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SPECTRAL METHODS FOR THE TRANSPORT EQUATION.

Ph.D. Thesis (DOCTORAT D'ETAT)

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Part I

INTRODUCTION

In recent years, a new class of equations has acquired great significance in mathematical physics in connection with the rapid development of neutron physics and its associated studies. These are the so-called kinetic (transport) equations which describe the process of neutron transport in a substance. These equations are linear, integro-differential equations in partial derivatives of the first order. Much of the efforts in transport theory are devoted to searching for methods that generate accurate results. In the stationary case they have the form

$$\sum_{i=1}^{3} v_i \frac{\partial N}{\partial x_i} + \alpha(P, \mid v \mid) N = \frac{1}{4\pi} \int \theta(P, v, v') N(v', P') dv' + F(v, P), \tag{1.1}$$

where the unknown function $N(v, P)$ is the density of neutrons moving with velocity $v = (v_1, v_2, v_3)$ at the point $P = (x_1, x_2, x_3)$.

With equation (1.1) we associate the boundary conditions

$$N(v, P) = F_1(v, P), \qquad P \in \Gamma, \ (v, n) < 0. \tag{1.2}$$

For simplicity we assume that the region G where neutron transport occurs is convex and bounded by a piece-wise smooth surface Γ. In (1.2) n is the outwards drawn normal vector at the point P to the boundary Γ.

The transport equations (1.1) describe different physical process in particle transport. Besides the above mentioned process of neutron scattering in a substance we have also such processes as the dispersion of light in the atmosphere, the passage of γ-rays through a dispersive medium, the transport of radiation in stellar atmospheres, etc.

Thus these equations have wide application in physics, geophysics and astrophysics.

A detailed solution of the equations for neutron transport can be found, for example, in an article by Davison [20].

Equation (1.1) is, in substance, Boltzmann's linearized transport equation for the distributed of molecules.

Since the problem (1.1) and (1.2) has an extremely complex structure different approximations to it have acquired importance as simplifications. Among the approximation theories we have, for example, multigroup approximation, age theory, the diffusion approximation, etc. However, approximate methods are not always sufficiently precise in practice. Therefore it is useful to examine the accuracy of the various approximations, particularly those constructed directly by the use of modern computer techniques. The construction of methods of solution and their substantiation demand, however, a preliminary qualitative study of the problem and, first of all, of such aspects of it as existence, uniqueness, the continuous dependence of the solution on the data of the problem, spectral properties, in particular, the properties of eigenvalues and eigenfunctions, variational principles, etc. Moreover, these things are mathematically interesting in themselves, since they are connected with the new class of problems which describe much more complex physical processes.

Thus the question arises of constructing a rigorous mathematical theory for transport equations. Among the mathematical works dealing with transport equations, the method proposed by Chandrasekhar [19] solves analytically the discrete equations , (S_N equations), the spherical harmonics method [22] expands the angular flux in Legendre polynomials, the F_N method [26] transforms the transport equation into an integral equation. The integral transform technique like the Laplace, Fourier and Bessel also have been applied to solve the transport equation in semi-infinite domain [24], [25], the SGF method [7], [8] is a numerical nodal method that generates numerical solution for the S_N equations in slab geometry that is completely free of spatial truncation error. The LTS_N method [48] solve analytically the S_N equations employing the Laplace Transform technique in the spatial variable (finite domain). Recently, following the idea encompassed by the LTS_N method, we have derived a generic method, prevailing the analyticity, for solving one-dimensional approximation that transform the transport equation into a set differential equations.

The version of this generic method are known as LTS_N [5], LTP_N [50], LTW_N [16], $LTCh_N$ [17], LTA_N [18], LTD_N [9].

The analytical character of this solution, in the sense that no approximation is made along its derivation, constitutes its main feature. The idea encompassed is threefold: application of the Laplace Transform to the set of ordinary equations resulting from the approximation, analytical solution of the resulting linear system depending on the complex parameter s and inversion of the transformed angular flux by the Heaviside expansion technique.

We remark that the second step was accomplished by the application of the procedures that we shall describe further ahead. For the LTS_N approach, exploiting the structure of the corresponding matrix, the inversion was performed by employing the definition of matrix inversion [5]. On the other hand, for the remaining approaches, the matrix inversion was performed by the *Trzaska's* method [46].

The series expansions method has been largely used in the solution of the differential equation. In particular, Legendre Polynomials [22] and the Walsh function [43] expansion have been employed to solve the one-dimensional linear transport.

During the following ten to fifteen years much effort, both native and foreign, has been expended on theoretical-physical and mathematical, particularly numerical, methods of approximate solution of the transport equations[1] especially an important work has been done in the context of multidimensional transport problems, based on analytical and numerical approaches, Fourier transforms [52] or the discrete-ordinates method and the transverse-integrated equations [39]. Even commercial codes are available [42]. However, we consider it still today a big challenge in the particle-transport theory in the sense of obtaining procedures that can be applied to a wide range of problems as well as getting high quality computational results As a rule the problems considered were of a practical nature and it was necessary to get answers quickly, albeit crudely. The methods used were either of minor importance or were not properly examined. Thus, pursuing the objective of attaining solutions, based on analytical procedures, for the multidimensional

[1]A detailled bibliographical index of publications related to these subjects can be found in the monographs [36], [20], [19].

transport problems, in this work we use the spectral method [27] to decompose the multidimensional problem into a set of one-dimensional problems, whose can then be solved by one of the well-known methods such as the P_N method [20], the F_N method [26], the discrete-ordinates method [19], [39], the LTS_N method and the other ones employ the Laplace transform [49] and so on.

According to Gottlieb and Orszag [27], spectral methods involve representation the solution to a problem as a truncated series of known functions of the independent variables. The determination of the expansion coefficients is, of course, a fundamental issue in this method and we can then recall some approximation to this end. But in regard to that, one should prefer to use orthogonal basis such that those coefficients could be determined by orthogonality properties.

The purpose of this thesis is to meet this problem to some extent, using the spectral method .

The principal results contained in this work, aside from those of part II and III, were published in our earlier papers [1]. We note also the works of Cardona [17] and Vilhena [47], which concern spectral investigations and the examination of solutions to stationary problems. These articles have some points in common with ours, however none overlap.

An outline of this thesis is as follows: In the part I of this thesis we present a new approximation for the one-group linear transport equation with anisotropic scattering in a slab, using Chebyshev polynomials. To this end, the angular flux is expanded in a truncated series of Chebyshev polynomials in the angular variable. Replacing this expression in the transport equation and taking moments like in the P_N method [22], leads to a new approximation. The resultant first-order linear differential system is solved for the spatial function coefficients by application of the Sumudu transform technique.

The inversion of the transformed coefficients is performed also analytically, using *Trzaska's* method and the heaviside technique.

In part II of this work is devoted to study a convergence of a combined spectral and (S_N) discrete approximation for a multidimensional, steady state, linear transport problem with isotropic scattering. The procedure is based on expansion of the angular flux in a truncated series of the Chebyshev polynomials in spatial variables that results in the transformation of the multidimensional problems into a set of one-dimensional problems. The convergence of this approach is studied in the context of the discrete-ordinates equations based on a special quadrature rule for the scattering integral. The discrete-ordinates and quadrature errors are expanded in truncated series of Chebyshev polynomials of degree$\leq L$, and the convergence is derived assuming $L \leq \sigma_t - 4\pi\sigma_s$ where σ_t and σ_s are total- and scattering cross-sections respectively.

Appendix I is devoted to certain properties of the Chebyshev and Legendre polynomials that are frequently used in this thesis, in Appendix II we derive the spectral equations in three dimensional setting.

For the convenience of the reader most of the ideas and results of Sumudu transform an *Trzaska's* method are consolidated in Appendix III and IV.

Part II

SOLVING THE ONE-DIMENSIONAL NEUTRON TRANSPORT EQUATION USING CHEBYSHEV POLYNOMIALS AND SUMUDU TRANSFORM.

Chapter 2

Introduction

As is well known, the study of a given transport equation is a quite important and interesting in transport theory. Various methods have been developed to investigate, and special attention has been given to the task of searching methods that generate accurate results to transport problems in the context of deterministic methods based on analytical procedures, for the multidimensional transport problems, one of the effective methods to treat linear transport equation is the spectral method [38] [36] [28] etc..., whose basic goals is to find exact solution for approximations of the transport equation, several approaches have been suggested.

According to Gottlieb [27], spectral method involve representation the solution to a problem as a truncated series of known functions of the independent variables, of course there exist other method to determine the coefficients of expansion, but in regard to that, we should prefer to use orthogonal basis such that those coefficients could be determined by orthogonality properties. Thereby, the orthogonal functions and polynomial series have received considerable attention in dealing with various problem. The main characteristic of this technique is that reduces this problems to those of solving a system of algebraic equations, thus greatly simplifying the problem and making it computational plausible.

Chebyshev spectral methods for radiative transfer problems are studied, e.g., by Kim and Ishimaru in [34] and by Kim and Moscoso in [35] and by Asadzadeh and Kadem in [1] and by Kadem [30] [31] [32] [33]. For more detailed study on Chebyshev spectral method and also approximations by the spectral methods we refer the reader to monographs by Body [13] and Bernardi and Maday [11].

In this part we present a new approximation for the one dimensional transport equation, using Chebyshev polynomials [40] combined with the Sumudu transform. The approach is based on expansion of the angular flux in a truncated series of Chebyshev polynomials in the angular variable. By replacing this development in the transport equation, this which will result a first-order linear differential system is solved for the spatial function coefficients by application of the Sumudu transform technique [10].

The inversion of the transformed coefficients is obtained using *Trzaska's* method [46] and the Heaviside expansion technique.

To our knowledge, the combination of the Chebyshev polynomials and the Sumudu transform solve the linear transport equation has not been considered before.

Chapter 3

Analysis

Let us consider the following mono-energetic $3-D$ transport equation:

$$\underline{\Omega}.\underline{\nabla}(\underline{r},\underline{\Omega}) + \sigma_t \Psi(\underline{r},\underline{\Omega}) = \int_{4\pi} \sigma_s(\underline{\Omega},\underline{\Omega'})\Psi(\underline{r},\underline{\Omega'})d\Omega' + \frac{1}{4\pi}Q(\underline{r}) \tag{3.1}$$

where

$$\underline{r} = (x, y, z) = \text{ spatial variable,} \tag{3.2}$$

$$\underline{\Omega} = (\eta, \xi) = \text{angular variable,} \tag{3.3}$$

and

$$\sigma_s(\mu_0) = \sum_{k=0}^{\infty} \frac{2k+1}{4\pi} \sigma_{sk} P_k(\mu_0) = \text{ differential scattering cross section,} \tag{3.4}$$

with $\mu_0 = \underline{\Omega}.\underline{\Omega'}$ and $P_k =$ the k^{th} Legendre polynomial.

Chapter 4

Planar Geometry

We consider a planar-geometry problem with spatial variation only in the $x-$direction:

$$Q(\underline{r}) = q(x), \tag{4.1}$$

$$\Psi(\underline{r}, \underline{\Omega}) = \frac{1}{2\pi}\Psi(x, \mu) \tag{4.2}$$

Eq. (3.1) simplifies to

$$\mu\frac{\partial \Psi}{\partial x}(x, \mu) + \sigma_t \Psi(x, \mu) = \int_{-1}^{1} \sigma_s(\mu, \mu')\Psi(x, \mu')d\mu' + \frac{q(x)}{2}, \tag{4.3}$$

with

$$\sigma_s(\mu, \mu') = \sum_{k=0}^{\infty} \frac{2k+1}{2}\sigma_{sk} P_k(\mu) P_k(\mu'). \tag{4.4}$$

So we consider Eq. (4.3) with $0 \leq x \leq a$ and $-1 \leq \mu \leq 1$, and subject to the boundary conditions

$$\Psi(x = a, \mu) = 0, \tag{4.5}$$

and

$$\Psi(x = 0, \mu) = f(\mu), \tag{4.6}$$

where $f(\mu)$ is the prescribed incident flux at $x = 0$; $\Psi(x, \mu)$ is the angular flux in the μ direction; σ_t is the total cross section; σ_{sl}, with $l = 0, 1, ..., L$ are the components of the differential scattering cross section, and $P_k(\mu)$ are the Legendre polynomials of degree k.

Theorem 4.1. *Consider the integro-differential equation (4.3) under the boundary conditions (4.5) and (4.6), then the function $\Psi(x, \mu)$ satisfies the following first-order linear differential equation system for the spatial component $g_n(x)$*

$$\sum_{n=0}^{N} \alpha^1_{n,m} g'_n(x) + \frac{\sigma_t \pi}{2 - \delta_{m,0}} g_m(x) = \sum_{l=0}^{L} \frac{2l+1}{2} \sigma_{sl} \alpha^2_{m,l} \sum_{n=0}^{N} \alpha^3_{n,l} g_n(x) + \frac{q(x)}{2}$$

where

$$\alpha^1_{n,m} := \int_{-1}^{1} \mu T_n(\mu) \frac{T_m(\mu)}{\sqrt{1-\mu^2}} d\mu,$$

$$\alpha^2_{n,l} := \int_{-1}^{1} T_n(\mu) P_l(\mu) d\mu,$$

$$\alpha^3_{n,l} := \int_{-1}^{1} \frac{T_n(\mu) P_l(\mu)}{\sqrt{1-\mu^2}} d\mu,$$

and $g_m(x)$ are the coefficients of the expansion of the $\Psi(x, \mu)$.

To prepare for the proof of the Theorem (4.1) we need the following result

Proposition 4.2. *Let*

$$T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0$$

and

$$P_{l+1}(x) = 2xP_l(x) - P_{l-1}(x) - \left[xP_l(x) - P_{l-1}(x)\right] / (l+1)$$

be the recurrence relations for the Chebyshev and the Legendre polynomials, respectively.

We have for $l > 2$ and $k = 2, 3$

$$\alpha^k_{n,l+1} := \frac{2l+1}{2l+2} \left[\alpha^k_{n+1,l} + \alpha^k_{n-1,l}\right] - \frac{l}{l+1} \alpha^k_{n,j-1}$$

Hence, in particular for $l = 0$ and 1 the coefficients $\alpha^2_{n,l}$ and $\alpha^3_{n,l}$ assume the values

$$\alpha^2_{n,l} = \begin{cases} 0 & \text{if } n+l \text{ odd,} \\ \frac{2}{(1+l)^2-n^2} & \text{if } n+l \text{ even,} \end{cases}$$

and

$$\alpha^3_{n,l} = \frac{\pi\delta_{n,l}}{2-\delta_{l,0}}$$

Proof

It easy to see that

$$\alpha^1_{n,m} = \frac{\pi\delta_{|n-m|}}{2(2-\delta_{n+m,1})}$$

For $k = 2$ by the multiplication of the Chebyshev and the Legendre recurrence formulas we have

$$\frac{2l+1}{2l+2}\left[P_l(\mu)T_{n+1}(\mu) + P_l(\mu)T_{n-1}(\mu)\right] - \frac{l}{2\mu\,(l+1)}P_{l-1}(\mu)\left[T_{n+1}(\mu) + T_{n-1}(\mu)\right]$$

it is known that

$$T_{n+1}(\mu) + T_{n-1}(\mu) = 2\mu T_n(\mu)$$

after doing some algebraic manipulations and integrating over $\mu \in [-1, 1]$ on the resulting equation we get

$$\alpha^2_{n,l+1} = \frac{2l+1}{2l+2}\left[\alpha^2_{n+1,l} + \alpha^2_{n-1,l}\right] - \frac{l}{l+1}\alpha^2_{n,j-1}$$

The case $k = 3$ is treated similarly but in this case we multiply the resulting expression by $\frac{1}{\sqrt{1-\mu^2}}$ and integrate over $\mu \in [-1, 1]$ we get the desired result.

Proof of Theorem 4.1

Expanding the angular flux in the μ variable in terms of the Chebyshev polynomials [40] leads to

$$\Psi(x,\mu) = \sum_{n=0}^{N} \frac{g_n(x)T_n(\mu)}{\sqrt{1-\mu^2}} \tag{4.7}$$

with $N = 0, 2, 4, \ldots$, where the expansions coefficients $g_n(x)$ should be determined.

Here $T_n(\mu)$ are the well known Chebyshev polynomials of order n which are orthogonal in the interval $[-1,1]$ with respect to the weight $w(t) = 1/\sqrt{1-t^2}$.

After replacing Eq. (4.7) into Eq. (4.3) it turns out

$$\sum_{n=0}^{N} \{\mu g_n'(x) + \sigma_t g_n(x)\} \frac{T_n(\mu)}{\sqrt{1-\mu^2}} =$$

$$\sum_{l=0}^{L} \frac{2l+1}{2} \sigma_{sl} P_l(\mu) \sum_{n=0}^{N} g_n(x) \int_{-1}^{1} P_l(\mu') \frac{T_n(\mu')}{\sqrt{1-\mu'^2}} d\mu' + \frac{q(x)}{2} \tag{4.8}$$

using the orthogonality of the Chebyshev polynomials [40], multiply the Eq. (4.8) by $T_m(\mu)$, considering $m = 0, 1, ..., N$, and integrated in the μ variable in the interval $[-1, 1]$. Thus we get the following first-order linear differential equation system for the spatial component $g_n(x)$

$$\sum_{n=0}^{N} \alpha_{n,m}^1 g_n'(x) + \frac{\sigma_t \pi}{2 - \delta_{m,0}} g_m(x) = \sum_{l=0}^{L} \frac{2l+1}{2} \sigma_{sl} \alpha_{m,l}^2 \sum_{n=0}^{N} \alpha_{n,l}^3 g_n(x) + \frac{q(x)}{2} \tag{4.9}$$

where

$$\alpha_{n,m}^1 = \frac{\pi \delta_{|n-m|}}{2(2 - \delta_{n+m,1})}, \tag{4.10}$$

$$\alpha_{n,l}^2 = \int_{-1}^{1} T_n(\mu) P_l(\mu) d\mu, \tag{4.11}$$

$$\alpha_{n,l}^3 = \int_{-1}^{1} \frac{T_n(\mu) P_l(\mu)}{\sqrt{1-\mu^2}} d\mu, \tag{4.12}$$

with $\delta_{n,m}$ denoting the delta of Kronecker. Here the coefficients $\alpha_{n,l}^2$ and $\alpha_{n,l}^3$ are evaluated by the multiplication of the Chebyshev and Legendre recurrence formulas and integration of the resulting equation (See proposition 2 from Appendix I).

So that we have

$$\alpha_{n,l+1}^k := \frac{2l+1}{2l+2} \left[\alpha_{n+1,l}^k + \alpha_{n-1,l}^k\right] - \frac{l}{l+1} \alpha_{n,j-1}^k \tag{4.13}$$

for $l > 2$, and $k = 2, 3$.

While for $l = 0$ and 1 the coefficients $\alpha^2_{n,l}$ and $\alpha^3_{n,l}$ assume the values

$$\alpha^2_{n,l} = \begin{cases} 0 & \text{if } n+l \text{ odd,} \\ \frac{2}{(1+l)^2-n^2} & \text{if } n+l \text{ even,} \end{cases} \tag{4.14}$$

and

$$\alpha^3_{n,l} = \frac{\pi\delta_{n,l}}{2-\delta_{l,0}} \tag{4.15}$$

we rewrite Eq. (4.9) in matrix form

$$A.\frac{dg}{dx}(x) + Bg(x) = C(x) \tag{4.16}$$

where $g(x)$ = Col. $[g_0(x), g_1(x), ..., g_N(x)]$ and A and B are squared matrices of order $N+1$ with the components

$$(A)_{i,j} = \alpha^1_{i-1,j-1}, \tag{4.17}$$

$$(B)_{i,j} = \frac{\pi\sigma_t}{2-\delta_{1,j}}\delta_{i,j} - \sum_{l=0}^{L}\frac{2l+1}{2}\sigma_{sl}\alpha^2_{i-1,l}\sum_{n=0}^{N}\alpha^3_{j-1,l} \tag{4.18}$$

and

$$C(x) = \frac{q(x)}{2} = \text{Col.}\left[C_0(x), C_1(x), ..., C_N(x)\right]. \tag{4.19}$$

we notice that this equation has the well known solution [44]

$$g(x) = e^{-A^{-1}Bx}g(0) + \int_0^x e^{-A^{-1}B(x-\xi)}C(\xi)d\xi, \tag{4.20}$$

that depends on vector $g(0)$. Having established an analytical formulation for the exponential appearing in equation (4.20), the $N+1$ unknown components of vector $g(0)$ for the boundary problem (4.3) can be readily obtained applying the boundary conditions (4.5) and (4.6) in the solution given by Eq. (4.7) and multiplying this expression by the Chebyshev polynomial $T_m(\mu)$, considering $m = 0, 2, 4, ..., N-1$, and integrating in the interval $[-1,1]$, this procedure gives

$$\sum_{n=0}^{N} g_n(0) \int_{-1}^{1} \frac{T_n(\mu)T_m(\mu)}{\sqrt{1-\mu^2}} d\mu = \int_{-1}^{1} g(\mu)T_m(\mu)d\mu \tag{4.21}$$

and

$$\sum_{n=0}^{N} (-1)^n g_n(a) \int_{-1}^{1} \frac{T_n(\mu)T_m(\mu)}{\sqrt{1-\mu^2}} d\mu = 0. \tag{4.22}$$

To derive an analytical formulation for the exponential of matrix $A^{-1}B$, appearing in equation (4.20), let us solve the homogeneous version of equation (4.16), namely

$$A.\frac{dg}{dx}(x) + Bg(x) = 0 \tag{4.23}$$

Now, following the idea of applying the Sumudu transform to equation (4.23) so by applying Theorem 4 Appendix III, we obtain an algebraic linear system that has the solution

$$G(u)\left[uB + A\right] = R \tag{4.24}$$

with

$$R = A.g(0), \tag{4.25}$$

Where $G(u) = S\left[g(x)\right]$ denotes the Sumudu transform of the vector $g(x)$ (See Appendix III). Solving equation (4.24) that has the solution

$$G(u) = \left[uB + A\right]^{-1} R \tag{4.26}$$

by *Trzaska's* method [46] the inverse of matrix $[uB + A]$ is readily obtained indeed

$$\left[uB + A\right]^{-1} = \sum_{k=1}^{M} \frac{1}{u - s_k} P_k \tag{4.27}$$

where the coefficients s_k denote the eigenvalues of matrix $B^{-1}A$ and the matrices P_k are the ones resulting from the application of *Trzaska's* method (See Appendix IV). The inversion of the transformed vector $G(u)$ is executed by the Heaviside expansion technique. Following this procedure, we obtain an analytical expression for the exponential of matrix $B^{-1}A$ [49].

$$e^{-B^{-1}Ax} = \sum_{k=1}^{M} P_k e^{s_k x}. \tag{4.28}$$

We substitute Eq. (4.28) into Eq. (4.20) then the transformed vector $g(x)$ by the Heaviside technique to get

$$g(x) = \sum_{k=1}^{M} e^{s_k x} P_k R + \sum_{k=1}^{M} P_k \int_0^x e^{s_k(x-\xi)} C(\xi) d\xi, \tag{4.29}$$

Replacing $g_n(0)$ and $g_n(a)$ by its values given by equation (4.20) in equation (4.21) and (4.22), it turns out

$$\sum_{l=1}^{N} \left[\sum_{k=1}^{M} P_k R_l + \sum_{k=1}^{M} P_k \int_0^x e^{A^{-1}B\xi} C(\xi) d\xi \right] \int_{-1}^{1} \frac{T_n(\mu)T_m(\mu)}{\sqrt{1-\mu^2}} d\mu = \int_{-1}^{1} g(\mu) T_m(\mu) d\mu \tag{4.30}$$

and

$$\sum_{l=1}^{N} \left[\sum_{k=1}^{M} P_k e^{s_k a} R_l + \sum_{k=1}^{M} P_k \int_0^x e^{-A^{-1}B(s_k-\xi)} C(\xi) d\xi \right] \int_{-1}^{1} \frac{T_n(\mu)T_m(\mu)}{\sqrt{1-\mu^2}} d\mu = 0 \tag{4.31}$$

with $m = 0, 2, 4, ..., N-1$, where R_l design the element of the unknown vector R.

After solving the linear system (4.30), (4.31) for the components of the vector R, the angular flux given by equation (4.7) is completely determined.

Chapter 5

Specific Application of the Method

Consider the three-dimensional neutron transport equation written as

$$\mu\frac{\partial}{\partial x}\Psi(\mathbf{x},\mu,\theta)+\sqrt{1-\mu^2}\left[\cos\theta\frac{\partial}{\partial y}\Psi(\mathbf{x},\mu,\theta)+\sin\theta\frac{\partial}{\partial z}\Psi(\mathbf{x},\mu,\theta)\right]$$

$$+\sigma_t\Psi(\mathbf{x},\mu,\theta)=\int_{-1}^{1}\int_{0}^{2\pi}\sigma_s(\mu^{'},\theta^{'}\to\mu,\theta)\Psi(\mathbf{x},\mu^{'},\theta^{'})d\theta^{'}d\mu^{'}+S(\mathbf{x},\mu,\theta) \tag{5.1}$$

where we assume that the spatial variable $\mathbf{x}:=(x,y,z)$ varies in the cubic domain $\Omega:=\{(x,y,z):-1\le x,y,z\le 1\}$, and $\Psi(\mathbf{x},\mu,\theta):=\Psi(x,y,z,\mu,\theta)$ is the angular flux in the direction defined by $\mu\in[-1,1]$ and $\theta\in[0,2\pi]$.

We seek for a solution of (5.1) satisfying the following boundary conditions:

For the boundary terms in x; for $0\le\theta\le 2\pi$,

$$\Psi(x=\pm 1,y,z,\mu,\theta)=\begin{cases} f_1(y,z,\mu,\theta),\ x=-1, & 0<\mu\le 1,\\ 0,\ x=1, & -1\le\mu<0.\end{cases} \tag{5.2}$$

For the boundary terms in y and for $-1\le\mu<1$,

$$\Psi(x,y=\pm 1,z,\mu,\theta)=\begin{cases} f_2(x,z,\mu,\theta),\ y=-1, & 0<\cos\theta\le 1,\\ 0,\ y=1, & -1\le\cos\theta<0.\end{cases} \tag{5.3}$$

Finally, for the boundary terms in z; for $-1 \leq \mu < 1$,

$$\Psi(x, y, z = \pm 1, \mu, \theta) = \begin{cases} f_3(x, y, \mu, \theta), \ z = -1, & 0 \leq \theta < \pi, \\ 0, \ z = 1, & \pi < \theta \leq 2\pi. \end{cases} \tag{5.4}$$

Here we assume that $f_1(y, z, \mu, \phi)$, $f_2(x, z, \mu, \phi)$ and $f_3(x, y, \mu, \phi)$ are given function.

Expanding the angular flux $\Psi(x, y, z, \mu, \phi)$ in a truncated series of Chebyshev polynomials $T_i(y)$ and $R_j(z)$ leads to

$$\Psi(x, y, z, \mu, \theta) = \sum_{i=0}^{I} \sum_{j=0}^{J} \Psi_{i,j}(x, \mu, \theta) T_i(y) R_j(z). \tag{5.5}$$

insert Eq. (5.5) into (5.1) Multiplying the resulting expressions by $\frac{T_i(y)}{\sqrt{1-y^2}} \times \frac{R_j(z)}{\sqrt{1-z^2}}$, and integrating over y and z we obtain $I \times J$ one-dimensional transport problems, viz

$$\mu \frac{\partial \Psi_{\alpha,\beta}}{\partial x}(x, \mu, \phi) + \sigma_t \Psi_{\alpha,\beta}(x, \mu, \phi) =$$

$$\int_{-1}^{1} \int_{-1}^{1} \sigma_s(\mu', \phi' \rightarrow \mu, \phi) \Psi_{\alpha,\beta}(x, \mu', \phi') d\phi' d\mu' + G_{\alpha,\beta}(x; \mu, \phi) \tag{5.6}$$

where

$$G_{\alpha,\beta}(x; \mu, \eta) = S_{\alpha,\beta}(x, \mu, \phi) - \sqrt{1 - \mu^2}$$

$$\times \left[\cos\phi \sum_{\alpha=i+1}^{I} A_i^{\alpha} \Psi_{\alpha,j}(x, \mu, \phi) + \sin\phi \sum_{\beta=j+1}^{J} B_j^{\beta} \Psi_{i,\beta}(x, \mu, \phi) \right], \tag{5.7}$$

with

$$S_{\alpha,\beta}(x, \mu, \phi) = \frac{(2 - \delta_{\alpha,0})(2 - \delta_{\beta,0})}{\pi^2} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \frac{T_\alpha(\mu)}{\sqrt{1 - \mu^2}} d\mu \frac{T_\alpha(y) T_\beta(z)}{\sqrt{(1 - y^2)(1 - z^2)}} S(x, y, z, \mu, \phi) dz dy \tag{5.8}$$

$$A_i^{\alpha} = \frac{2 - \delta_{\alpha,0}}{\pi} \int_{-1}^{1} \frac{d}{dy}(T_\alpha(y)) \frac{T_i(y)}{\sqrt{1 - y^2}} dy \tag{5.9}$$

$$B_j^{\beta} = \frac{2 - \delta_{\beta,0}}{\pi} \int_{-1}^{1} \frac{d}{dz}(T_\beta(z)) \frac{T_j(z)}{\sqrt{1 - z^2}} dz. \tag{5.10}$$

The corresponding discrete ordinates equation [19] is then

$$\mu_m \frac{\partial \Psi_{\alpha,\beta}}{\partial x}(x,\mu_m,\phi_m) + \sigma_t \Psi_{\alpha,\beta}(x,\mu_m,\phi_m) = \sum_{n=1}^{M} \omega_n \Psi_{\alpha,\beta}(x,\mu_m,\phi_m) + G_{\alpha,\beta}(x;\mu_m,\phi_m) \tag{5.11}$$

and also we expand $\Psi_{\alpha,\beta}(x,\mu_m,\phi_m)$ in a truncated series of Chebyshev polynomials i.e.

$$\Psi_{\alpha,\beta}(x,\mu_m,\phi_m) = \sum_{k=0}^{M} \frac{C_k(x,\phi_m)T_k(\mu_m)}{\sqrt{1-\mu_m^2}} \tag{5.12}$$

bringing the equation (5.12) in equation (5.11) to get

$$\mu_m \frac{\partial}{\partial x}\left[\sum_{k=0}^{M} \frac{C_k(x,\phi_m)T_k(\mu_m)}{\sqrt{1-\mu_m^2}}\right] + \sigma_t \left[\sum_{k=0}^{M} \frac{C_k(x,\phi_m)T_k(\mu_m)}{\sqrt{1-\mu_m^2}}\right] =$$

$$\sum_{n=1}^{M} \omega_n \left[\sum_{k=0}^{M} \frac{C_k(x,\phi_m)T_k(\mu_m)}{\sqrt{1-\mu_m^2}}\right] + G_{\alpha,\beta}(x;\mu_m,\eta_m) \tag{5.13}$$

with

$$G_{\alpha,\beta}(x;\mu_m,\eta_m) = S_{\alpha,\beta}(x,\mu,\phi) - \sqrt{1-\mu^2}$$

$$\times \left[\cos\phi \sum_{\alpha=i+1}^{I} A_i^{\alpha} \sum_{k=0}^{M} \frac{C_k(x,\phi_m)T_k(\mu_m)}{\sqrt{1-\mu_m^2}} + \sin\phi \sum_{\beta=j+1}^{J} B_j^{\beta} \sum_{k=0}^{M} \frac{C_k(x,\phi_m)T_k(\mu_m)}{\sqrt{1-\mu_m^2}}\right], \tag{5.14}$$

multiply the equation (5.13) par $T_l(\mu_m)$ and integrate over $\mu_m \in [-1,1]$ we find

$$\mu_m \frac{\partial}{\partial x} \sum_{k=0}^{M} C_k(x,\phi_m) \int_{-1}^{1} \frac{T_k(\mu_m)T_l(\mu_m)}{\sqrt{1-\mu_m^2}} d\mu_m + \sigma_t \sum_{k=0}^{M} C_k(x,\phi_m) \int_{-1}^{1} \frac{T_k(\mu_m)T_l(\mu_m)}{\sqrt{1-\mu_m^2}} d\mu_m =$$

$$\sum_{n=0}^{M} \omega_n \sum_{k=0}^{M} C_k(x,\phi_m) \int_{-1}^{1} \frac{T_k(\mu_m)T_l(\mu_m)}{\sqrt{1-\mu_m^2}} d\mu_m + \int_{-1}^{1} G_{\alpha,\beta}(x;\mu_m,\eta_m)T_l(\mu_m)d\mu_m \tag{5.15}$$

with

$$\int_{-1}^{1} G_{\alpha,\beta}(x;\mu_m,\eta_m)T_l(\mu_m)d\mu_m = \int_{-1}^{1} S_{\alpha,\beta}(x,\mu_m,\phi_m)T_l(\mu_m)d\mu_m - \cos\phi_m \sqrt{1-\mu_m^2} \sum_{i=\alpha+1}^{I} A_i^{\alpha}$$

$$\times \sum_{k=0}^{M} C_k(x,\phi_m) \int_{-1}^{1} \frac{T_k(\mu_m)T_l(\mu_m)}{\sqrt{1-\mu_m^2}}\ d\mu_m + \sin\phi_m \sqrt{1-\mu_m^2} \sum_{j=\beta+1}^{J} B_j^{\beta}$$

$$\times \sum_{k=0}^{M} C_k(x,\phi_m) \int_{-1}^{1} \frac{T_k(\mu_m)T_l(\mu_m)}{\sqrt{1-\mu_m^2}}\ d\mu_m \tag{5.16}$$

where

$$A_i^{\alpha} = \frac{2-\delta_{\alpha,0}}{\pi} \int_{-1}^{1}\int_{-1}^{1} \frac{d}{dy}(T_{\alpha}(y)) \frac{T_i(y)}{\sqrt{1-y^2}} T_l(\mu_m) dy d\mu_m \tag{5.17}$$

$$B_j^{\beta} = \frac{2-\delta_{\beta,0}}{\pi} \int_{-1}^{1}\int_{-1}^{1} \frac{d}{dz}(T_{\beta}(z)) \frac{T_j(z)}{\sqrt{1-z^2}} T_l(\mu_m) dz d\mu_m \tag{5.18}$$

by using the properties of Chebyshev polynomials to equation (5.16) to get

$$\int_{-1}^{1} G_{\alpha,\beta}(x;\mu_m,\eta_m)T_l(\mu_m)d\mu_m =$$

$$\int_{-1}^{1} S_{\alpha,\beta}(x,\mu_m,\phi_m)T_l(\mu_m)d\mu_m - \left[\frac{\pi}{2-\delta_{m,0}}\sqrt{1-\mu_m^2}C_k(x,\phi_m)\right]$$

$$\times \left[\sum_{i=\alpha+1}^{I} A_i^{\alpha}\cos\phi_m + \sum_{j=\beta+1}^{J} B_j^{\beta}\sin\phi_m\right] \tag{5.19}$$

then the equation (5.11) becomes

$$\mu_m \frac{\partial C_m}{\partial x} + \left[\sigma_t - \sum_{m=0}^{M}\sqrt{1-\mu_m^2}\left(\sum_{i=\alpha+1}^{I} A_i^{\alpha}\cos\phi_m - \sum_{j=\beta+1}^{J} B_j^{\beta}\sin\phi_m\right)\right] C_m$$

$$= \frac{\pi}{2-\delta_{m,0}} \int_{-1}^{1} S_{\alpha,\beta}(x,\mu_m,\phi_m)T_l(\mu_m)d\mu_m. \tag{5.20}$$

after written in vector and matrix notation and regrouping the coefficients C_m together in Eq. (5.13), we can derive the following differential equation

$$\frac{\partial C_m}{\partial x} + DC_m = E_m \tag{5.21}$$

where $D_m = \frac{1}{\mu_m} B_m$ and $E_m = \frac{1}{\mu_m} A_m$ with

$$A_m := \frac{\pi}{2 - \delta_{m,0}} \int_{-1}^{1} S_{\alpha,\beta}(x, \mu_m, \phi_m) T_l(\mu_m) d\mu_m \tag{5.22}$$

$$B_m := \left[\sigma_t - \sum_{m=0}^{M} \sqrt{1 - \mu_m^2} \left(\sum_{i=\alpha+1}^{I} A_i^{\alpha} \cos\phi_m - \sum_{j=\beta+1}^{J} B_j^{\beta} \sin\phi_m \right) \right] \tag{5.23}$$

the solution of differential equation for the vector C_m is thus constructed as follows:

$$C_m(x) = e^{-Dx} C_m(0) - \int_0^x e^{-(x-\xi)D} E_m(\xi) d\xi \tag{5.24}$$

equation (5.24) depend on vector $C_m(0)$. Having established an analytical formulation for the exponential appearing in equation (5.24), the unknown components of vector $C_m(0)$ for the boundary problem (5.1) can be readily obtained applying the boundary conditions (5.2), (5.3) and (5.4).

To derive an analytical formulation for the exponential of matrix D, appearing in equation (5.24), we use the Sumudu transform.

5.1 Study of the Spectral Approximation

now we expand

$$\Psi_{\alpha,\beta,N}(x, \phi_m) = \sum_{m=0}^{(N)} C_m^{(N)} \cos(m\phi_m) \tag{5.25}$$

where $C_m^{(N)}$ is the approximation to the coefficient C_m by the consideration of the truncated series $\Psi_{\alpha,\beta,N}$.

From spectral analysis, we know that when a function is infinitely smooth and all its derivatives exist, then the coefficients appearing in its sine or cosine series go to zero faster than $1/n$. Moreover, if the function and all its derivatives are periodic, then the decay is faster than any power of $1/n$.

However, as indicated by Canuto *et al.* (1988) [15], in practice this decay cannot be observed before enough coefficients that represent the essential structures of the function are considered.

In the calculation, one can test the convergence of the cosine truncated series defined in equation (5.25) by evaluating

$$\sup_k \left[\frac{| \Psi_{N+1}(k) - \Psi_N(k) |}{\Psi_N(k)} \right] \leq \epsilon \tag{5.26}$$

where ϵ is the required precision. In general, the few first coefficients of the series are enough to generate the angular flux.

If N is the chosen value, we can write

$$C_m^{(N)} = 0 \qquad \text{for all } n > N, \tag{5.27}$$

Combining therefore equations (5.27) and (5.24) we shall now describe the necessary algorithm to obtain all the cosine coefficients $C_m^{(N)}$

Step 0: $N = 0$; for $n = N = 0$

$$C_0^{(0)}(x) = e^{-A^{-1}Bx} C_0^{(0)}(0) - \int_0^x e^{-A^{-1}B(x-s)} A_0(x) dx, \tag{5.28}$$

with

$$A_0 := \pi \int_{-1}^{1} S_{\alpha,\beta}(x, \mu_0, \phi_0) T_l(\mu_0) d\mu_0 \tag{5.29}$$

which is well known, and thus $C_0^{(0)}(x)$ is completely determined. To finish the step, we apply equation (5.25) to obtain the first approximation to the angular flux, i.e., Ψ_0.

Step 1: $N = 1$; for $n = 0$,

$$C_0^{(1)}(x) = e^{-A^{-1}Bx} C_0^{(1)}(0) - \int_0^x e^{-A^{-1}B(x-s)} A_1(x) dx, \tag{5.30}$$

with

$$A_1 := \frac{\pi}{2} \int_{-1}^{1} S_{\alpha,\beta}(x, \mu_1, \phi_1) T_l(\mu_1) d\mu_1 \tag{5.31}$$

for $n = 1$

$$C_1^{(1)}(x) = e^{-A^{-1}Bx} C_1^{(1)}(0) - \int_0^x e^{-A^{-1}B(x-s)} A_1(x) dx, \tag{5.32}$$

with

$$A_1 := \frac{\pi}{2} \int_{-1}^{1} S_{\alpha,\beta}(x, \mu_1, \phi_1) T_l(\mu_1) d\mu_1 \tag{5.33}$$

Bringing the approximated solution for $C_0^{(0)}$ obtained at step 0 inside equation (5.32) and iterating with equation (5.28), we obtain immediately the approximated coefficients $C_0^{(1)}$and $C_1^{(1)}$. To finish the step, we evaluate through equation (5.25) the new approximation Ψ_1 and perform the precision condition defined in equation (5.26). If equation is verified, the calculation is stopped; if not, we go to step 2 and to likewise until the convergence condition in equation (5.26) is fulfilled.

With the above algorithm, we only need knowledge of the operator $e^{-A^{-1}Bx}$ (the problem was solved previously by using the Sumudu transform.)

Chapter 6

Conclusion

The Chebyshev spectral method combined with Sumudu transform should be general enough to consider higher spatial dimensions in a way similar to that presented in this paper, although we have not investigated this idea thoroughly. We will be considering more complicated geometries in future studies, during which we will ascertain this method's usefulness for larger spatial dimensional problems. In preparation for these problems, we are currently investigating the effectiveness of spectral methods combined with Sumudu transform in solving the linear system of differential equation analytically.

An adaptation of the method for the convergence of the spectral solution within the framework of the analytical solution to study and prove convergence by using the discrete ordinates method is relatively new. The methods employing Sumudu transforms represent very interesting new ideas for studying the convergence of many numerical methods and can be extended easily to general linear transport problems. In fact only some preliminary results have been obtained. In this context we intend to study the existence and uniqueness of its solution by using C_0 *semigroup approach*. Our attention will be focused in this direction.

Part III

CHEBYSHEV SPECTRAL-S_N METHOD FOR THE NEUTRON TRANSPORT EQUATION

Chapter 7

Introduction

In this part we develop spectral approximation for two and three dimensional, steady state, linear transport equation with isotropic scattering, in bounded domain. The procedure is based on the expansion of the angular flux in a truncated series of Chebyshev polynomials in the spatial variables. We study the convergence of this method in two dimensional case, where we use a special quadrature rule to discretise in the angular variables, approximating the scalar flux. The similarity of the spectral method to the finite element method is evident: the bases functions have a constant norm and the procedure is to represent the approximate solution as a linear combination of finite number of basis functions (truncated series of Chebyshev polynomials) and then use a variational formulation. The main difference is that: the finite element bases functions are locally supported, whereas the Chebyshev polynomials are global functions. See also [12] for further details.

In [47] this approach, with no convergence rate analysis, is considered for a truncated series of general orthogonal polynomials. The detailed study in [47] is carried out for the Legendre polynomials, where an index mix caused that a significant drift term is argued to be of lower and therefore its contribution is not included in the estimates.

We apply this procedure using Chebyshev polynomials with, e.g., the advantage of having constant weighted-L_2 norms, and give a full convergence study including estimates of the contribution from the whole drift term. The final estimations via an inverse iterative/induction argument, based on an estimate derived from some elementary properties of Chebyshev polynomials in Appendix I. In our knowledge convergence rate analysis, in this setting, is not considered in the literature.

Related problems, in different setting, are studied in the nuclear engineering literature, see, e.g., reference in Vilhena et al [47]. Barros and Larsen [6] carried out a spectral nodal method for certain discrete-ordinates problems. Chebyshev spectral methods for radiative transfer problems are studied, e.g., by Kim and Ishimaru in [34] and by Kim and Moscoso [35]. In, e.g., astrophysical aspects, spectral methods are considered for relativistic gravitation and gravitational radiation by Bonazzola et al [12]. A multi-domain spectral method is studied by Grangclément et al [29], for scalar and vectorial Poisson equation. C++ software library, developed for multi-domain, is available in public domain (GPL), **http://www.lorene.obspm.fr.** For more detailed study on Chebyshev spectral method and also approximations by the spectral methods we refer the reader to monographs by Boyd [13] and Bernardi and Maday [11].

An outline of this part is as follows: In Section 8 we derive the truncated spectral equations in 2 dimensions. In Section 9 we prove that a certain weighted-L_2 norm for the error in the discrete-ordinates approximation of the spectral solution is dominated by that of a quadrature approximation. In Section 10 we construct a special quadrature rule and derive convergence rates for the quadrature error. Combining the results of Section 9 and 10, we conclude the convergence of the discrete-ordinates for the spectral method. Appendix I is devoted to certain properties of the Chebyshev polynomials, that are frequently used in the paper, and also the proof of a crucial estimate used in the approximation of the contribution from the drift term. Finally in Appendix II we derive the spectral equations in a three dimensional setting.

Chapter 8

The Two-Dimensional Spectral Solution.

Consider the two-dimensional linear, steady state, transport equation given by

$$\mu\frac{\partial}{\partial x}\Psi(x,y,\mu,\phi)+\sqrt{1-\mu^2}\cos\phi\frac{\partial}{\partial y}\Psi(x,y,\mu,\phi)+\sigma_t\Psi(x,y,\mu,\phi)$$

$$=\int_{-1}^{1}\int_{0}^{2\pi}\sigma_s(\mu^{'},\phi^{'}\rightarrow\mu,\phi)\Psi(x,y,\mu^{'},\phi^{'})d\phi^{'}d\mu^{'}+S(x,y,\mu,\phi) \qquad (8.1)$$

in the rectangular domain $\Omega=\{\mathbf{x}:=(x,y):\quad -1\leq x\leq 1,\quad -1\leq y\leq 1\}$ and the direction in $D=\{(\mu,\theta):\quad -1\leq\mu\leq 1,\quad 0\leq\theta\leq 2\pi\}$. Here $\Psi(\mathbf{x},\mu,\phi)$ is the angular flux, σ_t and σ_s denote the total and the differential cross section, respectively, $\sigma_s(\mu^{'},\phi^{'}\rightarrow\mu,\phi)$ describes the scattering from an assumed pre-collision angular coordinates $(\mu^{'},\theta^{'})$ to a post-collision coordinates (μ,θ) and S is the source term. See [39] for the details.

Note that, in the case of one-speed neutron transport equation; taking the angular variable in a disc, this problem would corresponds to a three dimensional case with all functions being constant in the azimuthal direction of the z variable. In this way the actual spatial domain may be assumed to be a cylinder with the cross-section Ω and the axial symmetry in z. Then D will correspond to the projection of the points on the unit sphere (the "speed') onto the unit disc (which coincides with D.) See, [2] for the details.

Given the functions $f_1(y, \mu, \phi)$ and $f_2(x, \mu, \phi)$, describing the incident flux, we seek for a solution of (8.1) subject to the following boundary conditions:

For $0 \leq \theta \leq 2\pi$, let

$$\Psi(x = \pm 1, y, \mu, \theta) = \begin{cases} f_1(y, \mu, \phi), \ x = -1, & 0 < \mu \leq 1, \\ 0, \ x = 1, & -1 \leq \mu < 0. \end{cases} \tag{8.2}$$

For $-1 < \mu < 1$, let

$$\Psi(x, y = \pm 1, \mu, \theta) = \begin{cases} f_2(y, \mu, \phi), \ y = -1, & 0 < \cos\theta \leq 1, \\ 0, \ y = 1, & -1 \leq \cos\theta < 0. \end{cases} \tag{8.3}$$

Expanding the angular flux $\Psi(\mathbf{x}, \mu, \theta)$ in terms of the Chebyshev polynomials in the y variable, leads to

$$\Psi(\mathbf{x}, \mu, \theta) = \sum_{i=0}^{I} \Psi_i(x, \mu, \theta) T_i(y). \tag{8.4}$$

Below we determine the first component, i.e., $\Psi_0(x, \mu, \theta)$ explicitly, whereas the other components, $\Psi_i(x, \mu, \theta)$, $i = 1, ...I$, will appear as the unknowns in I one dimensional transport equations: We start to determine $\Psi_0(x, \mu, \theta)$, by inserting (8.4) into the boundary conditions (8.3) at $y = \pm 1$, to find that:

$$\Psi_0(x, \mu, \theta) = f_2(x, \mu, \phi) - \sum_{i=1}^{I} (-1)^i \Psi_i(x, \mu, \theta), \qquad 0 < \cos\theta \leq 1, \tag{8.5}$$

$$\Psi_0(x, \mu, \theta) = -\sum_{i=1}^{I} \Psi_i(x, \mu, \theta), \qquad -1 \leq \cos\theta < 0. \tag{8.6}$$

where $-1 \leq x \leq 1$, $-1 < \mu < 1$, and we have used the fact that for the Chebyshev polynomials $T_0(x) \equiv 0$, $T_i(1) \equiv 1$ and $T_i(-1) \equiv (-1)^i$. See Appendix I.

If we now insert Ψ from (8.4) into (8.1), multiply the resulting equation by $\frac{T_k(y)}{\sqrt{1-y^2}}$, $k = 1, ..., I$, and integrate over y we find that the components $\Psi_k(x, \mu, \theta)$, $k = 1, ..., I$, satisfy the following I one-dimensional equations:

$$\mu\frac{\partial}{\partial x}\Psi_k(x,\mu,\theta)+\sigma_t\Psi_k(x,\mu,\theta)$$

$$\int_{-1}^{1}\int_{0}^{2\pi}\sigma_s(\mu^{'},\phi^{'}\to\mu,\phi)\Psi_k(x,\mu^{'},\phi^{'})d\theta^{'}d\mu^{'}+G_k(x,\mu,\theta) \tag{8.7}$$

The same procedure with the boundary condition (8.2) at $x=-1$, and (8.4) yields

$$\Psi(-1,y,\mu,\theta)=f_1(y,\mu,\phi)=\sum_{i=0}^{I}\Psi_i(-1,\mu,\theta)T_i(y). \tag{8.8}$$

Now multiply (8.8) by $\frac{T_k(y)}{\sqrt{1-y^2}}$, $k=1,...,I$, and integrate over y we find that

$$\Psi_k(-1,\mu,\theta)=\frac{2}{\pi}\int_{-1}^{1}f_1(y;\mu,\theta)\frac{T_k(y)}{\sqrt{1-y^2}}dy. \tag{8.9}$$

Similarly, (note the sign of μ below), the boundary condition at $x=1$ is written as

$$\sum_{i=0}^{I}\Psi_i(1,-\mu,\theta)T_i(y)=0 \qquad 0<\mu\leq 1. \tag{8.10}$$

Multiplying (8.10) by $\frac{T_k(y)}{\sqrt{1-y^2}}$, $k=1,...,I$ and integrating over y, we get

$$\Psi_k(1,-\mu,\theta)=0 \qquad 0<\mu\leq 1, \qquad 0\leq\theta\leq 2\pi. \tag{8.11}$$

We can easily check that G_k in (8.7) is written as

$$G_k(x,\mu,\theta)=S_k(x,\mu,\theta)-\sqrt{1-\mu^2}\cos\theta\sum_{i=k+1}^{I}A_i^k\Psi_k(x,\mu,\theta) \tag{8.12}$$

where

$$A_i^k=\frac{2}{\pi}\int_{-1}^{1}\frac{d}{dy}(T_i(y))\frac{T_k(y)}{\sqrt{1-y^2}}dy \tag{8.13}$$

and

$$S_k(x,\mu,\theta) = \frac{2}{\pi}\int_{-1}^{1} S(x,y,\mu,\theta)\frac{T_k(y)}{\sqrt{1-y^2}}dy. \tag{8.14}$$

Note that the solutions to the one-dimensional problems given through the equation (8.7)-(8.14) define the components $\Psi_k(x,\mu,\theta)$, for $k = I,...,1$, in this decreasing order to avoid the coupling of the equations. Once this is done, the angular flux given by (8.4) is completely determined. Here we have used the convention $\sum_{i=I+1}^{I} ... = 0$. Hence the starting $G_I(x,\mu,\theta) \equiv S_I(x,\mu,\theta)$. Note also that although the solution, developed in here, rely on specific boundary conditions the procedure is quite general in the sense that the expression for the first component, $\Psi_0(x,\mu,\theta)$, keeps the information from the boundary conditions in the y variable, while the other components are derived based on the boundary conditions in x.

Chapter 9

Convergence of the Spectral Solution.

In the sequel we focus on the two dimensional, steady state linear transport process with isotropic scattering, i.e., $\sigma_s(\mu', \phi' \to \mu, \phi) \equiv \sigma_s =$ constant. For this problem we show, using a weighted-L_2 norm, convergence of the spectral solution defined for the spatial variables. More specifically we show that: in a certain weighted-L_2 norm, the (truncated) *discrete ordinate* approximation error for the spectral solution is dominated by that of a special quadrature approximation error. The study of convergence of this quadrature approximation is the matter of the next section.

Assuming isotropic scattering, the equation (8.1) is written as

$$\mu \frac{\partial}{\partial x} \Psi(\mathbf{x}, \mu, \theta) + \sqrt{1-\mu^2} \cos\theta \frac{\partial}{\partial y} \Psi(\mathbf{x}, \mu, \theta) + \sigma_t \Psi(\mathbf{x}, \mu, \theta)$$

$$= \sigma_s \int_{-1}^{1} \int_{0}^{2\pi} \Psi(\mathbf{x}, \mu', \theta') d\theta' d\mu' + S(\mathbf{x}, \mu, \theta) \tag{9.1}$$

for $\mathbf{x} \in \Omega := \{(x, y): \ -1 \leq x \leq 1, \ \ -1 \leq y \leq 1\}$, $\mu \in [-1, 1]$ and $\theta \in [0, 2\pi]$. the study of the problem with the anisotropic scattering is a rather involved task. See, e.g., [4] for an approach involving anisotropic scattering. Consider now the discrete ordinate

(S_N) approximation of the equation (9.1): for $m = 1, ..., M$, let

$$\mu_m \frac{\partial}{\partial x}\Psi_m(\mathbf{x}) + \eta_m \frac{\partial}{\partial y}\Psi_m(\mathbf{x}) + \sigma_t \Psi_m(\mathbf{x}) = \sigma_s \sum_{n=1}^{M} \omega_n \Psi_n(\mathbf{x}) + S_m(\mathbf{x}), \tag{9.2}$$

where

$$\eta_m = \sqrt{1 - \mu_m^2}\cos\theta_m \tag{9.3}$$

and $\Psi_m(\mathbf{x}) := \Psi_m(x, y)$ is the angular flux in the directions defined by μ_m and η_m and associated with the quadrature weights ω_m. Finally $S_m(\mathbf{x})$ is the corresponding inhomogeneous source term defined in the discrete directions $(\mu_m, \eta_m) \in [-1, 1]^2$.

We assume a quadrature mesh $(\mu_m, \eta_m) \neq (0, 0)$,

$$\begin{cases} \mu_1 < \mu_2 < .... < \mu_M, \\ \eta_1 < \eta_2 < ... < \eta_{M,} \end{cases} \tag{9.4}$$

satisfying the following conditions:

$$\omega_m \sim 4\pi/M, \qquad \sum_{m=1}^{M} \omega_m \sim 4\pi, \qquad m = 1, ..., M \tag{9.5}$$

Further, we assume that the discrete-ordinates equation (9.2) satisfy the same boundary conditions, in the discrete directions, as the continuous one, i.e., (9.1) (as stated in Section 8). We shall prove that, under certain assumptions, the solution of the equation (9.2) would converge to that of the equation (9.1) as $N \to \infty$.

To this approach we define the error in the *approximate flux* by

$$\epsilon_m(\mathbf{x}) = \Psi(\mathbf{x}, \mu_m, \eta_m) - \Psi_m(\mathbf{x}), \qquad m = 1, ..., M, \tag{9.6}$$

and the *truncation error in the quadrature formula* as

$$\tau(\mathbf{x}) = \int_{-1}^{1}\int_{0}^{2\pi} \Psi(\mathbf{x}; \mu', \theta')d\mu' d\theta' - \sum_{n=1}^{M} \omega_n \Psi(\mathbf{x}, \mu_m, \eta_m). \tag{9.7}$$

Subtracting the discrete-ordinates equation (9.2) from the continuous equation (9.1) in the discrete directions, for each $m = 1, ..., M$, an equation relating the discrete-ordinates approximation error to the quadrature error, viz,

$$\mu_m \frac{\partial \epsilon_m(\mathbf{x})}{\partial x} + \eta_m \frac{\partial \epsilon_m(\mathbf{x})}{\partial y} + \sigma_t \epsilon_m(\mathbf{x}) = \sigma_s \sum_{n=1}^{M} \omega_n \epsilon_m(\mathbf{x}) + \sigma_s \tau(\mathbf{x}). \tag{9.8}$$

We expand both the approximation and the quadrature errors in a truncated series of Chebyshev polynomials in y,

$$\epsilon_m(x, y) = \sum_{l=0}^{L} \epsilon_m^l(x) T_l(y), \tag{9.9}$$

$$\tau(x, y) = \sum_{l=0}^{L} \tau^l(x) T_l(y) \tag{9.10}$$

and define the $l-th$ moments of the errors by

$$\left\| \epsilon^l \right\| = \left[\frac{2 - \delta_{l,0}}{\pi} \int_{-1}^{1} \sum_{m=1}^{M} \omega_m (\epsilon_m^l(x))^2 dx \right]^{1/2} \tag{9.11}$$

$$\left\| \tau^l \right\| = \left[\frac{2 - \delta_{l,0}}{\pi} \int_{-1}^{1} (\tau^l(x))^2 dx \right]^{1/2}. \tag{9.12}$$

Remark. Note that (9.9) and (9.10) involve further, truncated, approximations of $\tau(\mathbf{x})$, in (9.7) and the solution $\epsilon(\mathbf{x})$ of (9.6). We keep using the same notation as before the truncation. Also, despite the recent truncation in y, we use equalities in (9.9), (9.10), as well as in the subsequent relation below.

The main result of this paper is as follows:

Theorem 9.1. *Let $L = \mathcal{O}(\sigma)$ where $\sigma = \sigma_t - 4\pi\sigma_s$, then for $l = 0, 1, ..., L$*

$$\left\| \epsilon^l \right\| \to 0, \qquad as \quad M \to \infty.$$

In the remaining part of this section we show that, for $\omega_m \sim 4\pi/M$, $m = 1, ..., M$

the L_2 norm of the truncated spectral error $\left\|\epsilon^l\right\|$, counted in a reverse order on $l = L, L-1, ..., 0$, is dominated by that quadrature error $\left\|\tau^l\right\|$.

The next section is devoted to proof of the following result:

Theorem 9.2. *For $\omega_m \sim 4\pi/M$, $m = 1, ..., M$, if $\Psi \in L_2(\mu, \theta)$, then*

$$\left\|\tau^l\right\| \to 0, \qquad \text{as} \quad M \to \infty.$$

To prepare for the proof of the Theorem 9.1, we substitute (9.9) and (9.10) into the equation (9.8) to get

$$\mu_m \sum_{l=0}^{L} \frac{d\epsilon_m^l(x)}{dx} T_l(y) + \eta_m \sum_{l=0}^{L} \epsilon_m^l(x) \frac{dT_l}{dy}(y) + \sigma_t \sum_{l=0}^{L} \epsilon_m^l(x) T_l(y)$$

$$= \sigma_s \sum_{n=1}^{M} \omega_n \sum_{l=0}^{L} \epsilon_n^l(x) T_l(y) + \sigma_s \sum_{l=0}^{L} \tau^l(x) T_l(y), \tag{9.13}$$

Multiplying (9.13) by $\frac{T_j(y)}{\sqrt{1-y^2}}$, $j = 0, ..., L$ and integrating over y yields

$$\frac{\pi}{2-\delta_{j,0}} \mu_m \frac{d\epsilon_m^j(x)}{dx} + \eta_m \sum_{l=0}^{L} \gamma_j(l) \epsilon_m^l(x) + \frac{\pi}{2-\delta_{j,0}} \sigma_t \epsilon_m^j(x) =$$

$$\frac{\pi}{2-\delta_{j,0}} \sigma_s \sum_{n=1}^{M} \omega_n \epsilon_m^l(x) + \frac{\pi}{2-\delta_{j,0}} \sigma_s \tau^j(x), \tag{9.14}$$

where

$$\gamma_j(l) = \int_{-1}^{1} \frac{dT_l(y)}{dy}(y) \frac{T_j(y)}{\sqrt{1-y^2}} dy. \tag{9.15}$$

Finally, we multiply the equation (9.14) by $\epsilon_m^j(x)$ and integrate over x to obtain

$$\frac{\pi}{2-\delta_{j,0}} \mu_m \int_{-1}^{1} \epsilon_m^j(x) \frac{d\epsilon_m^j(x)}{dx} dx + \eta_m \sum_{l=0}^{L} \gamma_j(l) \int_{-1}^{1} \epsilon_m^j(x) \epsilon_m^l(x) dx$$

$$+ \frac{\pi}{2-\delta_{j,0}} \sigma_t \int_{-1}^{1} \left[\epsilon_m^j(x)\right]^2 dx \tag{9.16}$$

$$= \frac{\pi}{2-\delta_{j,0}} \sigma_s \sum_{n=1}^{M} \omega_n \int_{-1}^{1} \epsilon_m^j(x) \epsilon_n^j(x) dx + \frac{\pi}{2-\delta_{j,0}} \sigma_s \int_{-1}^{1} \epsilon_m^j(x) \tau^j(x) dx$$

Now we rewrite the first term in equation (9.16) as

$$\mu_m \int_{-1}^{1} \epsilon_m^j(x) \frac{d\epsilon_m^j(x)}{dx} dx = \frac{\mu_m}{2} \left[(\epsilon_m^j(-1))^2 - (\epsilon_m^j(1))^2 \right]. \tag{9.17}$$

Note that $(\mu_m(\epsilon_m^j(1))^2 - (\epsilon_m^j(-1))^2) > 0$. Indeed, for $\mu_m > 0$, using the boundary condition $\epsilon_m(-1, y) = 0$ and the identity

$$\epsilon_m^j(x) = \frac{2-\delta_{j,0}}{\pi} \int_{-1}^{1} \epsilon_m(x,y) T_j(y) \frac{1}{\sqrt{1-y^2}} dy, \tag{9.18}$$

we find that $\epsilon_m^j(-1) = 0$. The same is valid for $x = 1$, when $\mu_m < 0$. Consequently,

$$\frac{2-\delta_{j,0}}{\pi} \eta_m \sum_{l=0}^{L} \gamma_j(l) \int_{-1}^{1} \epsilon_m^j(x) \epsilon_m^l(x) dx + \sigma_t \int_{-1}^{1} \left[\epsilon_m^j(x) \right]^2 dx$$

$$\leq \sigma_s \sum_{n=1}^{M} \omega_n \int_{-1}^{1} \epsilon_m^j(x) \epsilon_n^j(x) dx + \sigma_s \int_{-1}^{1} \epsilon_m^j(x) \tau^j(x) dx \tag{9.19}$$

To proceed we multiply the inequality (9.19) by ω_m and sum over m to obtain

$$\sigma_t \int_{-1}^{1} \sum_{m=1}^{M} \omega_m \left[\epsilon_m^j(x) \right]^2 dx \leq \sigma_s \int_{-1}^{1} \left[\sum_{n=1}^{M} \omega_n \epsilon_m^j(x) \right]^2 dx$$

$$+ \sigma_s \int_{-1}^{1} \left[\sum_{m=1}^{M} \omega_m \epsilon_m^j(x) \right] \tau^j(x) dx \tag{9.20}$$

$$- \frac{2-\delta_{j,0}}{\pi} \sum_{m=1}^{M} \omega_m \left[\eta_m \sum_{l=0}^{L} \gamma_j(l) \int_{-1}^{1} \epsilon_m^j(x) \epsilon_m^l(x) dx \right]$$

$$:= I + II + III.$$

The crucial part is now to estimate the $\gamma-$term III using the elementary properties of the Chebyshev polynomials. We start with the simpler terms I and II :

Lemma 9.3. *With* $\omega_m \sim 4\pi/M$, $m = 1, ..., M$, *we have, for* $j = 0, ..., L$, *that*

$$| I |\leq 4\pi\sigma_s \frac{2-\delta_{j,0}}{\pi} \left\| \epsilon^j(x) \right\|^2$$

$$| II |\leq \sqrt{4\pi}\sigma_s \frac{2-\delta_{j,0}}{\pi} \left\| \epsilon^j(x) \right\| \left\| \tau^j \right\| \tag{9.21}$$

Proof. We use the elementary relation

$$(a_1 + a_2 + ... + a_M)^2 \leq M \left(a_1^2 + a_2^2 + ... + a_M^2 \right),$$

to write

$$\left[\sum_{m=1}^{M} \omega_m \epsilon_m^j(x) \right]^2 \leq M \max_{1\leq m \leq M} | \omega_m | \sum_{m=1}^{M} \omega_m \left[\epsilon_m^j(x) \right]^2. \tag{9.22}$$

integrating (9.22) over x and using $\omega_m \sim 4\pi/M$ we get

$$\int_{-1}^{1} \left[\sum_{m=1}^{M} \omega_m \epsilon_m^j(x) \right]^2 dx \leq 4\pi \int_{-1}^{1} \sum_{m=1}^{M} \omega_m \left[\epsilon_m^j(x) \right]^2 dx, \tag{9.23}$$

and hence the first estimate follows recalling (9.11). As for the second estimate, applying the Cauchy-Schwarz inequality, (9.23), (9.11) and (9.12) we get

$$\int_{-1}^{1} \left[\sum_{m=1}^{M} \omega_m \epsilon_m^j(x) \right] \tau^j(x) dx$$

$$\leq \left(\int_{-1}^{1} \left[\sum_{m=1}^{M} \omega_m \epsilon_m^j(x) \right]^2 dx \right)^{1/2} \times \left(\int_{-1}^{1} \left| \tau^j(x) \right|^2 dx \right)^{1/2} \tag{9.24}$$

$$\sqrt{4\pi} \left[\int_{-1}^{1} \sum_{m=1}^{M} \omega_m \left(\epsilon_m^j(x) \right)^2 dx \right]^{1/2} \times \sqrt{\frac{\pi}{2-\delta_{j,0}}} \left\| \tau^j \right\|$$

$$\leq \sqrt{4\pi} \frac{\pi}{2-\delta_{j,0}} \left\| \epsilon^j \right\| \left\| \tau^j \right\|,$$

which gives the desired estimate for II and the proof is complete.

Next using the proposition 3 from the Appendix I we estimate the contribution from the γ-term III and derive the following key estimate:

Proposition 9.4. *For $k = 0, 1, 2, ..., L$, we have the recursive estimates*

$$\left\|\epsilon^{L-k}\right\| \leq \sum_{j=0}^{k} \frac{\left(1-(-1)^{j+k}\right)}{\sigma}(L-j)\left\|\epsilon^{L-j}\right\| + \frac{\sqrt{4\pi}\sigma_s}{\sigma}\left\|\epsilon^{L-k}\right\|. \tag{9.25}$$

Hence, in particular the starting estimate, for $k = 0$, is:

$$\left\|\epsilon^{L}\right\| \leq \frac{\sqrt{4\pi}\sigma_s}{\sigma}\left\|\tau^{L}\right\|. \tag{9.26}$$

With these estimates we can easily prove our main result:

Proof of Theorem 9.1. Proposition 9.4 and Theorem 9.2 give the desired result.

Proof of Theorem 9.4. By the Proposition 3 (see Appendix I) we have that

$$\gamma_j(l) = 0, \qquad \text{for} \quad j \geq l, \tag{9.27}$$

whereas for $j \leq l$,

$$\gamma_j(l) = \begin{cases} 0 & \text{for } j+l \quad \text{even} \\ l\pi & \text{for } j+l \quad \text{odd.} \end{cases} \tag{9.28}$$

Therefore if we start with $j = L$, then $\gamma_j(L) = 0$ and hence (9.20) combined with the definition (9.11) and Lemma 9.3 yields

$$\sigma_t\frac{\pi}{2}\left\|\epsilon^{L}\right\|^2 \leq 4\pi\sigma_s\frac{\pi}{2}\left\|\epsilon^{L}\right\|^2 + \sqrt{4\pi}\left\|\epsilon^{L}\right\|\left\|\tau^{L}\right\|. \tag{9.29}$$

Now rearranging the terms and recalling that $\sigma := \sigma_t - 4\pi\sigma_s$ we obtain (9.26).

The proof of (9.25) is a reversed inductive argument as follows:

For $j = L-1$ we have that $\gamma_j(L) = \gamma_{L-1}(L) = L\pi$, whereas $\gamma_{L-1}(l) = 0$, for $l < L$.

Hence, using (9.27) we get

$$\sum_{l=0}^{L}\gamma_j(l)\epsilon_m^l(x) = \sum_{l=0}^{L}\gamma_{L-1}(l)\epsilon_m^l(x) = \gamma_{L-1}(L)\epsilon_m^L(x) = L\pi\epsilon_m^L(x). \tag{9.30}$$

Thus using the Cauchy-Shwarz inequality

$$| \, III \, |=| -\frac{2-\delta_{j,0}}{\pi} \sum_{m=1}^{M} \omega_m \left[\eta_m \int_{-1}^{1} \sum_{l=0}^{L} \gamma_{L-1}(l) \epsilon_m^l(x) \epsilon_m^{L-1}(x) dx \right] |$$

$$\leq \frac{2}{\pi} L\pi \int_{-1}^{1} | \sum_{m=1}^{M} \eta_m \omega_m \epsilon_m^L(x) \epsilon_m^{L-1}(x) \, | \, dx$$

$$\leq 2L(\max_m | \, \eta_m \, |) \left[\int_{-1}^{1} \sum_{m=1}^{M} \omega_m \left[\epsilon_m^L(x) \right]^2 dx \right]^{1/2} \times \tag{9.31}$$

$$\left[\int_{-1}^{1} \sum_{m=1}^{M} \omega_m \left[\epsilon_m^{L-1}(x) \right]^2 dx \right]^{1/2}$$

$$2L\sqrt{\frac{\pi}{2}} \left\| \epsilon^L \right\| \sqrt{\frac{\pi}{2}} \left\| \epsilon^{L-1} \right\| = L\pi \left\| \epsilon^L \right\| \left\| \epsilon^{L-1} \right\|.$$

Inserting in (9.20) and using also (9.11) and Lemma 9.3, with $j = L - 1$, we get

$$\sigma_t \frac{\pi}{2} \left\| \epsilon^{L-1} \right\|^2 \leq 4\pi\sigma_s \frac{\pi}{2} \left\| \epsilon^{L-1} \right\|^2 + \sqrt{4\pi}\sigma_s \frac{\pi}{2} \left\| \epsilon^{L-1} \right\| \left\| \tau^{L-1} \right\|$$

$$+ L\pi \left\| \epsilon^L \right\| \left\| \epsilon^{L-1} \right\|, \tag{9.32}$$

or equivalently using the notation $\sigma = \sigma_t - 4\pi\sigma_s$,

$$\sigma \left\| \epsilon^{L-1} \right\| \leq 2L \left\| \epsilon^L \right\| + \sqrt{4\pi}\sigma_s \left\| \epsilon^{L-1} \right\|. \tag{9.33}$$

The same procedure applied to $j = L - 2$ yields $\gamma_j(L) = \gamma_{L-2}(L) = 0$, (note that here $j + L$ is even), $\gamma_{L-2}(L-1) = (L-1)\pi$ and $\gamma_{L-2}(l) = 0$, for $l < L - 1$. Thus

$$\sum_{l=0}^{L} \gamma_{L-2}(l) \epsilon_m^l(x) = \gamma_{L-2}(L-1) \epsilon_m^{L-1}(x) = (L-1)\pi \epsilon_m^{L-1}(x), \tag{9.34}$$

so that, as in the previous step

$$\sigma \left\| \epsilon^{L-2} \right\| \leq 2(L-1) \left\| \epsilon^{L-1} \right\| + \sqrt{4\pi}\sigma_s \left\| \tau^{L-2} \right\|. \tag{9.35}$$

Similarly since for $j = L-3$; we have $\gamma_{L-3}(L) = L\pi$, $\gamma_{L-3}(L-1) = 0$, $\gamma_{L-3}(L-2) = (L-2)\pi$ and $\gamma_{L-3}(l) = 0$ for $l < L-2$, we get

$$\sum_{l=0}^{L} \gamma_{L-3}(l)\epsilon_m^l(x) = \gamma_{L-3}(L-2)\epsilon_m^{L-2}(x) + \gamma_{L-3}(L)\epsilon_m^L(x)$$

$$= 2(L-2)\epsilon_m^{L-2}(x) + 2L\epsilon_m^L(x), \tag{9.36}$$

which using the same procedure as before yields

$$\sigma \left\|\epsilon^{L-3}\right\| \leq 2L \left\|\epsilon^L\right\| + 2(L-2)\left\|\epsilon^{L-2}\right\| + \sqrt{4\pi}\sigma_s \left\|\tau^{L-3}\right\|. \tag{9.37}$$

Now the formula (9.25) is proved by an induction argument.

Chapter 10

The Quadrature Rule and Proof of Theorem 9.2.

In this section we construct a special quadrature mesh satisfying the conditions in (9.5) and prove the Theorem 9.2 in this setting. This would provide us the remaining step in the proof of the Theorem 9.1 and complete the convergence analysis. We also derive convergence rates for the quadrature error (9.7) where we identify the angular domain

$$D = \{(\mu, \theta) : \quad -1 \le \mu \le 1, \quad 0 \le \theta \le 2\pi\}, \tag{10.1}$$

by

$$\widetilde{D} = \Big\{(\mu, \eta) : \quad -1 \le \mu, \eta \le 1, \quad \eta = \sqrt{1-\mu^2}\cos\theta.\Big\} \tag{10.2}$$

Then the quadrature (cubature) rule, for the multiple integral in (9.1) can be constructed using (10.2) as in (9.7), see [23]. To derive convergence rates, below we construct an equivalent rule, directly discretizing D given by (10.1), and with a somewhat general features:

$$\int_0^{2\pi}\int_{-1}^{1} \Psi(\mathbf{x}, \mu, \theta)\, d\mu d\theta \sim \sum_{\Delta} \omega_{kj} \Psi(\mathbf{x}, \mu, \theta), \tag{10.3}$$

where $\Delta := \{(\mu_k, \theta_j),\ k = 1, ..., K \text{ and } j = 1, ..., J,\ J \sim K\} \subset D$ is a $M = JK$, discrete set of points in D consisting of the Gauss quadrature points $\mu_k \in [-1, 1]$ associated with the equally spaced $\theta_j = \frac{2\pi}{J}$, $j = 1, ..., J$, and weights $\omega_{kj} = A_k W_j$ where $W_j = \frac{2\pi}{J}$, $j = 1, ..., J$, and A_k are given below. Thus the error in (10.3) can be split into two decoupled quadrature error:

$$| e_M(\Psi) | := | \int_0^{2\pi} \int_{-1}^1 \Psi(\mathbf{x}, \mu, \theta) d\mu d\theta - \sum_\Delta \omega_{kj} \Psi(\mathbf{x}, \mu_k, \theta_j) |$$

$$\leq \int_0^{2\pi} | \int_{-1}^1 \Psi(\mathbf{x}; \mu, \theta) d\mu - \sum_{k=1}^K A_k \Psi(\mathbf{x}, \mu_k, \theta) | d\theta$$

$$+ \sum_{k=1}^K A_k \left[| \int_0^{2\pi} \Psi(\mathbf{x}, \mu_k, \theta) d\theta - \sum_{j=1}^J W_j \Psi(\mathbf{x}, \mu_k, \theta_j) | \right] \tag{10.4}$$

$$:= \int_0^{2\pi} | e_K [\Psi(\mathbf{x}; \theta)] | d\theta + \sum_{k=1}^K A_k | e_J [\Psi(\mathbf{x}, \mu_k)] |,$$

with the obvious notations for the two quadrature errors:

$$e_J [\Psi(\mathbf{x}; \mu)] := \int_0^{2\pi} \Psi(\mathbf{x}, \mu, \theta) d\theta - \sum_{j=1}^J W_j \Psi(\mathbf{x}, \mu, \theta_j), \tag{10.5}$$

$$e_K [\Psi(\mathbf{x}; \mu)] := \int_{-1}^1 \Psi(\mathbf{x}; \mu, \theta) d\mu - \sum_{k=1}^K A_k \Psi(\mathbf{x}, \mu_k, \theta), \tag{10.6}$$

Below we derive error estimates for the quadrature rules (10.5) and (10.6), with optimal convergence rates with respect to the assumed regularity of Ψ in μ and θ.

Lemma 10.1. *Let $e_J [\Psi]$ denote the error in (10.5), with J equally spaced quadrature points $\theta_j \in [0, 2\pi]$. Suppose that $| \frac{\partial^r \Psi(\mathbf{x}, \mu, \theta)}{\partial \theta^r} |$ is integrable on $[0, 2\pi]$, then*

$$| e_J [\Psi] | \leq \frac{C_r}{J^r} \int_0^{2\pi} | \frac{\partial^r \Psi(\mathbf{x}, \mu, \theta)}{\partial \theta^r} | d\theta, \tag{10.7}$$

where C_r is independent of J and Ψ.

Lemma 10.2. *Let $e_K [\Psi]$ denote the error on K-point Gaussian quadrature approxi-*

mation of the integral of Ψ *on* $\mu \in [-1,1]$ *. Suppose that* $(1-\mu^2) \mid \frac{\partial^r \Psi(\mathbf{x},\mu,\theta)}{\partial \theta^r} \mid$ *is integrable on* $[-1,1]$ *, then*

$$\mid e_K[\Psi] \mid \leq \frac{C_s}{K^s} \int_{-1}^{1} \mid \frac{\partial^s \Psi(\mathbf{x},\mu,\theta)}{\partial \mu^s} \mid .(1-\mu^2)^{s/2} d\mu, \tag{10.8}$$

where C_s *is independent of* K *and* Ψ*.*

We postpone the proofs of these lemmas and first derive of the proof of Theorem 9.2 from them. For the transport equation (9.1), in polygonal domains, the regularity requirements in the lemmas 10.1 and 10.2 are proved for $r = s = 1$ in [2]:

Proposition 4.3. *Let* $\frac{\partial \Psi}{\partial \theta} \in L_1[0,2\pi]$ *and* $\frac{\partial \Psi}{\partial \mu} \in L_1^{\widetilde{\omega}}[0,2\pi]$ *, where* $\widetilde{\omega} := (1-\mu^2)^{1/2}$*. Then for the quadrature error* $\tau(\mathbf{x})$ *of the approximation (4.3) we have,*

$$\|\tau\|_{L_2(\Omega)} \leq C \left(\frac{1}{J} + \frac{1}{K}\right) \|g\|_{H(\Omega)}, \tag{10.9}$$

where g is the right hand side of (9.1), i.e. $g = \sigma_s \widetilde{\Psi} + S$ with $\widetilde{\Psi} = \int_{-1}^{1} \int_0^{2\pi} \Psi$, and $H^1(\Omega)$ is the usual L_2-based Sobolev space of order one on Ω.

Now we are ready to derive our final error estimate:

Proof of Thorem 9.2. We multiply (9.10) by $\frac{T_k(y)}{\sqrt{1-y^2}}$, $k = 0, ..., L$ integrate over $y \in [-1,1]$ and use the Cauchy-Shwarz inequality to get for $l = 0, ..., L$,

$$\tau^l(x) = \frac{2-\delta_{l,o}}{\pi} \int_{-1}^{1} \tau(\mathbf{x}) \frac{T_l(y)}{\sqrt{1-y^2 dy}}$$

$$\leq \frac{2-\delta_{l,o}}{\pi} \left[\int_{-1}^{1} \tau(\mathbf{x})^2 \frac{T_l(y)}{\sqrt{1-y^2 dy}}\right]^{1/2} \left[\int_{-1}^{1} \tau(\mathbf{y})^2 \frac{T_l(y)}{\sqrt{1-y^2 dy}}\right]^{1/2} \tag{10.10}$$

$$= \left[\frac{2-\delta_{l,o}}{\pi} \int_{-1}^{1} \tau(\mathbf{x})^2 \frac{T_l(y)}{\sqrt{1-y^2 dy}}\right]^{1/2}.$$

Now recalling (9.12) it follows that

$$\|\tau\| \leq \frac{2-\delta_{l,o}}{\pi} \left[\int_{-1}^{1} \int_{-1}^{1} \tau(\mathbf{x})^2 \frac{dy}{\sqrt{1-y^2 dy}} dx\right]^{1/2} \leq C \|\tau\|_{L_2(\Omega)}. \tag{10.11}$$

Combining with (10.9), recalling also $M \sim J^{1/2} \sim K^{1/2}$ we get the desired result.

Remark. The convergence rate in the Lemmas 10.1 and 10.2, as well as the rates in Proposition 10.3, can be improved up to the optimal order $\mathcal{O}(J^{2-\epsilon}) \sim \mathcal{O}(K^{2-\epsilon})$, ϵ arbitrarily small, for the neutron transport equation, in polygonal domains using, e.g., post processing procedure cf. Asadzadeh [3].

Now it remains to verify the estimates in Lemmas 10.1-10.2.

Proof of Lemma 10.1. We assume that Ψ is 2π-periodic in θ and in the quadrature formula

$$\int_0^{2\pi} \Psi(\mathbf{x}, \mu, \theta) d\theta \sim \sum_{j=1}^{J} W_j \Psi(\mathbf{x}, \mu, \theta_j), \tag{10.12}$$

approximate Ψ by trigonometric polynomials in θ. Then we can easily check that: on matter how we choose the quadrature points θ_j and weights W_j, the formula (10.12) can not be exact for trigonometric polynomials of degree J, (see, e.g., [37] for the details). It turns out that the highest degree of precision $J-1$ is achieved just for our simplest quadrature formula: equally spaced nodes $\theta_j = \frac{2\pi j}{J}$ and constant weights $W_j = \frac{2\pi}{J}$, $j = 1, 2, ..., J$. Thus we have

$$\int_0^{2\pi} \Psi(\theta) d\theta \sim \frac{2\pi}{J} \sum_{j=1}^{J} \Psi\left((j-1)\frac{2\pi}{J}\right). \tag{10.13}$$

We can easily verify (10.13) is exact for the functions $e^{imx}, m = 0, 1, ..., J-1$. Further a trigonometric polynomial of degree J, with the Fourier series expansion

$$T_j(x) \equiv \frac{a_0}{2} + \sum_{j=1}^{J} (a_j \cos jx + b_j \sin jx), \tag{10.14}$$

having $2J+1$ degrees of freedom $(a_0, a_j, b_j, j = 1, ..., J)$ corresponds to an algebraic polynomials of degree $2J$. Thus (10.13) is exact for algebraic polynomials of degree $2J-1$, so that for $\Psi \in C^r[0, 2\pi]$, $r = 2J$, (Ψ is $2J$ times continuously differentiable in θ), using Taylor expansion up to degree $2J-1$, in both sides of (10.12), we obtain the desired result.

Lemma 10.2 is a special case of the classical result due to DeVore and Scott (Theorem 3 in [21], Proposition 10.4 below): Consider, for positive integer s, the function space

$$\Psi \in Y_{\omega}^{s} := \left\{ u \in L_{loc}^{1}\left(]-1,1[\right) : \|u\|_{\omega,s} < \infty \right\} \tag{10.15}$$

with ω being a weight function and

$$\|u\|_{\omega,s} = \int_{-1}^{1} \left[\mid u(\mu) \mid + \mid u^{(s)}(\mu) \mid (1-\mu^2)^s \right] \omega(\mu) d\mu, \tag{10.16}$$

where $u^{(s)}$ is interpreted as a weak derivative.

Proposition 10.4. (DeVore and Scott). *Let $e_K[\Psi]$ denote the error in K-point Gaussian quadrature approximation of the integral of Ψ on $[-1,1]$. Suppose that $(1-\mu^2)^s \mid \frac{\partial^s \Psi(\mathbf{x},\mu,\theta)}{\partial \mu^s} \mid$ (weak derivative) is integrable on $[-1,1]$, i.e., $\Psi \in Y_1^s$, where s is any positive integer such that $1 \leq s \leq 2K$. Then*

$$\mid e_K[\Psi] \mid \leq C_s \int_{-1}^{1} \mid \frac{\partial^s \Psi(\mathbf{x},\mu,\theta)}{\partial \mu^s} \mid \min\left\{ \left[\frac{\sqrt{1-\mu^2}}{K}\right]^s, (1-\mu^2)^s \right\} d\mu, \tag{10.17}$$

where C_s is independent of K and Ψ.

Proof of Lemma 10.2. This follows, evidently, from the Proposition 10.4.

Below we review a procedure, based on analyzing the Peano kernel for the quadrature error (10.6), and establish the bound (10.8) for $s=1$, see [2] or [21]. This would suffices to justify the use of Proposition 10.3. The full proof of (10.8), or (10.17), for $s \geq 1$ is treated as in [21]. Consider the Gauss quadrature rule

$$\int_{-1}^{1} \Psi(\mathbf{x};\mu,\theta) d\mu \sim \sum_{k=1}^{K} A_k \Psi(\mathbf{x},\mu_k,\theta), \tag{10.18}$$

where

$$\mu_k := -\cos\alpha_k, \quad \alpha_k \in \left[\frac{(2k-1)\pi}{2K+1}, \frac{2k\pi}{2K+1},\right], \quad k = 1, ...K, \tag{10.19}$$

are zeros of Legendre polynomials and

$$A_k := \int_{-1}^{1} \prod_{l \neq k} \frac{x - x_l}{x_k - x_l} dx, \quad k = 1, ...K, \tag{10.20}$$

are the integrals of the associated Lagrange interpolation polynomials. Now using the Peano kernel theorem we can write

$$e_K[\Psi] = \int_{-1}^{1} \Lambda(\zeta)\Psi'(\zeta)d\zeta, \tag{10.21}$$

where $\Lambda(\zeta) = e_K[H_\zeta]$, $\mid \zeta \mid \leq 1$ and H_ζ is the Heaviside function

$$H_\zeta(\mu) := \begin{cases} 0, & \mu < \zeta, \\ 1, & \mu \geq \zeta \end{cases} \tag{10.22}$$

it follows that

$$\Lambda(\zeta) = 1 - \zeta - \sum_{\mu_k > \zeta} A_k = \sum_{\mu_k < \zeta} A_k - \zeta - 1. \tag{10.23}$$

By the Chebyshev-Markov-Stieltjes (cf. [45] p. 50) inequality we have

$$1 + \mu_k \leq \sum_{k=1}^{K} A_i \leq 1 + \mu_{k+1}, \qquad k = 1, ..., K. \tag{10.24}$$

Thus with $-1 = \mu_0 < \mu_1 < ... < \mu_K < \mu_{K+1} = 1$ we get for $k = 1, ..., K$ that

$$\mu_{k-1} - \mu_k \leq \Lambda(\mu_{k^-}) \leq 0 \leq \Lambda(\mu_{k^+}) \leq \mu_{k+1} - \mu_k. \tag{10.25}$$

Since Λ vanishes on each interval $\left[\mu_{k-1}, \mu_k\right]$ and has the slope one almost everywhere, we have

$$\max\left\{\mid \Lambda(\mu) \mid : \mu \in \left[\mu_{k-1}, \mu_k\right]\right\} \leq \mu_k - \mu_{k-1}, \qquad k = 1, ..., K. \tag{10.26}$$

To bound $\mu_k - \mu_{k-1}$, we define $I_k := [\alpha_{k-1}, \alpha_k]$, then

$$\mu_k - \mu_{k-1} = \cos\alpha_{k-1} - \cos\alpha_k = \int_{\alpha_{k-1}}^{\alpha_k} \sin\alpha d\alpha$$

$$\leq (\alpha_k - \alpha_{k-1}) \max_{\alpha\in I_k} \{\sin\alpha\} \leq \frac{3\pi}{2K} \max_{\alpha\in I_k} \{\sin\alpha\} . \tag{10.27}$$

Now since $(\sin\alpha)/\alpha$ is decreasing in $[0, \pi]$, using (10.19) we get

$$\sin\alpha \leq \left[\frac{\alpha}{\alpha_{k-1}}\right] \sin\alpha_{k-1} \leq \left[\frac{\alpha_k}{\alpha_{k-1}}\right] \sin\alpha_{k-1} \leq 4\sin\alpha_{k-1}, \qquad \alpha \in I_k, \tag{10.28}$$

for $k = 2, ..., K$. By the symmetry properties of α_j (cf. [45]) we also get

$$\sin\alpha \leq 4\sin\alpha_{k-1}, \qquad \alpha \in I_k, \qquad k = 2, ..., K. \tag{10.29}$$

Thus for $k = 2, ..., K$,

$$\max_{\alpha\in I_k} \{\sin\alpha\} \leq 4 \min_{\alpha\in I_k} \{\sin\alpha_{k-1}\} \equiv 4 \min_{\alpha\in I_k} \left\{\sqrt{1-\cos^2\alpha}\right\} . \tag{10.30}$$

Hence, combining (10.27) and (10.30), and using (10.19) we have for $k = 2, ..., K$,

$$\mu_k - \mu_{k-1} = \frac{6\pi}{K} \min_{\alpha\in I_k} \left\{\sqrt{1-\cos^2\alpha}\right\} = \frac{6\pi}{K} \min_{\alpha\in I_k} \left\{\sqrt{1-\mu^2}\right\} . \tag{10.31}$$

Thus, by (10.26), for $\mu \in [\mu_1, \mu_N]$,

$$| \Lambda(\mu) | \leq \frac{6\pi\sqrt{1-\mu^2}}{K}. \tag{10.32}$$

The corresponding estimate for $\mu \in [-1, \mu_1]$ and $\mu \in [\mu_1, 1]$ is (see [21]):

$$| \Lambda(\mu) | \leq \frac{\pi\sqrt{1-\mu^2}}{\sqrt{2}K}. \tag{10.33}$$

Summing up we have shown

$$| e_K [\Psi] | \leq \frac{6\pi}{K} \int_{-1}^{1} | \frac{\partial \Psi}{\partial \mu} | . \sqrt{1-\mu^2} d\mu. \tag{10.34}$$

This proves (10.8) for $s = 1$. For further details we refer to [2] and [21].

Chapter 11

Concluding remarks.

We believe that the idea of using the spectral method for searching solutions to the multidimensional transport problems, in addition to leads us to a solution for all values of the independent variables, is promising for two reasons: first the proposed decompositions reduce the solution of the multidimensional problem into a set of one-dimensional ones that have well established deterministic solutions. Furthermore, in the framework of the analytical solution it may be possible to study and to prove the convergence, that implies numerical stability, and the estimation of the error for the proposed solution. Of course the question left to be answered concerns the investigation of the approximating basis functions to be considered in the expansion as well as other aspects like computational implementations and performances. This is an important issue to be investigated from now. In regard to that, just some preliminary results were obtained by application of the spectral method.

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Part IV

APPENDIX I: Elementary properties of Chebyshev polynomials and Legendre polynomials

Chebyshev polynomials are weighted orthogonal polynomials defined by

$$T_n(x) = \cos(n \ \arccos(x)), \tag{11.1}$$

with the weight function $\omega(x) = \frac{1}{\sqrt{1-x^2}}$. Thus Chebyshev polynomials are a subclass of Jacobi polynomials, where the Jacobi weights $\omega_J = (1+x)^a(1-x)^b$, $a, b > -1$ are restricted to $a = b = -1/2$. It follows that

$$\int_{-1}^{1} T_i(x)T_j(x)\omega(x)dx = \begin{cases} 0 & i \neq j \\ \pi/2 & i = j \neq 0 \\ \pi & i = j = 0. \end{cases} \tag{11.2}$$

Hence

$$\| T_i \|_\omega = \frac{\pi}{2 - \delta_{i,0}}, \qquad i = 0, 1, ... \tag{11.3}$$

$T_n(x)$ is a polynomial of degree n, orthogonal to all polynomials of degree$\leq n-1$. On differentiating $T_n(x) = \cos n\beta$ with respect to $x (= \cos\beta)$ we obtain a polynomial of degree $n-1$ called the *Chebyshev polynomials of second kind:*

$$U_{n-1} = \frac{1}{n}T_n'(x) = \frac{\sin n\beta}{\sin\beta}, \qquad x = \cos\beta. \tag{11.4}$$

Further we can easily verify the following properties (see [41] for the details):

For even (odd) n only even (odd) powers of x occur in $T_n(x)$.

$$T_n(-x) = (-1)^n T_n(x). \tag{11.5}$$

$$\frac{1}{2} + T_2(x) + T_4(x) + ... + T_{2k}(x) = \frac{U_{2k}(x)}{2}, \qquad k = 0, 1, ..., \tag{11.6}$$

$$T_1(x) + T_3(x) + ... + T_{2k+1}(x) = \frac{U_{2k+1}(x)}{2}, \qquad k = 0, 1, ..., . \tag{11.7}$$

We have also the recurrence relations for the Chebyshev polynomials.

$$T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0 \qquad (11.8)$$

$$U_{n+1}(x) - 2xU_n(x) + U_{n-1}(x) = 0 \qquad (11.9)$$

and for the Legendre polynomials

$$P_{n+1}(x) = 2xP_n(x) - P_{n-1}(x) - \left[xP_n(x) - P_{n-1}(x)\right]/(n+1) \qquad (11.10)$$

Below we formulate and prove the formulae (4.10), (4.11), (4.12).

Proposition 1 *Let*

$$T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0 \qquad (11.11)$$

and

$$P_{l+1}(x) = 2xP_l(x) - P_{l-1}(x) - \left[xP_l(x) - P_{l-1}(x)\right]/(l+1) \qquad (11.12)$$

we have for $l > 2$ *and* $k = 2$

$$\alpha^2_{n,l+1} := \frac{2l+1}{2l+2}\left[\alpha^2_{n+1,l} + \alpha^2_{n-1,l}\right] - \frac{l}{l+1}\alpha^2_{n,j-1} \qquad (11.13)$$

where

$$\alpha^2_{n,l+1} := \int_{-1}^{1} T_n(\mu)P_{l+1}(\mu)d\mu \qquad (11.14)$$

and for $l > 2$ *and* $k = 3$

$$\alpha^3_{n,l+1} = \frac{2l+1}{2l+2}\left[\alpha^3_{n+1,l} + \alpha^3_{n-1,l}\right] - \frac{l}{l+1}\alpha^3_{n,j-1} \qquad (11.15)$$

where

$$\alpha^3_{n,l+1} := \int_{-1}^{1} \frac{T_n(\mu)P_{l+1}(\mu)}{\sqrt{1-\mu^2}}d\mu \qquad (11.16)$$

proof. for $k = 2$

by the multiplication of the Chebyshev and the Legendre recurrence formulas we have

$$\frac{2l+1}{2l+2}\left[P_l(\mu)T_{n+1}(\mu)+P_l(\mu)T_{n-1}(\mu)\right]-\frac{l}{2\mu\,(l+1)}\left[P_{l-1}(\mu)T_{n+1}(\mu)+P_{l-1}(\mu)T_{n-1}(\mu)\right] \tag{11.17}$$

we can rewrite this equation as

$$\frac{2l+1}{2l+2}\left[P_l(\mu)T_{n+1}(\mu)+P_l(\mu)T_{n-1}(\mu)\right]-\frac{l}{2\mu\,(l+1)}P_{l-1}(\mu)\left[T_{n+1}(\mu)+T_{n-1}(\mu)\right] \tag{11.18}$$

it is known that

$$T_{n+1}(\mu)+T_{n-1}(\mu)=2\mu T_n(\mu) \tag{11.19}$$

after doing some algebraic manipulations and integrating over $\mu \in [-1,1]$ on the resulting equation we get

$$\alpha^2_{n,l+1}=\frac{2l+1}{2l+2}\left[\alpha^2_{n+1,l}+\alpha^2_{n-1,l}\right]-\frac{l}{l+1}\alpha^2_{n,j-1} \tag{11.20}$$

The case $k=3$ is treated similarly but in this case we multiply the resulting expression by $\frac{1}{\sqrt{1-\mu^2}}$ and integrate over $\mu \in [-1,1]$ we get the desired result.

Below we formulate and prove the property that has been essential in deriving the basic estimate in section 9 (proposition 9.3):

Proposition 2 *Let*

$$\gamma_j(l)=\int_{-1}^{1}\frac{dT_l(y)}{dy}(y).\frac{T_j(y)}{\sqrt{1-y^2}}dy, \tag{11.21}$$

we have that

$$\gamma_j(l)=0 \qquad for \quad j\geq l, \tag{11.22}$$

and for $j<l$,

$$\gamma_j(l)=\begin{cases}0 & j\ +l \quad even\\ l\pi & j+l \quad odd.\end{cases} \tag{11.23}$$

Proof. The first assertion is a trivial consequence of the fact that T_j is orthogonal to all polynomials of degree $\leq j-1$. As for the second assertion we note that

$$T_j'(x) = lU_{l-1}(x).$$

Thus if l is odd then $l-1$ is even, say $l-1=2k$, hence using (11.6)

$$\gamma_j(l) = 2l \int_{-1}^{1} \left[\frac{1}{2} + T_2(x) + T_4(x) + ... + T_{l-1}(x) \right] . \frac{T_j(y)}{\sqrt{1-y^2}} dy$$

$$= \begin{cases} 0 & j \quad odd, \quad i.e., \ j+l \quad even \\ 2l\frac{\pi}{2-\delta_{j,0}} & j \quad even, \quad i.e., \ j+l \quad odd. \end{cases} \tag{11.24}$$

The case l is even is treated similarly and using (11.7) and the proof is complete.

Part V

APPENDIX II: the three-dimensional spectral solution.

We extend now the approach presented in Section 8 to the transport process in three dimension,

$$\mu\frac{\partial}{\partial x}\Psi(\mathbf{x},\mu,\theta)+\sqrt{1-\mu^2}\left[\cos\theta\frac{\partial}{\partial y}\Psi(\mathbf{x},\mu,\theta)+\sin\theta\frac{\partial}{\partial z}\Psi(\mathbf{x},\mu,\theta)\right]$$

$$+\sigma_t\Psi(\mathbf{x},\mu,\theta)=\int_{-1}^{1}\int_{0}^{2\pi}\sigma_s(\mu',\theta'\to\mu,\theta)\Psi(\mathbf{x},\mu',\theta')d\theta' d\mu'+S(\mathbf{x},\mu,\theta) \qquad (11.25)$$

where we assume that the spatial variable $\mathbf{x} :=(x,y,z)$ varies in the cubic domain $\Omega := \{(x,y,z) : -1 \le x,y,z \le 1\}$, and $\Psi(\mathbf{x},\mu,\theta) := \Psi(x,y,z,\mu,\theta)$ is the angular flux in the direction defined by $\mu \in [-1,1]$ and $\theta \in [0,2\pi]$,

We seek for a solution of (11.25) satisfying the following boundary conditions:

For the boundary terms in x; for $0 \le \theta \le 2\pi$,

$$\Psi(x=\pm1,y,z,\mu,\theta)=\begin{cases} f_1(y,z,\mu,\theta), \ x=-1, & 0<\mu\le 1,\\ 0, \ x=1, & -1\le\mu<0.\end{cases} \qquad (11.26)$$

For the boundary terms in y and for $-1 \le \mu < 1$,

$$\Psi(x,y=\pm1,z,\mu,\theta)=\begin{cases} f_2(x,z,\mu,\theta), \ y=-1, & 0<\cos\theta\le 1,\\ 0, \ y=1, & -1\le\cos\theta<0.\end{cases} \qquad (11.27)$$

Finally, for the boundary terms in z; for $-1 \le \mu < 1$,

$$\Psi(x,y,z=\pm1,\mu,\theta)=\begin{cases} f_3(x,y,\mu,\theta), \ z=-1, & 0\le\theta<\pi,\\ 0, \ z=1, & \pi<\theta\le 2\pi.\end{cases} \qquad (11.28)$$

Here we assume that $f_1(y,z,\mu,\phi)$, $f_2(x,z,\mu,\phi)$and $f_3(x,y,\mu,\phi)$are given function.

Expanding the angular flux $\Psi(x,y,z,\mu,\phi)$in a truncated series of Chebyshev polynomials $T_i(y)$and $R_j(z)$ leads to

$$\Psi(x,y,z,\mu,\theta)=\sum_{i=0}^{I}\sum_{j=0}^{J}\Psi_{i,j}(x,\mu,\theta)T_i(y)R_j(z). \qquad (11.29)$$

We repeat the procedure in Section 8 and insert $\Psi(x,y,z,\mu,\theta)$ given by (11.29) into the boundary condition in (11.27), for $y=\pm 1$. Multiplying the resulting expressions by $\frac{R_j(z)}{\sqrt{1-z^2}}$ and integrating over z, we get the components $\Psi_{0,j}(x,\mu,\theta)$ for $j=0,...J$:

$$\Psi_{0,j}(x,\mu,\theta)=f_2^j(x,\mu,\theta)-\sum_{i=1}^{I}(-1)^j\Psi_{i,j}(x,\mu,\theta);\qquad 0<\cos\theta\leq 1, \tag{11.30}$$

and

$$\Psi_{0,j}(x,\mu,\theta)=-\sum_{i=1}^{I}\Psi_{i,j}(x,\mu,\theta);\qquad -1\leq\cos\theta<0. \tag{11.31}$$

Similarly, we substitute $\Psi(x,y,z,\mu,\theta)$ from (11.29) into the boundary conditions for $z=\pm 1$, multiply the resulting expression by $\frac{T_i(y)}{\sqrt{1-y^2}}$, $i=0,...I$ and integrating over y, to define the components $\Psi_{i,0}(x,\mu,\theta)$: For $-1\leq x\leq 1$, $-1<\mu<1$,

$$\Psi_{i,0}(x,\mu,\theta)=f_3^i(x,\mu,\theta)-\sum_{j=1}^{J}(-1)^j\Psi_{i,j}(x,\mu,\theta);\qquad 0\leq\theta<\pi, \tag{11.32}$$

$$\Psi_{i,0}(x,\mu,\theta)=-\sum_{j=1}^{J}\Psi_{i,j}(x,\mu,\theta);\qquad \pi<\theta\leq 2\pi, \tag{11.33}$$

where

$$f_2^{\beta}(x,\mu,\theta)=\frac{2-\delta_{0,j}}{\pi}\int_{-1}^{1}f_2(x,z,\mu,\theta)\frac{R_j(z)}{\sqrt{1-z^2}}dz \tag{11.34}$$

$$f_3^i(x,\mu,\theta)=\frac{2-\delta_{i,0}}{\pi}\int_{-1}^{1}f_3(x,y,\mu,\theta)\frac{T_i(y)}{\sqrt{1-y^2}}dy. \tag{11.35}$$

To determine the components $\Psi_{i,j}(x,\mu,\theta)$, $i=1,...I$, and $j=1,...J$, we substitute $\Psi(x,\mu,\theta)$, from (11.29) into (11.25) and the boundary conditions for $x=\pm 1$. Multiplying the resulting expressions by $\frac{T_i(y)}{\sqrt{1-y^2}}\times\frac{R_j(z)}{\sqrt{1-z^2}}$, and integrating over y and z we obtain $I\times J$ one-dimensional transport problems, viz

$$\mu\frac{\partial\Psi_{i,j}}{\partial x}(x,\mu,\theta)+\sigma_t\Psi_{i,j}(x,\mu,\theta)=G_{i,j}(x;\mu,\theta)$$

$$\int_{-1}^{1}\int_{-1}^{1}\sigma_s(\mu',\theta'\to\mu,\theta)\Psi_{i,j}(x,\mu',\theta')d\theta' d\mu' \tag{11.36}$$

with the boundary conditions

$$\Psi_{i,j}(-1,\mu,\eta) = f_1^{i,j}(\mu,\theta), \tag{11.37}$$

where

$$f_1^{i,j}(\mu,\theta) = \frac{4}{\pi^2}\int_{-1}^{1}\int_{-1}^{1}\frac{T_i(y)R_j(z)}{\sqrt{(1-y^2)(1-z^2)}}f_1(y,z,\mu,\theta)dzdy, \tag{11.38}$$

and

$$\Psi_{i,j}(1,-\mu,\theta) = 0, \tag{11.39}$$

for $0 < \mu \leq 1$, and $0 \leq \theta \leq 2\pi$. Finally

$$G_{i,j}(x;\mu,\theta) = S_{i,j}(x,\mu,\theta) -$$

$$\sqrt{1-\mu^2}\left[\cos\theta\sum_{k=i+1}^{I}A_i^k\Psi_{k,j}(x,\mu,\theta) + \sin\theta\sum_{l=j+1}^{J}B_j^l\Psi_{i,l}(x,\mu,\theta)\right], \tag{11.40}$$

with

$$S_{i,j}(x,\mu,\theta) = \frac{4}{\pi^2}\int_{-1}^{1}\int_{-1}^{1}\frac{T_i(y)R_j(z)}{\sqrt{(1-y^2)(1-z^2)}}S(\mathbf{x},\mu,\theta)dzdy, \tag{11.41}$$

$$A_i^k = \frac{2}{\pi}\int_{-1}^{1}\frac{d}{dy}(T_k(y))\frac{T_i(y)}{\sqrt{1-y^2}}dy \tag{11.42}$$

$$B_j^l = \frac{2}{\pi}\int_{-1}^{1}\frac{d}{dy}(R_l(y))\frac{R_j(z)}{\sqrt{1-z^2}}dz. \tag{11.43}$$

Now, starting from the solution of the problem given by equations (11.36)-(11.43) for $\Psi_{I,J}(x,\mu,\theta)$, we then solve the problems for the other components, in the decreasing order in i and j. Recall that $\sum_{i=I+1}^{I}\cdots = \sum_{j=J+1}^{J} \equiv 0$. Hence, solving $I \times J$ one-dimensional problems, the angular flux $\Psi(\mathbf{x},\mu,\theta)$ is now completely determined through (11.29).

Remark: If we have to deal with different type of boundary conditions, we have to keep in mind that the first components $\Psi_{i,0}(x,\mu,\theta)$ and $\Psi_{0,j}(x,\mu,\theta)$ for $i = 1,...,I$ and $j = 1,...,J$ will satisfy one-dimensional transport problems subject to the same of boundary conditions of the original problem in the variable x.

Part VI

APPENDIX III: the SUMUDU transform.

The Sumudu transform is a new integral transform [51], which is a little known and not widely used whose defined for the functions of exponential order. So we consider functions in the set A, defined by

$$A = \left\{ f(t) \mid \exists M, \tau_1, \text{ and/or } \tau_2 > 0, \text{such that } \mid f(t) \mid < M e^{|t|/\tau_j}, \text{ if } t \in (-1)^j \times [0, \infty) \right\} \tag{11.44}$$

For a given function in the set A, the constant M must be finite, while τ_1 and τ_2 need not simultaneously exist, and each may be infinite. Instead of being used as a power to the exponential as in the case of the Laplace transform, the variable u in the Sumudu transform is used to factor the variable t in the argument of the function f. Specifically, for $f(t)$ in A, the Sumudu transform is defined by

$$G(u) = \mathbf{S}[f(t)] = \begin{cases} \int_0^\infty f(ut)e^{-t}dt, & 0 \leq u \leq \tau_2, \\ \\ \int_0^\infty f(ut)e^{-t}dt, & -\tau_1 \leq u \leq 0. \end{cases} \tag{11.45}$$

Albeit similar in expression, the two parts in the previous definition arise because in the domain of f, the variable t may not change sign. For further details and properties of Sumudu transform we refer to [10] and [51].

Theorem 3 *Let $n \geq 1$, and let $G_n(u)$ and $F_n(u)$ be the Sumudu and Laplace transform of the nth derivative of $f^{(n)}(t)$, of the function $f(t)$, respectively. Then*

$$G_n(u) = \frac{G(u)}{u^n} - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{u^{n-k}} \tag{11.46}$$

proof. By definition, the Laplace transform for $f^{(n)}(t)$ is given by

$$F_n(s) = s^n F(u) - \sum_{k=0}^{n-1} s^{n-(k+1)} f^{(k)}(0). \tag{11.47}$$

Therefore

$$F_n\left(\frac{1}{u}\right) = \frac{F(\frac{1}{u})}{u^n} - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{u^{n-(k+1)}}. \tag{11.48}$$

Now, since $G_k(u) = F_k(1/u)/u$, *for* $0 \leq k \leq m$, *we have*

$$G_n(u) = \frac{G(u)}{u^n} - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{u^{n-k}} = \frac{1}{u^n}\left[G(u) - \sum_{k=0}^{n-1} u^k f^{(k)}(0)\right] \tag{11.49}$$

In particular, this means that the Sumudu transform of the second derivative of the function f *is given by*

$$G_2(u) = \mathbf{S}\left[f^{''}(t)\right] = \frac{G(u)}{u^2} - \frac{f(0)}{u^2} - \frac{f^{'}(0)}{u^2}. \tag{11.50}$$

Part VII

APPENDIX IV: the TRZASKA'S method.

Trzaska's method consist to compute the inverse of regular matrix $M(s) = sA_0 - A$ called the linear matrix pencil [14] where only A_0 is singular, or both A_0 and A are singular. Nonsingular systems are considered as a particular case of singular systems.

We expand the linear matrix pencil inverse as follows

$$M^{-1}(s) = \frac{D(s)}{d(s)} = \frac{P_1}{s - p_1} + \frac{P_2}{s - p_2} + ... + \frac{P_{n-1}}{s - p_{n-1}} + P_n \qquad (11.51)$$

where $d(s) = \det M(s)$ and $D(s)$ =adj $M(s)$ denote the determinant and the adjoint matrix of regular matrix pencil $M(s)$, respectively and $P_1, P_2, ..., P_n$ called the partial matrices.

To develop an efficient formula for the determination of the matrix $D(s)$ and characteristic polynomial $d(s)$, we apply the Cayley-Hamilton theorem to $M(s)$, so that we have

$$M^n(s) + a_1(s)M^{n-1}(s) + ... + a_{n-1}(s)M(s) + a_n(s)I = 0 \qquad (11.52)$$

where I and 0 denote the $n \times n$ unit matrix and zero matrix respectively.

It follows from Eq. (11.52) that

$$I = -\frac{1}{a_n(s)}\left[M^n(s) + a_1(s)M^{n-1}(s) + ... + a_{n-1}(s)M(s)\right] \qquad (11.53)$$

Premultiplying both sides of Eq .(11.53) by $M^{-1}(s)$, gives

$$M^{-1}(s) = -\frac{1}{a_n(s)}\left[M^{n-1}(s) + a_1(s)M^{n-2}(s) + ... + a_{n-1}(s)I\right] \qquad (11.54)$$

This equation states that the inverse of the linear matrix pencil $M(s)$ can be expressed in terms of its successive integer powers of $n - k$ $(k = 1, 2, ..., n)$ orders premutiplied by the corresponding coefficients $a_{k-1}(s), a_0(s) = 1$. The coefficients $a_k(s)$, $(k = 0, 1, ..., n)$ can be represented in the following form

$$a_k(s) = a_{k,k}s^k + a_{k,k-1}s^{k-1} + ... + a_{k,1}s + a_{k,0} \qquad (11.55)$$

where $a_{k,l}$ are real numbers with $l = 0, 1, ..., k$ such that

$$a_{k,k} = \frac{1}{k}\text{trace}\left[M_{k,k} + \sum_{l=1}^{k-1} a_{l,l} M_{k-l,k-l}\right], k = 1, 2, ..., n \tag{11.56}$$

$$a_{k,0} = -\frac{1}{k}\text{trace}\left[M_{k,0} + \sum_{l=1}^{k-1} a_{l,0} M_{k-l,0}\right], k = 1, 2, ..., n \tag{11.57}$$

$$a_{k,l} = -\frac{1}{k}\text{trace}\left[M_{k,l} + \sum_{\substack{h=1, j<h \\ q=1, r<q}}^{k-1} a_{h,j} M_{q,r}\right], k = 1, 2, ..., n \tag{11.58}$$

with $h + q = k$ and $j + r = 1 < k$.

The matrix $M_{k,l}$ will be compute by using the Matrix Pascal Triangle [46].

For example, the coefficients of the polynomials $a_3(s)$ can be computed by applying the above rules as follows

$$a_{3,3} = -\frac{1}{3}\text{trace}\left(M_{3,3} + a_{1,1}M_{2,2} + a_{2,2}M_{1,1}\right) \tag{11.59}$$

$$a_{3,2} = -\frac{1}{3}\text{trace}\left(M_{3,2} + a_{1,0}M_{2,2} + a_{1,1}M_{2,1} + a_{2,1}M_{1,1} + a_{2,2}M_{1,0}\right) \tag{11.60}$$

$$a_{3,1} = -\frac{1}{3}\text{trace}\left(M_{3,1} + a_{1,0}M_{2,1} + a_{1,1}M_{2,0} + a_{2,0}M_{1,1} + a_{2,1}M_{1,0}\right) \tag{11.61}$$

$$a_{3,0} = -\frac{1}{3}\text{trace}\left(M_{3,0} + a_{1,0}M_{2,0} + a_{2,0}M_{1,0}\right). \tag{11.62}$$

Moreover the kth power of the linear matrix pencil $M(s)$ can be expressed in the following manner

$$M^k(s) = s^k M_{k,k} + s^{k-1} M_{k,k-1} + ... + sM_{k,1} + M_{k,0} \tag{11.63}$$

tacking into account Eqs. (11.55) and (11.56) we can state that

$$D(s) = s^{n-1} D_{n-1} + s^{n-2} D_{n-2} + ... + sD_1 + D_0 \tag{11.64}$$

and D_k $(k = 0, 1, 2, ..., n-1)$ is a $n \times n$ constant matrix determined by

$$D_k = M_{n-1,k} + \sum_{\substack{h=1, j<h \\ q=1, r<q}}^{k-1} a_{h,j} M_{q,r} \tag{11.65}$$

with $k = 0, 1, 2, ..., n-1$ and $h + q = k$.

The partial matrices $P_1, P_2, ..., P_n$ in expression (11.51) are independent of s and are expressed by

$$P_k = q_k \left[p_k^{n-1} D_{n-1} + p_k^{n-2} D_{n-2} + ... + p_k D_1 + D_0 \right] \tag{11.66}$$

where

$$q_k = - \left[\dot{a}_n \left(p_k \right) \right]^{-1} \qquad \text{with } k = 1, 2, ..., n-1 \tag{11.67}$$

and

$$\dot{a}_n \left(p_k \right) = \left[\frac{d}{ds} a_n(s) \right]_{s=p_k} \tag{11.68}$$

Thus knowing $D(s)$ and $d(s)$ we can easily find the matrices $P_1, P_2, ..., P_n$.

For $k = n$, we have

$$P_n = D_{n-1} \tag{11.69}$$

Thus for all matrices, $P_1, P_2, ..., P_n$ we give the following fundamental equation:

$$\begin{bmatrix} P_1 \\ P_2 \\ . \\ . \\ . \\ P_{n-1} \end{bmatrix} = \begin{bmatrix} q_1 & & & & 0 \\ 0 & q_2 & & & \\ & & . & & \\ & & & . & \\ & & & . & 0 \\ 0 & & & & q_{n-1} \end{bmatrix} \times$$

$$\times \begin{bmatrix} I & p_1 I & . & . & . & p_1^{n-1} I \\ I & p_2 I & . & . & . & p_2^{n-1} I \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ I & p_{n-1} I & & & . & p_{n-1}^{n-1} I \end{bmatrix} \begin{bmatrix} D_0 \\ D_1 \\ . \\ . \\ D_{n-1} \end{bmatrix} \tag{11.70}$$

$$P_n = D_{n-1} \tag{11.71}$$

or in more compact form

$$[P] = [\mathrm{diag}\ q_k]_1^{n-1} [V] [D] , \ P_n = D_{n-1} \tag{11.72}$$

where $[V]$ denotes the Kronecker product of the Vandermonde and unit matrices of appropriate dimensions. For further details we refer to [46].

Part VIII

Illustrative Examples

EXAMPLE 1

Consider the matrices

$$A_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; A = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

the linear matrix pencil $M(s) = sA_0 - A$ (is regular).

$$M^{-1}(s) = -\frac{1}{a_3(s)}\left[s^2 D_2 + s^1 D_1 + D_0\right]$$

where

$$a_3(s) = a_{3,3}s^3 + a_{3,2}s^2 + a_{3,1}s + a_{3,0}.$$

Applying (11.60) and the rule of the Matrix Pascal Triangle we obtain

$$D_2 = M_{2,2} + a_{1,1}M_{1,1} + a_{2,2}I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$D_1 = M_{2,1} + a_{1,1}M_{1,0} + a_{1,0}M_{1,1} + a_{2,1}I = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$D_0 = M_{2,0} + a_{1,0}M_{1,0} + a_{2,0}I = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 2 \\ 1 & -1 & -2 \end{bmatrix}$$

and

$$a_{3,3} = 0, \quad a_{3,2} = 1, \quad a_{3,1} = -3 \quad a_{3,0} = 1,$$

so that

$$a_3(s) = s^2 - 3s + 1 = -d(s).$$

Evaluating the zeros of $a_3(s)$, we obtain $p_1 = \frac{1}{2}(3+\sqrt{5})$ and $p_2 = \frac{1}{2}(3-\sqrt{5})$. Now using expression (8.1), we obtain the following partial, fraction expansion of the linear matrix pencil inverse:

$$M^{-1}(s) = \frac{P_1}{s-p_1} + \frac{P_2}{s-p_2} + P_3$$

where by Eqs. (11.70), we have

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix} \begin{bmatrix} I & p_1 I & p_1^2 I \\ I & p_2 I & p_2^2 I \end{bmatrix} \begin{bmatrix} D_0 \\ D_1 \\ D_2 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2}-\frac{\sqrt{5}}{2} & -\frac{3}{2}-\frac{\sqrt{5}}{2} & -1 \\ 1 & \frac{1}{2}+\frac{\sqrt{5}}{2} & -\frac{1}{2}+\frac{\sqrt{5}}{2} \\ -1 & -\frac{1}{2}-\frac{\sqrt{5}}{2} & \frac{1}{2}-\frac{\sqrt{5}}{2} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \frac{1}{2}-\frac{\sqrt{5}}{2} & \frac{3}{2}-\frac{\sqrt{5}}{2} & 1 \\ -1 & -\frac{1}{2}+\frac{\sqrt{5}}{2} & \frac{1}{2}+\frac{\sqrt{5}}{2} \\ 1 & \frac{1}{2}-\frac{\sqrt{5}}{2} & -\frac{1}{2}-\frac{\sqrt{5}}{2} \end{bmatrix}$$

and

$$P_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

EXAMPLE 2.

This example deals with a singular system with the following singular matrices A and B

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$$

The matrix pencil takes the form

$$M(s) = s \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$$

this matrix pencil is regular, so that by applying (11.55), (11.56), (11.57), (11.58) and (11.54), we obtain

$$a_2(s) = s \quad \text{and} \quad M^{-1}(s) = \frac{P_1}{s} + P_2$$

where

$$P_1 = \begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix} \quad \text{and} \quad P_1 = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}.$$

EXAMPLE 3.

Consider the nonsingular system where

$$A = I, \qquad B = \begin{bmatrix} 3 & -1 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

the matrix pencil takes the form

$$M(s) = s \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & -1 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

so that

$$\begin{aligned} M^{-1}(s) &= \frac{1}{d(s)} \left[s^2 D_2 + s D_1 + D_0 \right] \\ &= \frac{P_1}{s - p_1} + \frac{P_2}{s - p_2} + \frac{P_3}{s - p_3} \end{aligned}$$

where $-d(s) = a_3(s) = a_{3,3}s^3 + a_{3,2}s^2 + a_{3,1}s + a_{3,0}$ applying (11.55) and the rule of the Matrix Pascal Triangle we obtain

$$a_{3,3} = -1, \quad a_{3,2} = 5, \quad a_{3,1} = -8, \quad a_{3,0} = 4,$$

Evaluating the zeros of $a_3(s) = -s^3+5s^2-8s+4 = 0 \iff a_3(s) = -(s-1)(s^2-4s+4) = 0 \iff s_1 = s_2 = 2$; $s_3 = 1$ so $p_1 = p_2 = 2$ and $p_3 = 1$

$$D_2 = M_{2,2} + a_{1,1}M_{1,1} + a_{2,2}I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$D_1 = M_{2,1} + a_{1,1}M_{1,0} + a_{1,0}M_{1,1} + a_{2,1}I = \begin{bmatrix} -2 & -1 & 1 \\ 2 & -5 & 1 \\ 1 & -1 & -3 \end{bmatrix},$$

$$D_0 = M_{2,0} + a_{1,0}M_{1,0} + a_{2,0}I = \begin{bmatrix} 1 & 1 & -1 \\ -3 & 5 & -1 \\ -2 & 2 & 2 \end{bmatrix},$$

we compute the partial fraction matrices P_1, P_2 and P_3 by expression (11.72), we get

$$P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix}, \; P_2 = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \; P_3 = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix}.$$

Résumé.
Il y a beaucoup de littérature disponible qui traite la résolution l'équation de transport stationnaire bidimensionnelle à tel point qu'il nous est pratiquement impossible de les mentionner toutes. En revanche, la littérature concernant la convergence et l'estimation de l'erreur est rarissime. Dans cette thèse nous concentrons notre attention dans cette direction. Dans la première partie nous étudions l'expansion spectrale en polynôme de Chebyshev combinée avec les transformations de Sumudu pour résoudre, analytiquement, l'équation de transport neutronique dans des milieux isotropique mono-dimensionnels. Ensuite nous nous intéressons à l'étude de la convergence de la solution spectrale ainsi qu'à l'estimation de son erreur en dimension deux moyennant une règle de quadrature spéciale en discrétisant les variables angulaires, ce qui nous permettra d'approximer le flux scalaire. Finalement l'équation spectrale est prouvée en dimension 3.

Mots et phrases clés : Convergence, équation de transport linéaire, dispersion isotopique , méthode spectrale de Chebyshev, méthode de l'ordonnée discrète.
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Abstract.
There is much literature available regarding the subject of solving the two-dimensional steady-state transport equation that it would be impossible to mention all of them. Nevertheless, the literature concerning convergence and the estimative of the error is scarce. Therefore in this thesis we focus our attention in this direction. In the first part we study the spectral Chebyshev polynomial expansion combined with the Sumudu transform leading to solve, analytically , the neutron transport equation in isotropic one-dimensional media. Next we study the convergence as well as an estimative of error for the spectral solution of the isotropic two-dimenssional discrete ordinates problem where a special quadrature rule is used to discretize in the angular variables, approximating the scalar flux. Finally the spectral equation is derived in a three dimensional setting.

Key words and phrases : Convergence analysis, linear transport equation, isotropic scattering, Chebyshev spectral method, discrete-ordinates method.
1991 AMS Subject Classification: 65N35, 65D32, 82D75,40A10,41A50.

ملخص.
كثير من الأبحاث تطرقت و بشكل وافي عن حلول معادلة التنقل دات البعد 2 ألى درجة أنه لا يمكننا سردها كلها بينما الأبحاث التي تتطرق الى دراسة تقارب الحلول و تقدير خطأها تكاد تكون منعدمة.
في هده الأطروحة نركز أهتمامنا في هدا الأتجاه ففي الجزء الأول منها نتطرق الى حل معادلة التنقل دات البعد 1 تحليليا باستعمال نشر كثير حدود تشبيشاف مع تحويلات سومودو. ثم نهتم بدراسة تقارب الحل الطيفي و كدا لك تقييم الخطأ في البعد 2 باستعمال قاعدة تربيعية خاصة. أخيرا نبرهن المعادلة الطيفية في البعد 3.

كلمات و جمل فاتحة : تقارب تحليلي- معادلة تنقل خطية- تبديد متجانس- الطريقة الطيفية لتشبيشاف- طريقة ترتيب الغير المتصل.
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