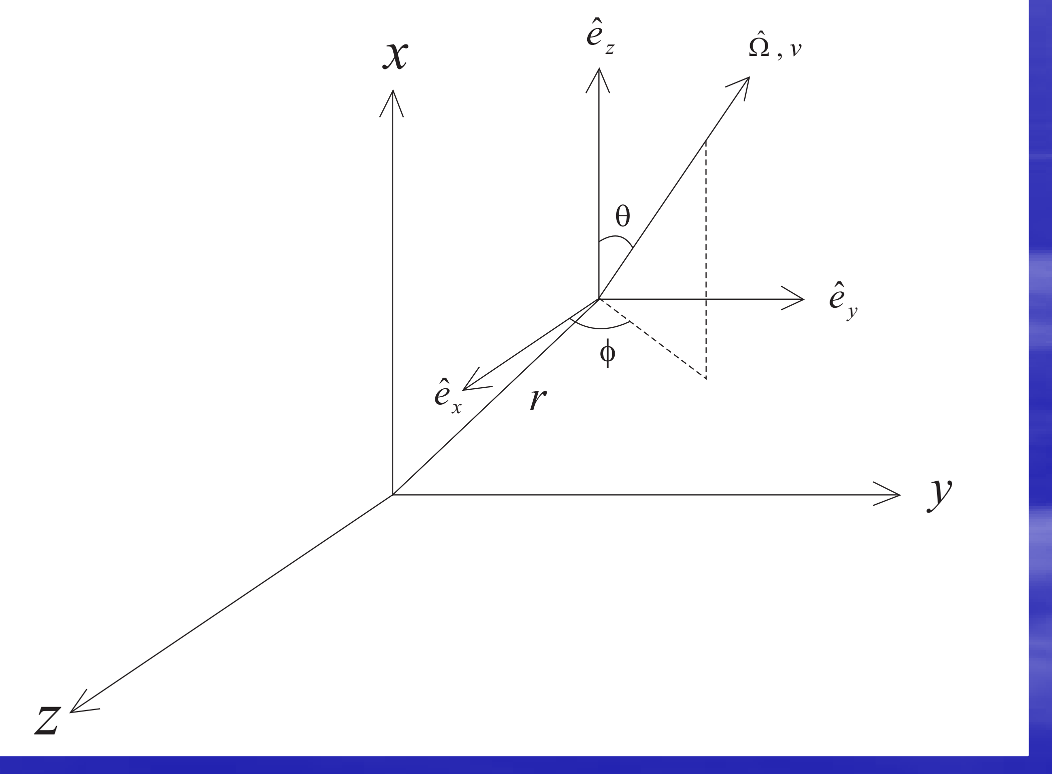
$$u \cdot \nabla f = T(E)\left[\frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial f}{\partial \mu} + \frac{1}{1-\mu^2}\frac{\partial^2 f}{\partial \phi^2}\right] + \frac{\partial f S}{\partial E} + \frac{\partial^2 f R}{\partial E^2}$$

$$T(E) = T_1(E) + T_2(E)$$

$$S(E) = S_1(E) + S_2(E)$$

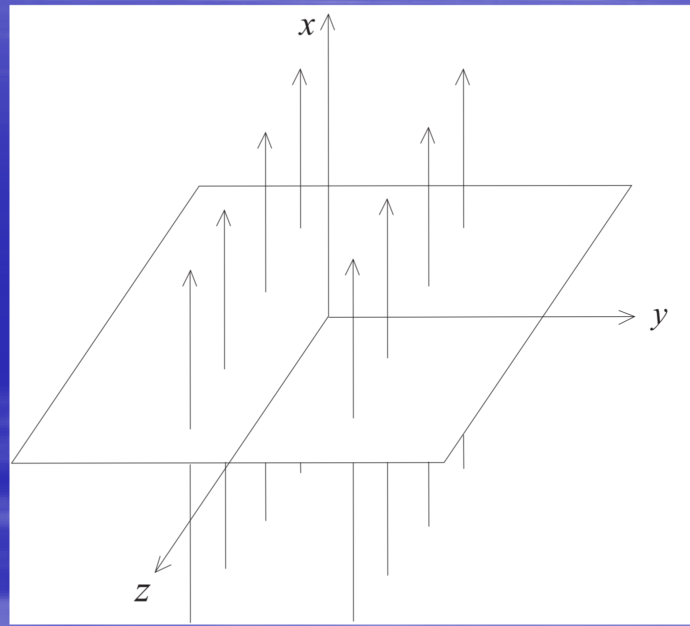
$$R(E) = R_1(E) + R_2(E)$$

# Broad beam and 2D pencil beam models



# Broad beam and 2D pencil beam models

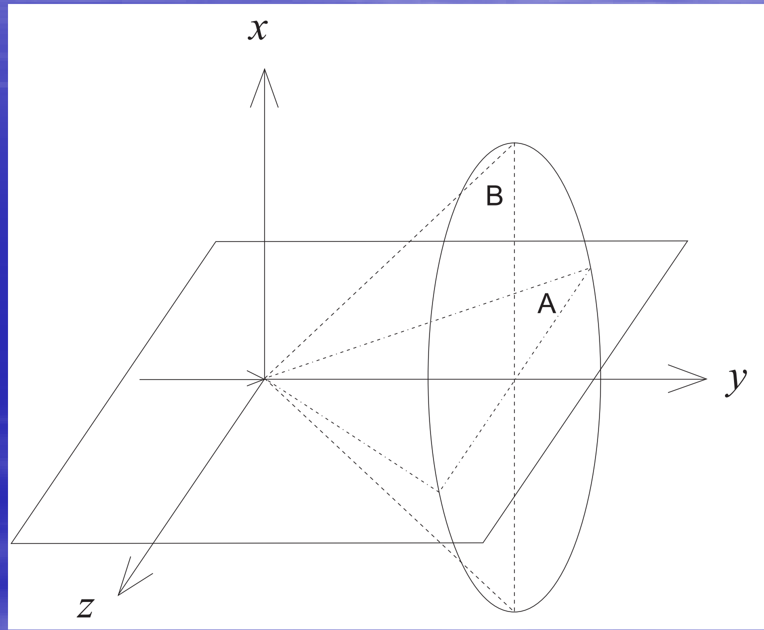
We could consider it as a monoenergetic and monodirectional plane source embedded in an infinite homogeneous medium. We should emphasize that the emission direction of the source is paralled to x-axis. So by the symmetry f(r,u.E) is independent of y, z and phi, and we may simplify the Fokker Planck equation to obtain the broad beam equation:




$$\mu \frac{\partial f(x,\mu,E)}{\partial x} = T(E)\frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial}{\partial \mu} f(x,\mu,E)\right] + \frac{\partial}{\partial E}\left[S(E) f(x,\mu,E)\right] + \frac{\partial^2}{\partial E_2}\left[R(E) f(x,\mu,E)\right]$$

# Broad beam and 2D pencil beam models

The 2D pencil beam model has been used in Fermi-Eyges theory in [30,40]. We may view it as a projection of 3Dpencil beam model on yz-plane. Note that the emission direction of the source is now paralled to the y-axis. Because if we use the same emission direction as the broad beam model, then only mu will not be sufficient to characterize the particle phase density and we should still keep phi. But if we use y-axis as the emission direction, we may neglect mu and just keep phi. Finally we assume that 2PBM is independent of energy. Then we simplify the Fokker-Planck equation to get:




$$\cos\phi \frac{\partial f(y,z,\phi)}{\partial y} + \sin\phi \frac{\partial f(y,z,\phi)}{\partial z} = T(E)\frac{\partial^2 f(y,z,\phi)}{\partial \phi^2}$$

Mohammad: [19,20,21,22,25,27,26]

# Bipartition model

Bipartition model was presented in 1967 by Luo in [3]. The main ideaof this model is to separate the beam into a diffusion group and a straight forward group and deal with them separately. [4,7] are the two most important papers about bipartition model for electron transport. In [4] Luo used bipartition model for 20 keV-1 MeV electron transport which takes into account both elastic and inelastic scatterings. In [7] Luo extended bipartition model into the energy range 1-50 MeV, considered the influence of the energy-loss straggling, secondary-electron production, and bremsstrahlung. The applications of bipartition model for inhomogeneous problems and ion transport are discussed in [5,6]. The history and development of the transport theory of charged particles and bipartition model are summarized in [8,9]. Bipartiton model could also be combined with Fermi-Eyges theory to produce the hybrid electron pencil beam model for 3D problems.

# Bipartition model

Lewis transport equation (broad beam model, only Fokker-Planck approximation for energy, the asymptotic approximation for screened Rutherford cross section not accurate)

$$-\frac{\partial}{\partial E}(\rho_c f)+\mu\frac{\partial f}{\partial x}-\frac{1}{2}\frac{\partial^2}{\partial E^2}(\Omega_c f)+\varphi_r f=\int_E^{E_0} f(x,\mu,E')\frac{N_A D}{A}\sigma_r(E',E'-E)dE'$$
$$+C_f(x,\mu,E)+S(x,\mu,E)$$

$$C_f(x,\mu,E)=\int_{4\pi}[f(x,\mu',E)-f(x,\mu,E)]\frac{N_A D}{A}\sigma_{MF}(E,u'\cdot u)du'$$

# Bipartition model

$$f(x,\mu,E) = f_s(x,\mu,E) + f_d(x,\mu,E)$$

$$-\frac{\partial}{\partial E}(\rho_c f_s) + \mu\frac{\partial f_s}{\partial x} - \frac{1}{2}\frac{\partial^2}{\partial E^2}(\Omega_c f_s) + \varphi_r f_s =$$

$$\int_{4\pi}\left[f_s(x,\mu',E) - f_s(x,\mu,E)\right]\frac{N_A D}{A}\sigma_{MF}(E,u'\cdot u)du' - S_d + S(x,\mu,E)$$

$$-\frac{\partial}{\partial E}(\rho_t f_d) + \mu\frac{\partial f_d}{\partial x} - \frac{1}{2}\frac{\partial^2}{\partial E^2}(\Omega_t f_d) = \int_E^{E_0} f_s(x,\mu,E')\frac{N_A D}{A}\sigma_r(E',E'-E)dE'$$

$$+\int_{4\pi}\left[f_d(x,\mu',E) - f_d(x,\mu,E)\right]\frac{N_A D}{A}\sigma_{MF}(E,u'\cdot u)du' + S_d + S(x,\mu,E)$$

# Bipartition model

$$f_s(x,\mu,E)=\sum_{l=0}^{\infty}\frac{2l+1}{4\pi}p_l(\mu)A_l(x,E)$$

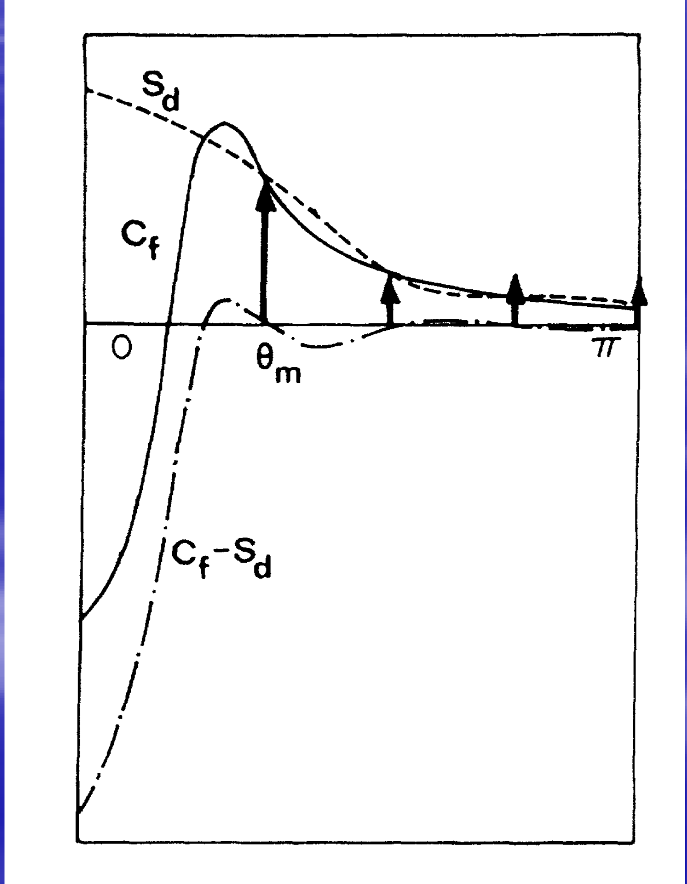
$$f_d(x,\mu,E)=\sum_{l=0}^{\infty}\frac{2l+1}{4\pi}p_l(\mu)N_l(x,E)$$

$$S_d(x,\mu,E)=\sum_{l=0}^{m}\frac{2l+1}{4\pi}p_l(\mu)S_l(x,E)$$

$$\left.\begin{aligned}&C_f=-\sum_{n=0}^{\infty}\frac{2n+1}{4\pi}P_n(\mu)A_n(x,t)\frac{ad_n}{t(t+\alpha)}\qquad\left(t=1-\frac{1}{R_0}\int_E^{E_0}\left[\frac{dE'}{dx}\right]^{-1}dE'\right)\\&C_{f_s}(x,\mu_i,E)=S_d(x,\mu_i,E),\qquad i=0,1,\cdots m.\end{aligned}\right\}$$

Kernel for bipartition model

# Bipartition model



# Bipartition model

$$-\frac{\partial}{\partial E}\left[L(E,\Delta)A_l(x,E)\right]+\mu_a\frac{\partial A_l}{\partial x}-\frac{1}{2}\frac{\partial^2}{\partial E^2}\left[\Omega_c A_l(x,E)\right]+\varphi_r A_l(x,E)$$
$$=-\varphi_l A_l(x,E)+\delta(x)\delta(E-E_0),\quad l>m$$

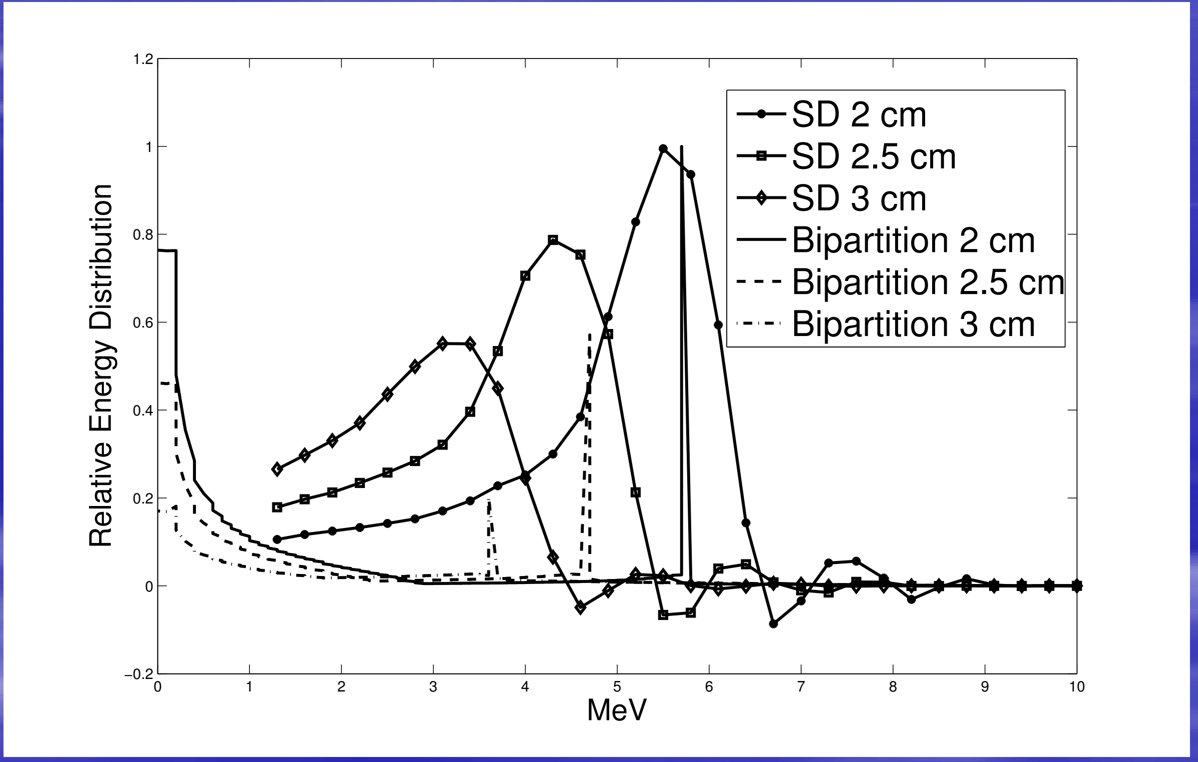
FT

$$-\frac{\partial}{\partial E}\left[L(E,\Delta)A_l(x,E)\right]+\mu_a\frac{\partial A_l}{\partial x}-\frac{1}{2}\frac{\partial^2}{\partial E^2}\left[\Omega_c A_l(x,E)\right]+\varphi_r A_l(x,E)$$
$$=\sum_{l'=m+1}^{\infty}D_{ll'}\varphi_{l'}A_{l'}(x,E)+\delta(x)\delta(E-E_0),\quad l\le m$$

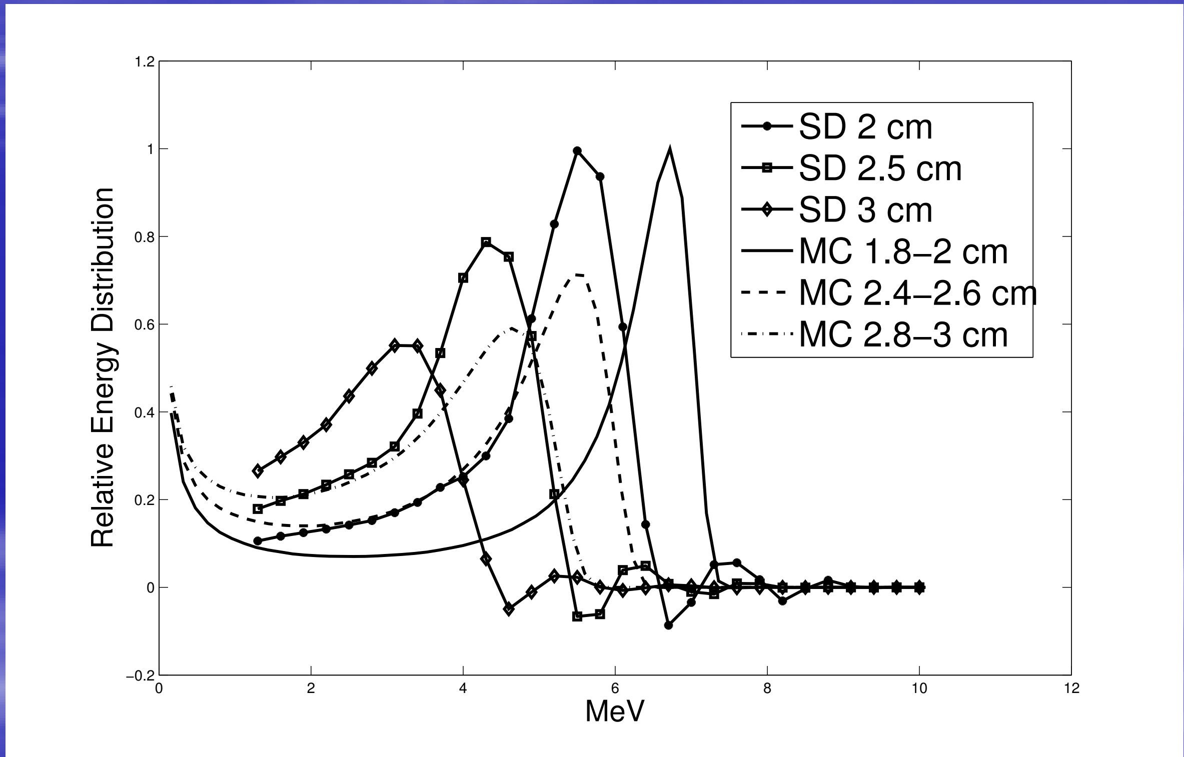
$$-\frac{\partial\rho_t N_l}{\partial E}+\left[\frac{l+1}{2l+1}\frac{\partial N_{l+1}}{\partial x}+\frac{l}{2l+1}\frac{\partial N_{l-1}}{\partial x}\right]$$
$$=\frac{1}{2}\frac{\partial^2\Omega_t N_l}{\partial E^2}-\varphi_l N_l+\frac{N_A D}{A}\int_{E+\varepsilon}^{E_0}A_l(x,E')\sigma_r(E',E'-E)\,dE'+S_l$$

FD

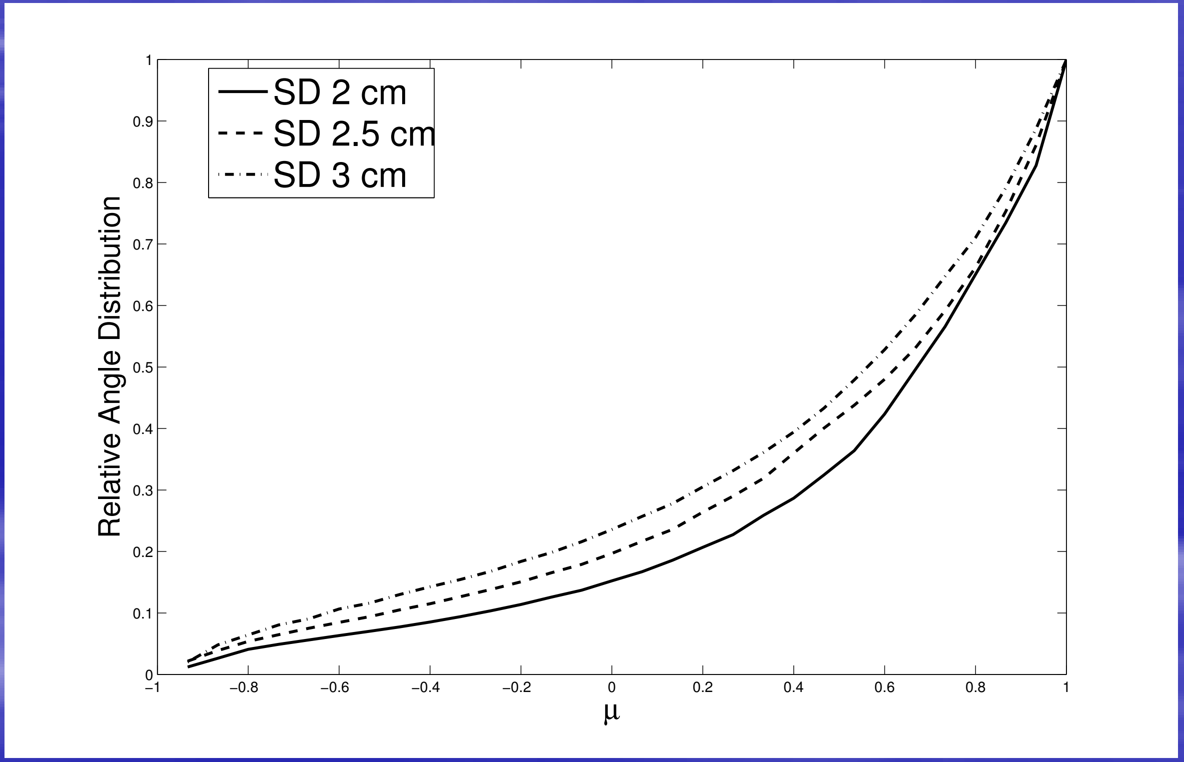
# Results by Sd-method



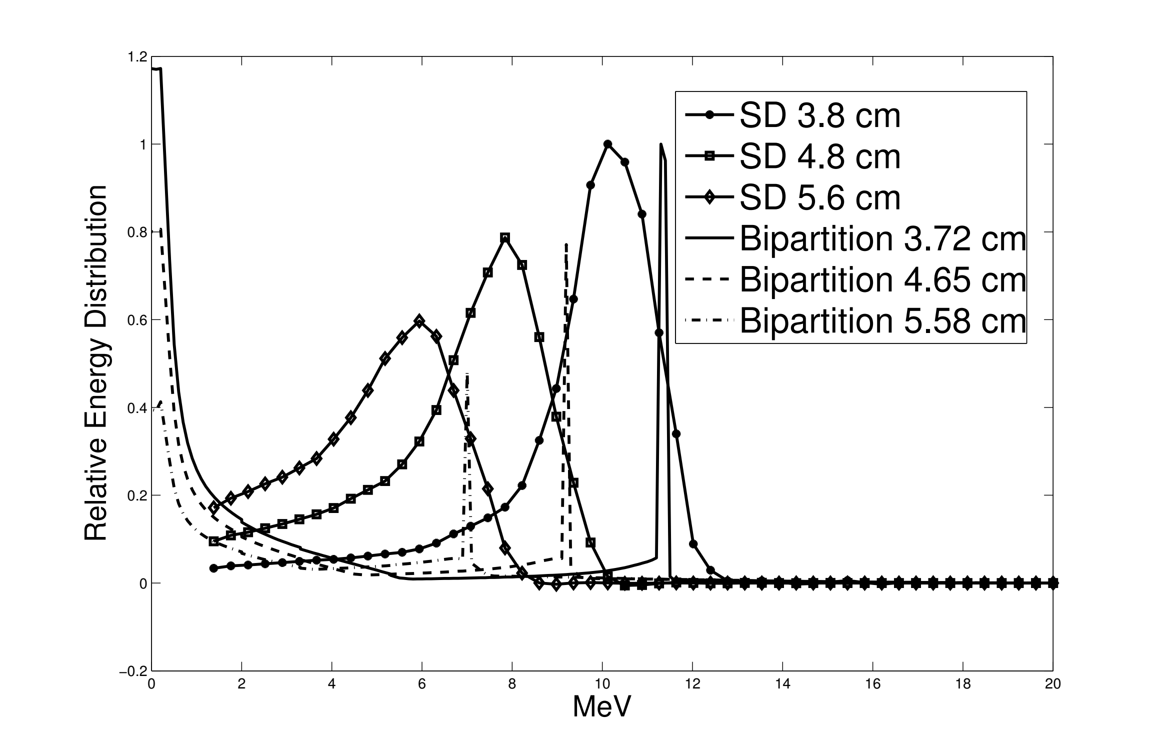
# Results by Sd-method



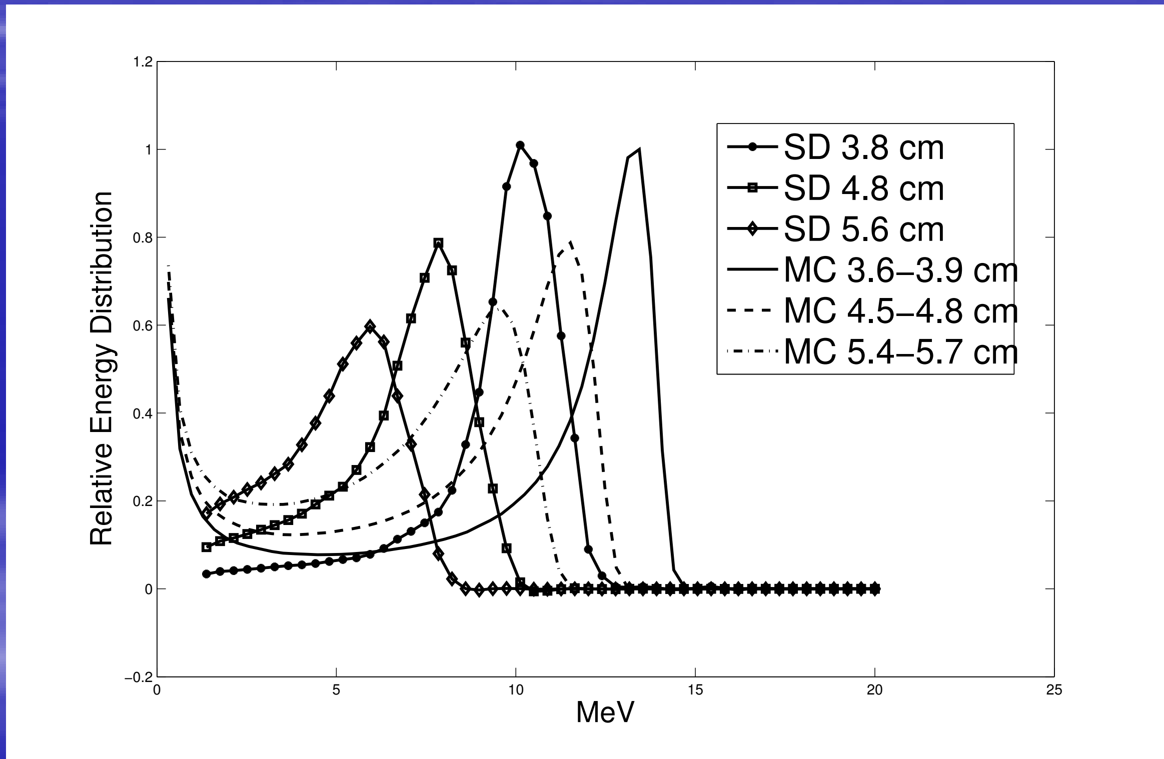
# Results by Sd-method



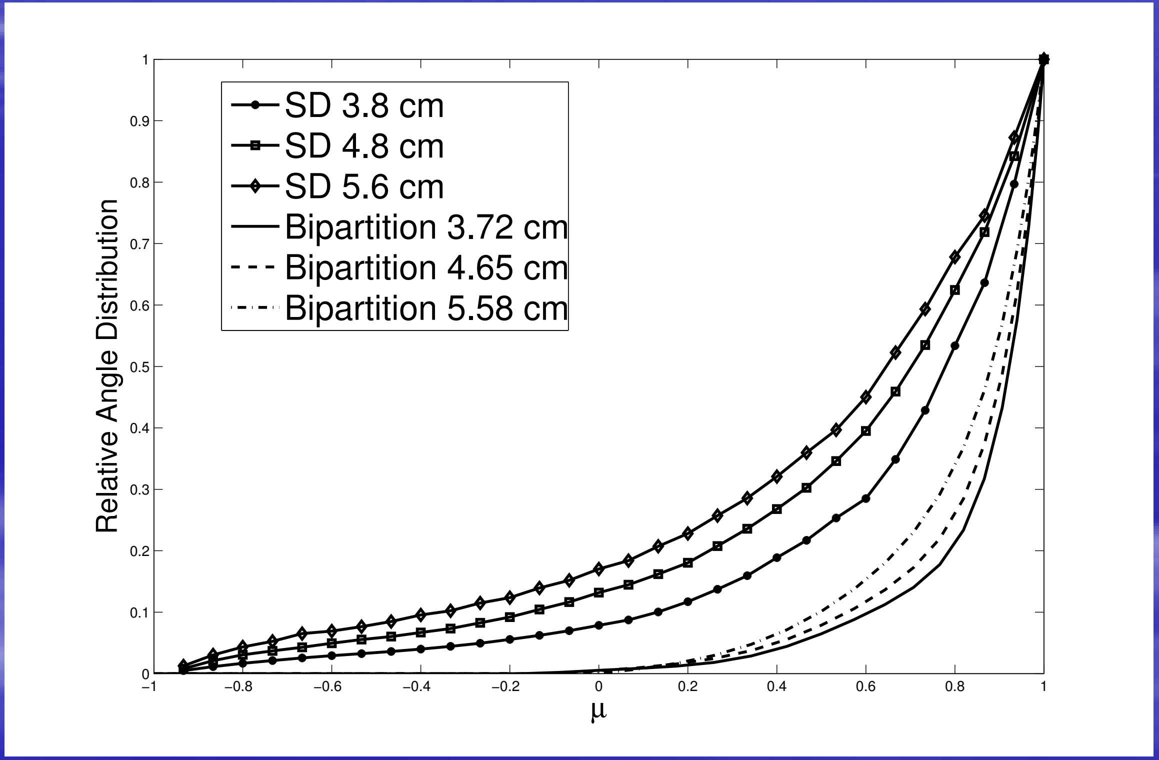
# Results by Sd-method



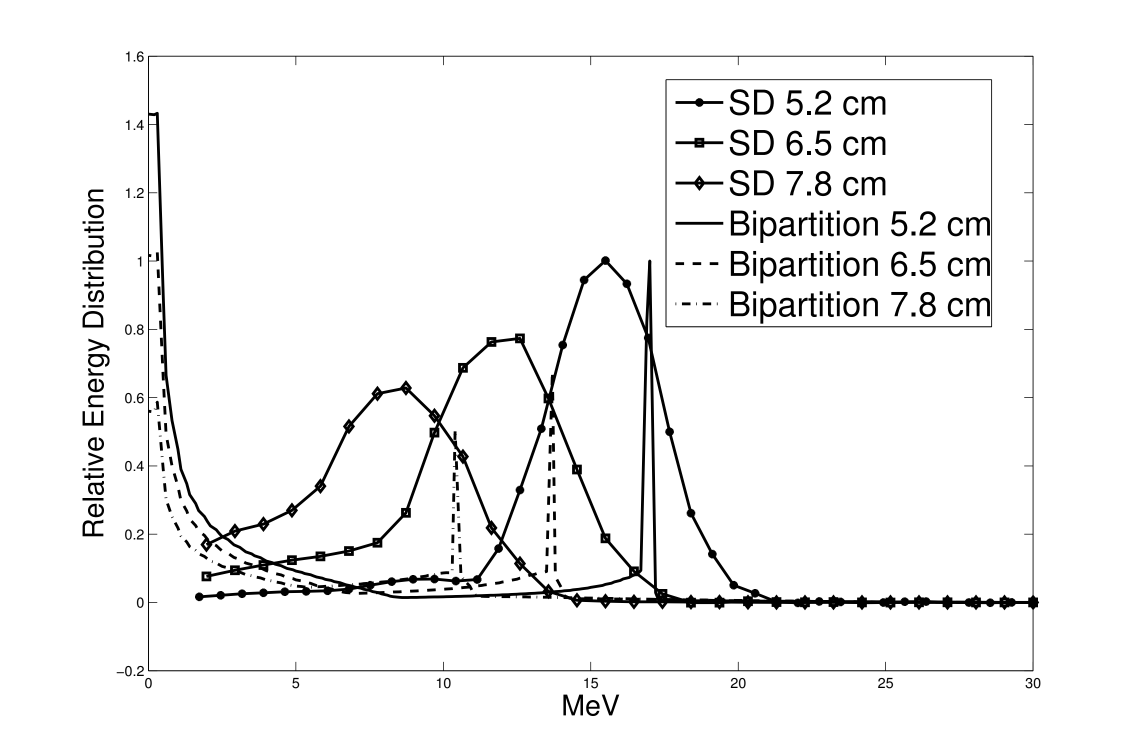
# Results by Sd-method



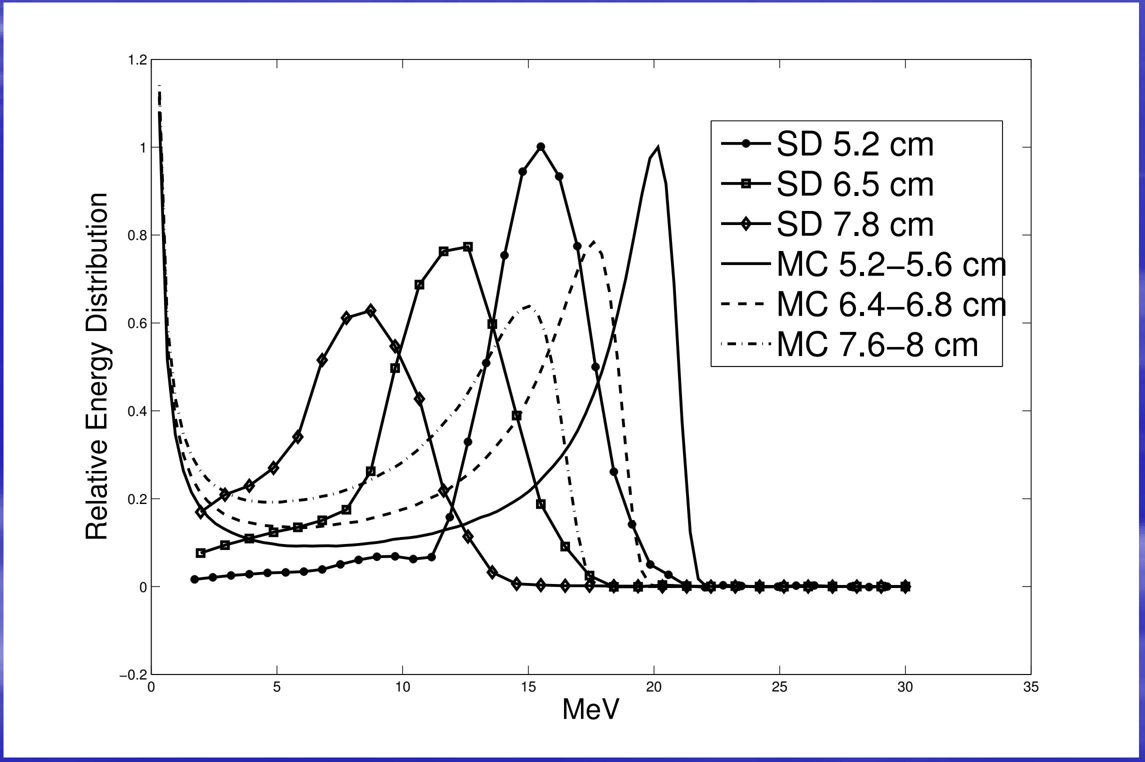
# Results by Sd-method



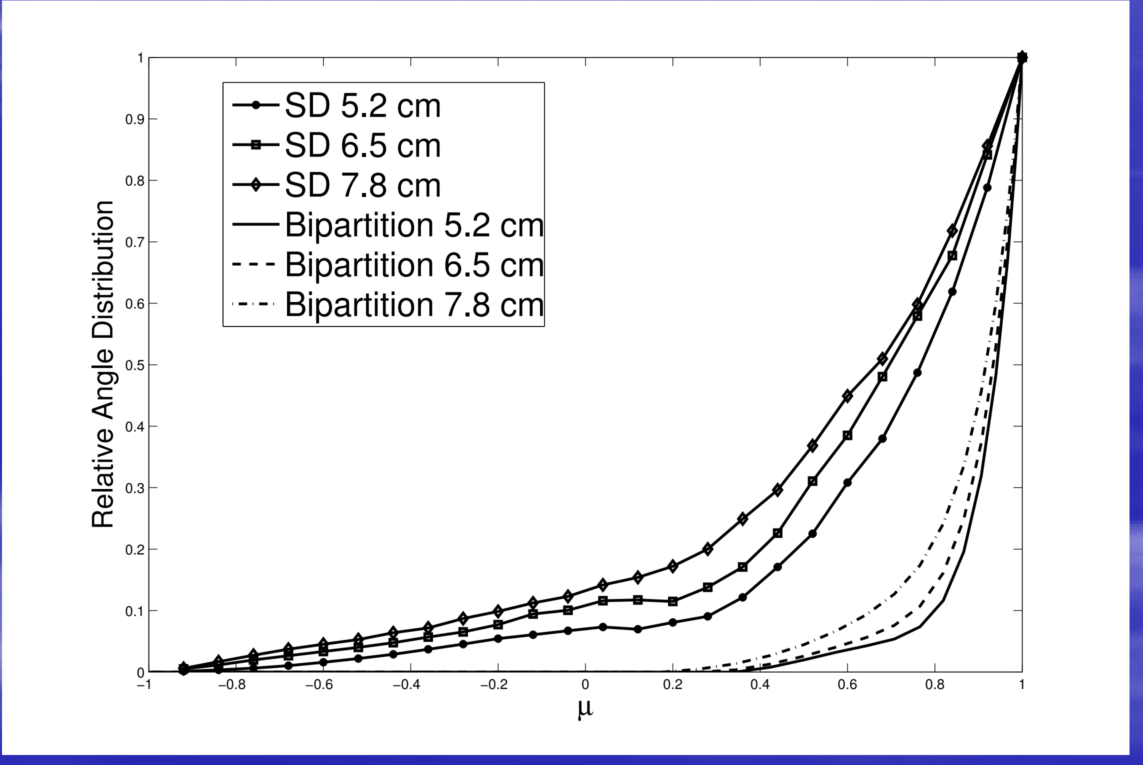
# Results by Sd-method



# Results by Sd-method



# Results by Sd-method



# Future work

- Energy deposition
- 3D pencil beam model
- Inhomogeneous medium and irregular geometry
- Ion transport
- More finite element methods
- Error estimates

# People in this field

- C. BORGERS (Tufts University)
- E. W. Larsen (Univ. of Michigan, Ann Arbor)
- G. C. Pomraning (UCLA)
- Lawrence H. Lanzl (University of Chicago, the Manhattan Project )
- David Jette (Dep. of Med. Phy., Rush-Presbyterian-St Luke's Medical Center, Chicago, Illinois)
- Anders Brahme (Karolinska Institute)
- Zhengming Luo (Sichuan University)
- Mohammad Asadzadeh (CTH)

Thank you very much
for your listening!