Numerical Algorithms for Electron Beams

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Outline

- Radiation Oncology
  - Electron Beams: Physical properties
  - Radiation Therapy planning
- Algorithms for Electron Beams
  - A model Problem
  - The fully discrete scheme
  - Characteristic Schemes
- Implementation
- Results and conclusions

Motivations

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- 1 person in 5 will die from that cancer.
- Cancer is the second leading cause of death, but exceeds all other diseases in terms of years of working life lost.
- About half of all cancer patients receive radiation therapy.

Motivations

Any improvements to radio therapy will benefit a great number of people.

Radiation Oncology

- Use of radiation to kill diseased cells
- The ultimate goal:

  Destroy the tumor with minimal damage to the normal tissue.

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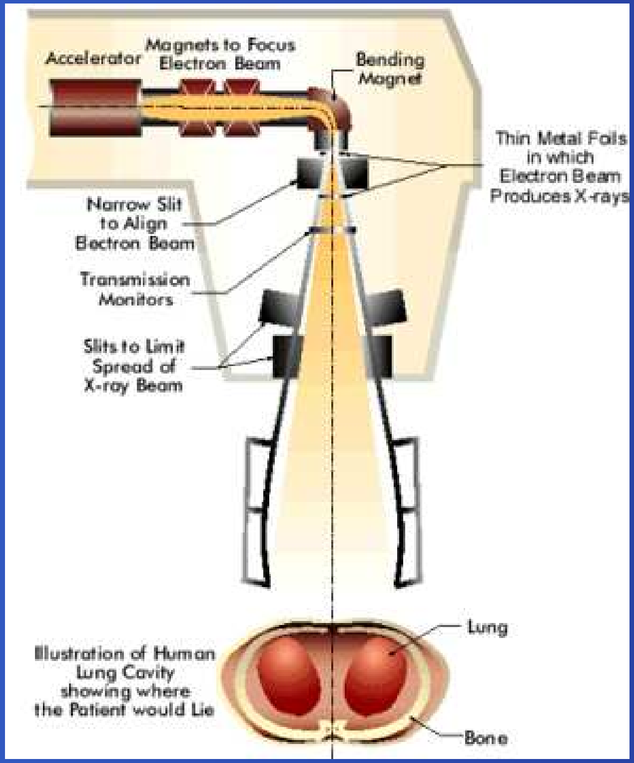
- Electrons, photons (X-ray), protons, neutrons

  . . .

Radiation Oncology



- They typically deliver 4 to $5MeV$ electron beams.
- Electron therapy is used to treat superficial tumor.

Radiation therapy planning

Calculate dose, or energy deposited per unit mass.

- 3D image of the tumor and surronding tissue.
- choose a set of beams (intuitively).
- calculate the dose.
- generate a graph showing the dose volume distribution.
- Decision weither the chosen set is optimal.

Electron beams: Physical and clinical

- Interactions, ionization.
- Energy deposition.
  - *Collisional losses*
  - *Radiative losses*
- Biological effects.

Electron beams: Mathematical models

- Variants of Boltzmann transport equation.
- The Fermi pencil beam equation.
  - Determine quatitatively the broadening of the beam.

The monoenergetic transport equation

let $\tilde{Q} := [0, L] \times \mathbf{R} \times \mathbf{R}$ be a homogeneous slab.

$$\omega \cdot \nabla_{\boldsymbol{X}} \psi(\boldsymbol{X}, \omega) + \sigma_t(\boldsymbol{X}) \psi(\boldsymbol{X}, \omega) = \int_{S^2} \sigma_s(\boldsymbol{X}, \omega \cdot \omega') \psi(\boldsymbol{X}, \omega) \, d\omega' \quad \text{in} \quad \tilde{Q} \times S^2,$$

(0)

and associated with the boundary conditions

(0)

$$\begin{cases} \psi(L, y, z, \omega) = 0 & \text{if } \xi < 0, \\ \psi(0, y, z, \omega) = \frac{1}{2\pi} \delta(1 - \xi) \delta(y) \delta(z) & \text{if } \xi > 0, \end{cases}$$

The monoenergetic transport equation

with $\boldsymbol{X} = (x, y, z) \in \tilde{Q}, \omega = (\xi, \eta, \zeta) \in S^2$. describing the spreading of a pencil beam of particles normally incident at the boundary $(0, y, z)$ of the slab $\tilde{Q}$. ψ is the density of particles at the point $\boldsymbol{X}$ moving in the direction of ω. σ_t is the total cross section, whereas σ_s is the scattering cross section.

The Fermi pencil beam equation

Derived from the Fokker-Planck equation.

$$(0) \quad \begin{cases} \omega_0 \cdot \nabla_{\mathsf{x}} \psi^F = \sigma \Delta_{\eta\zeta} \psi^F, \\ \psi^F(0, y, z, \eta, \zeta) = \delta(y)\delta(z)\delta(\eta)\delta(\zeta), & \text{if } \xi > 0, \\ \psi^F(L, y, z, \eta, \zeta) = 0, & \text{if } \xi < 0, \end{cases}$$

here $\omega_0 = (1, \eta, \zeta)$, where $(\eta, \zeta) \in \mathbf{R} \times \mathbf{R}$ and $\Delta_{\eta,\zeta} = \partial^2/\partial\eta^2 + \partial^2/\partial\zeta$.

A model problem

(0)
$$\begin{cases} u_x + z u_y = \epsilon u_{zz}, & (x, x_\perp) \in Q, \\ u_z(x, y, \pm z_0) = 0, & \text{for} (x, y) \in I_x \times I_y, \\ u(x, \pm y_0, z) = 0, & \Gamma^-_{\tilde{\beta}} \setminus \{suppf\}, \\ u(0, x_\perp) = f(x_\perp), & \end{cases}$$

where $\Gamma^-_{\tilde{\beta}} := \{(x, x_\perp) \in \partial Q : \tilde{\beta} \cdot \boldsymbol{n} < 0\}$, $\tilde{\beta} = (1, z, 0)$, and $x_\perp \equiv (y, z)$ is the transversal variable. Further, $\boldsymbol{n} := n(x, x_\perp)$ is the outward unit normal to Γ at $(x, x_\perp) \in \Gamma$.

A model problem

Our model problem corresponds to a forward-backward (z changes the sign), convection dominated (ϵ is small), convection-diffusion equation of degenerate type (convection in (x, y) and diffusion in z).

Standard Galerkin

Find $u_h \in V_{h,\beta}$, such that

(0)
$$\begin{cases} (u_{h,x}, \chi)_\perp + (z u_{h,y}, \chi) + (\epsilon u_{h,z}, \chi_z)_\perp = 0, & \forall \chi \in V_{h,\beta}, \\ u_h(0, x_\perp) = f_h(x_\perp), \end{cases}$$

where f_h is a finite element approximation of f. The mesh size h is related to ϵ according to:

(0)
$$h^2 \leq \epsilon \leq h.$$

A Semi Streamline Diffusion Method

The *test* function has the form $v + \delta v_\beta$ with $\delta \geq \epsilon$, $\beta = (z, 0)$, $v_\beta = \beta \cdot \nabla_\perp v$ and $\nabla_\perp = (\partial/\partial y, \partial/\partial z)$, and v satisfi es the boundary conditions.

$$(u_x + u_\beta - \epsilon u_{zz}, v + \delta v_\beta)_\perp = (u_x, v)_\perp + \delta(u_x, v_\beta)_\perp + (u$$

The fully discrete problem

We split the SSD variational formulation as follows:

(-1) $$a(u,v) = (u_\beta, v)_{I_\perp} + \delta(\epsilon u_\beta, u_\beta)_{I_\perp} + (\epsilon u_z, v_z)_{I_\perp} + \delta(\epsilon u_z, (v_\beta)_z)_{I_\perp},$$

(-1) $$b(u,v) = \delta(u, v_\beta)_{I_\perp} + (u,v)_{I_\perp},$$

and rewrite the problem as

(-1) $$\begin{cases} \text{find a solution } u \in H^1_\beta(I_\perp) \text{ such that} \\ b(u_x, v) + a(u,v) = 0, \qquad \forall v \in H^1_\beta(I_\perp). \end{cases}$$

The fully discrete problem

We consider the "space-time-discrete" ansatz

(-1) $$u_h(x,y,z) = \sum_{i=1}^{M} \xi_i(x)\phi_i(y,z),$$

where $M \sim 1/h$. We replace v by ϕ_j for $j = 1, \cdots, M$. This gives the discretization method

(-1) $$\sum_{i=1}^{M} \xi_i'(x) b(\phi_i, \phi_j) + \sum_{i=1}^{M} \xi_i(x) a(\phi_i, \phi_j), \qquad j = 1, \cdots, M.$$

Or in matrix form,

(-1) $$B\xi_i'(x) + A\xi_i(x) = 0.$$

The fully discrete problem

Now we discretize in x using Backward Euler to get

(-1) $$B(U_h^n - U_h^{n-1}) + k_n A U_h^n = 0.$$

$B = (b_{ij})$ is the matrix with entries $b_{ij} = b(\phi_i, \phi_j)$ and $A = (a_{ij})$ with entries $a_{ij} = a(\phi_i, \phi_j)$.

Characteristic schemes

The idea is *exact transport + projection*.

Characteristic schemes

(-1)
$$\begin{cases} J_x + zJ_y = \epsilon J_{zz}, & (x, x_\perp) \in Q, \\ J_z(x, y, \pm z_0) = 0, & \text{for}(x, y) \in I_x \times I_y, \\ J(x, \pm y_0, z) = 0, & \Gamma^-_{\tilde{\beta}} \setminus \{suppf\}, \\ J(0, x_\perp) = f(x_\perp), \end{cases}$$

where Q is a bounded domain $Q \equiv I_x \times I_y \times I_z = [0, L] \times [-y_0, y_0] \times [-z_0, z_0]$, $\Gamma^-_{\tilde{\beta}} := \{(x, x_\perp) \in \partial Q : \tilde{\beta} \cdot \boldsymbol{n} < 0\}$, $\tilde{\beta} = (1, z, 0)$, $x_\perp \equiv (y, z)$ is the transversal variable and $\boldsymbol{n} := \boldsymbol{n}(x, x_\perp)$ is the outward unit normal to Γ at $(x, x_\perp) \in \Gamma$.

Characteristic schemes

(-1) $$\mathcal{L}(J) := J_x + \beta \cdot \nabla_\perp J - \epsilon \Delta_\perp J = 0,$$

where $\epsilon \approx C\sigma$. $\Delta_\perp := \partial^2/\partial y^2 + \partial^2/\partial z^2$ is the transversal Laplacian operator, and $\beta \equiv (z, 0)$.

Characteristic Galerkin

For $n = 1, 2 \ldots, N$

$$\begin{cases} \text{find } J^{h,n} \in \mathcal{V}_n \text{ such that} \\ \int_{I_y \times I_z} J^{h,n}(x_\perp) v(x_\perp)\, dx_\perp = \int_{I_y \times I_z} J^{h,n-1}(x_\perp - \hbar_n \beta) v(x_\perp)\, dx_\perp, \end{cases}$$

(-1)

where $\hbar_n = x_n - x_{n-1}$ and $J^{h,0} = f$.

Characteristic Galerkin

(-1) $$J^{h,n} = \mathcal{P}_n T_n J^{h,n-1},$$

Characteristic Streamline Diffusion

For $n = 1, 2, \ldots, N$, find $\hat{J}^h \equiv \hat{J}^h|S_n \in \hat{\mathcal{V}}_n$ such that

$$\int_{S_n} (\hat{J}^h_x + \beta \cdot \nabla_\perp \hat{J}^h)(v + \delta(v_x + \beta \cdot \nabla_\perp v))\, dx dx_\perp + \int_{S_n} \hat{\epsilon} \nabla_\perp \hat{J}^h \cdot \nabla_\perp v \, dx dx_\perp + \int_{I_\perp}$$

(-1)

where $v^n_\pm(x_\perp) = \lim_{\Delta x \to 0} v(x \pm \Delta x, x_\perp)$,
$\hat{\epsilon} = max(\epsilon, \mathcal{F}(Ch^\alpha \mathcal{R}(\hat{J}^h))/M_n$, with

$$\mathcal{R}(\hat{J}^h) = |\hat{J}^h_x + \beta \cdot \nabla_\perp \hat{J}^h| + |[\hat{J}^h]|/\hbar. \quad \text{(-1)}$$

Characteristic Streamline Diffusion

The streamline diffusion modification is given by $\delta(v_x + \beta \cdot \nabla_\perp v)$. If β is approximated by piecewise constants on each slab, the streamline diffusion modification will disappear in the CSD-method.

Characteristic Streamline Diffusion

For $n = 1, 2, \dots, N$, find $J^{h,n} \in \mathcal{V}_n$ such that

$$\int_{I_\perp} \hat{\epsilon}\nabla_\perp J^{h,n}\cdot\nabla_\perp v\, dx dx_\perp + \int_{I_\perp} J^{h,n} v\, dx_\perp = \int_{I_\perp} T_n J^{h,n-1} v\, dx_\perp, \qquad \forall \hat{v} \in \mathcal{V}$$

(-1)

where $J^{h,0} = f$ and $\hat{\epsilon} = \mathcal{F}(Ch_n^\alpha |J^{h,n} - T_n J^{h,n-1}|)/M_n$.

Characteristic Streamline Diffusion

We introduce

(-1)
$$(\hat{\mathcal{P}}_n w, v) = (\hat{\epsilon} \nabla_\perp \hat{\mathcal{P}}_n w, \nabla_\perp v) = (w, v), \qquad \forall v \in \hat{\mathcal{P}}_n,$$

to get

(-1)
$$J^{h,n} = \hat{\mathcal{P}}_n T_n J^{h,n-1}.$$

Implementation

- DOLFIN
  - user level.
  - module level.
  - kernel level.
  - Object Orientedness.

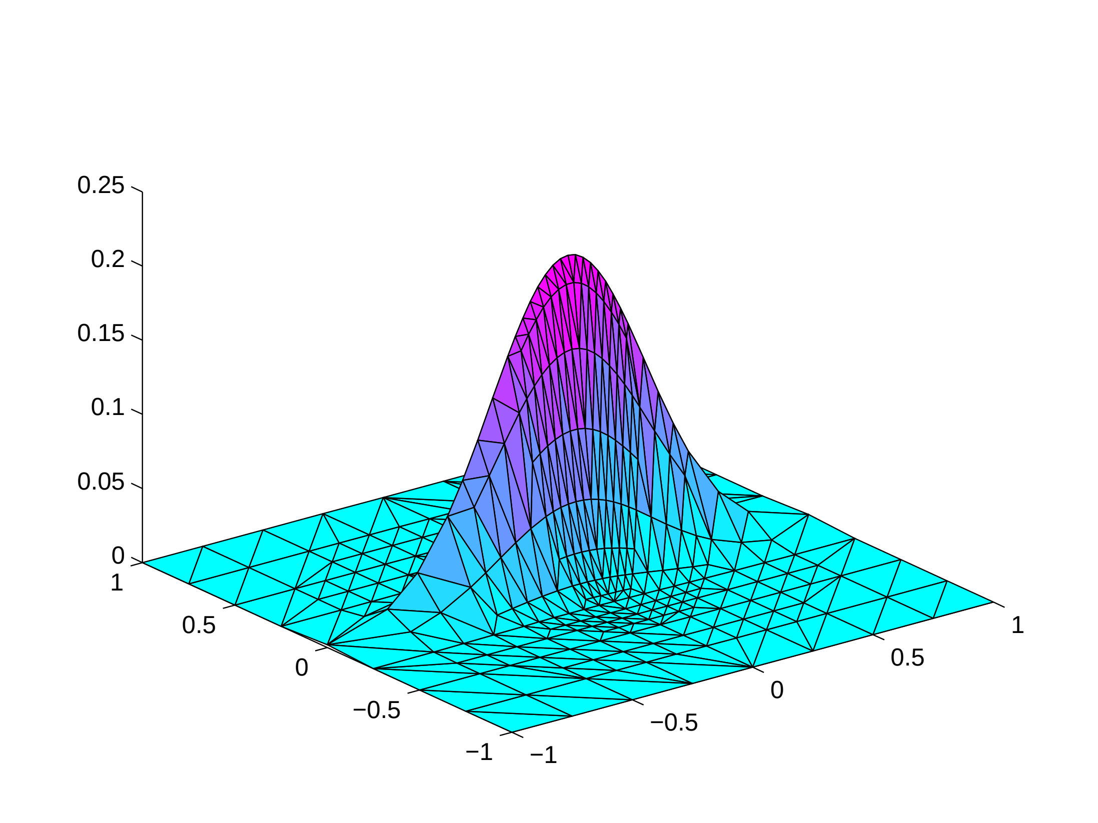
Implementation

Exact Solution.

(-1) $$J(x,y,z) = \frac{\sqrt{3}}{\pi\epsilon x^2} e^{-2(3(y/x)^2 - 3(y/2)z + z^2)/(\epsilon x)}.$$

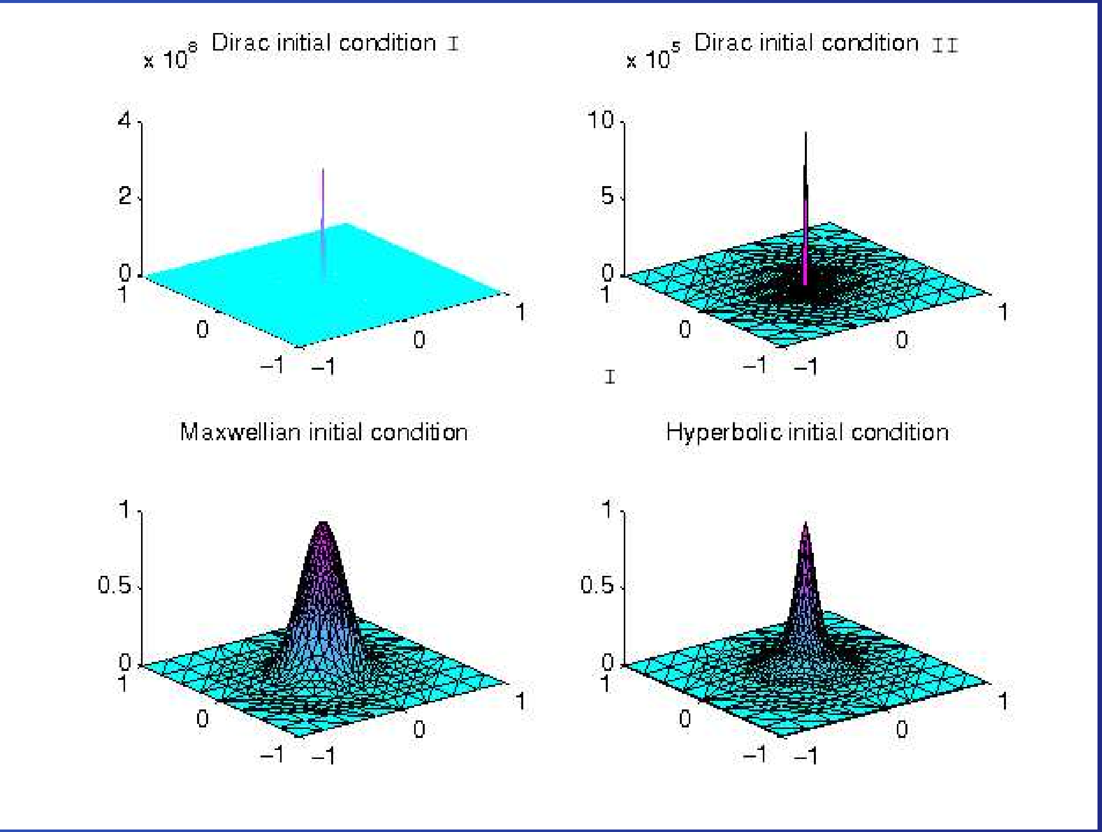
Implementation

Exact Solution.



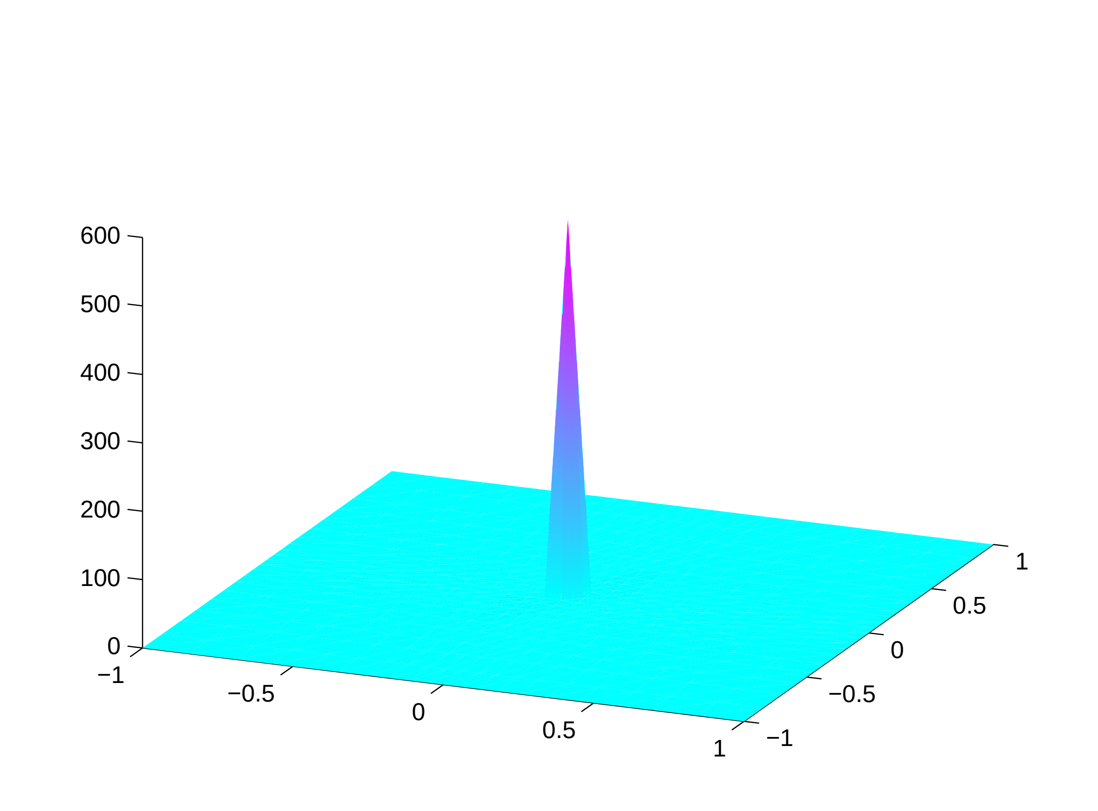
Results

Initial conditions used



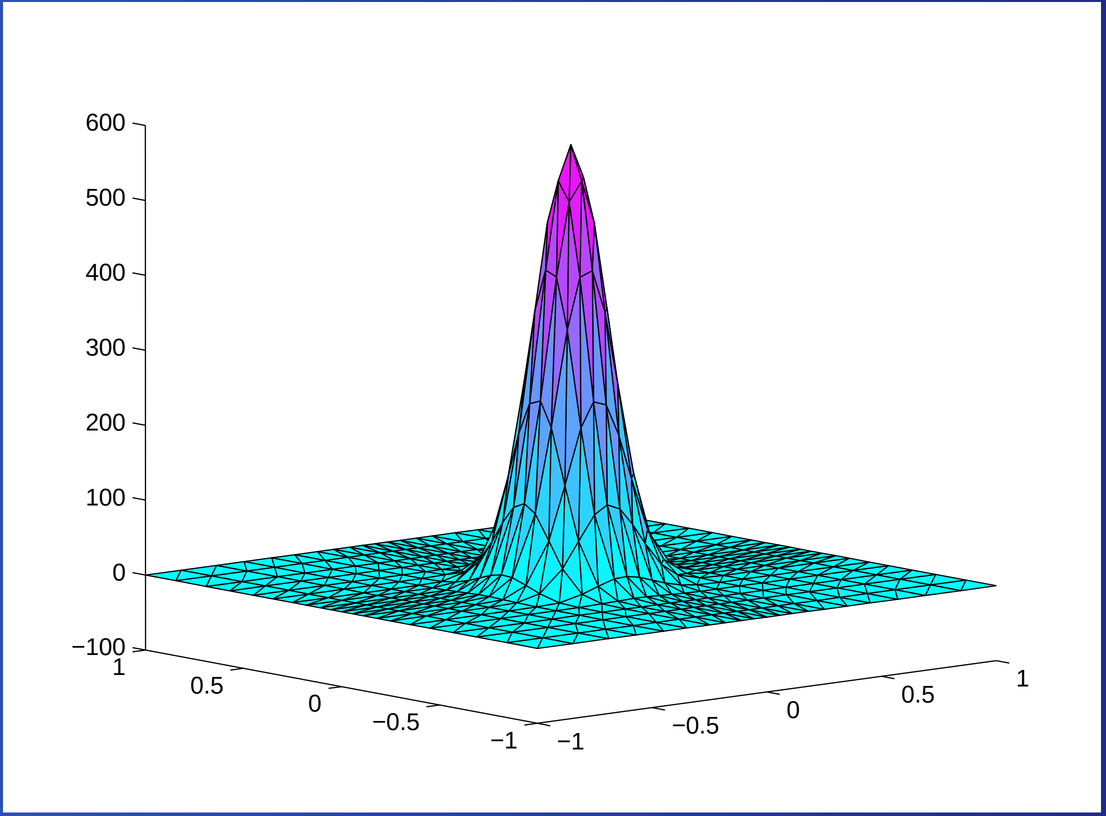
Results

Exact solution at $x = 1.0$



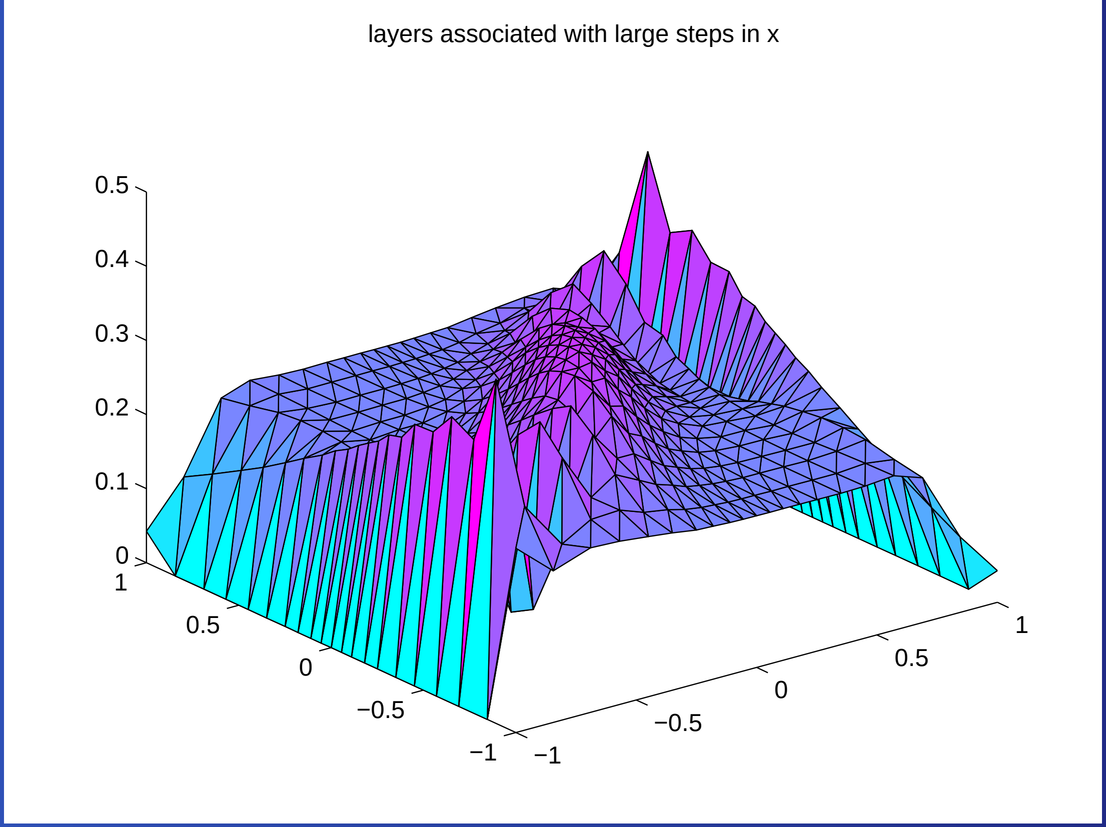
Results

Computed solution using CSD.



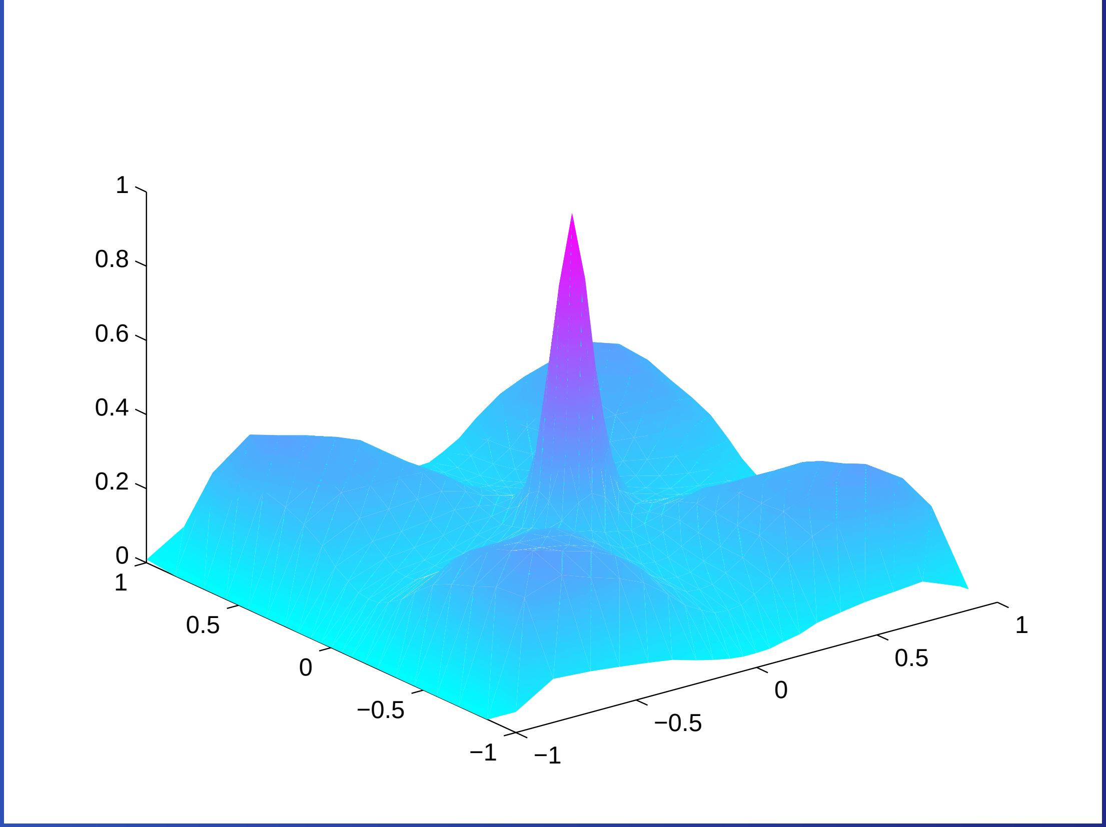
Results

Formation of layers.



Results

Oscillatory bahaviour.



Results

Streamline Diffusion and Characteristic Streamline Diffusion are stable and accurate methods for similar problems.