ASYMPTOTIC DIFFUSION AND SIMPLIFIED P_N APPROXIMATION FOR DIFFUSIVE AND DEEP PENETRATION PROBLEMS IN CHARGED RADIATION PARTICAL BEAM

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Abstract

The classical diffusion approximation to the linear Boltazmann transport equation is known to be accurate if the underlying physical system is (i) optically thick and (ii) scattering dominated. This approximation has been mathematically justified by an asymptotic analysis having a scaling that is consistent with these two conditions. Also the simplified $P_N(SP_N)$ equation has been shown to be higher-order asymptotic correction to the diffusion equation for the same class of the physical problems. In this study, we alter the asymptotic scaling that yields the standard diffusion and $(SP_N), N = 1, 2$ approximation to obtained *modified diffusion* and $(SP_N), N = 1, 2$ approximation. A successful study in this direction yields approximations that can be significantely more accurate for deep penetration problems, which are not scattering-dominated and hence we skip assumption (ii) that is required in classical approximation. This theoratical approach is valuable in madical physics, in the study of forward-peaked (convection-dominated, convection diffusion) radiation beams penetrating malignant cancer tumors.

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Introduction

In this thesis, we relay on an article by E. Larsen [1], where the asymptotic analysis is used to generate alternate forms of the SP_N equations, and derive SP_1 and SP_2 approximations. The diffusion and $SP_N(N = 1, 2, 3)$ approximations are extensively used in the neutron and photon transport problems which occur in the Nuclear Engineering field [1].
The simplified $P_N(SP_N)$ method was developed by E. Gelbard[18]-[20], where computer resources were used to solve $3 - D$ diffusion problems.
For the past 50 years, spherical harmonic mathodes are considered as a standard approximation techniques to the transport equation, see [13]. Gelbard considered a very old method as a simplification of the spherical harmonics method: the simplified P_N or SP_N method. 1960, Gelbard presented a simplification to the full spherical harmonics method that greatly reduced the number of unknowns, and also provided an analysis when the method was equivalent to the full P_N equations, see [3], [4], [5]. The P_N equations in slab geometry can be written as a system of 1-D diffusion equations as illustrated by the Gelbard's derivation but this is not possible in general geometry. It was by the formation of writing these 1-D equations in a $3-D$ form that go along with Gelbard's formal derivation of the simplified P_N or SP_N equations.
The SP_N equations were expressed as higher-order asymptotic corrections to the diffusion approximation for the same physical type of problems in late 1990's. In 1993 Pomraning and Larsen et al independentely, showed that SP_N equations as asymptotic correction to the standard diffusion theory. In these studies, one can say that Gelberd's equation helped to drive the asymptotic equation.
Recently, the SP_N equations accuracy are considered, and these are come up for the transport equation with the higher-order asymptotic approximation in the physical system where P_1 (conventional equations) is the leading order approximation, see [10], [13], [17]. We investigate the asymptotic analysis to drive the diffusion and SP_N approximations. To this approach, we have used Boltzmann transport equation with dominating asymptotic scaling collision term of order $0(\epsilon^{-1})$, leakage term of order $0(1)$, which absorption and source term of small order $0(\epsilon)$, where ϵ is a small dimensionless parameter. These scaling is valid for a physical system with the two above mentioned conditions (i) optically thick and (ii) scattering dominated.
By an insignificant change in the scaling or by the general asymptotic scaling we can derived the modified diffusion and SP_N approximations which is different from the standard diffusion and SP_N equations. For more accurate approximations the transport elements of physics are important. The goal of this study is to show that the modified diffusion and $SP_N(SP_1, SP_2)$ approximations will be significantly more accurate for the deep penetration problems. Our fours in the asymptotic analysis is to steady state, $3 - D$ anisotropically scattering particle transport in homogeneous, monoenergetic,

optically thick medium. This study is testing the theoretical foundations of modified diffusion and $SP_N(SP_1, SP_2)$ approximations equations of the standard notation of the Boltzmann transport equations.
In Section 1, the modified diffusion equation is derived using the standard notation of Boltzmann transport equation. Here, we study the asymptotic scaling and expand the solution in term of the small dimensionless parameter (ϵ), and we also get the standard diffusion equation.
In Section 2, we use alternate asymptotic analysis and expand the Boltzmann transport problem in terms of small ϵ that yields the same modified diffusion equation as in Section 1.
Section 3 is devoted to calcultes of the exponential decay rate and try to find the exact information about the solution and a justification about the fact that the standard diffusion equation have a different result than that of the transport equation.
In Section 4, asymptotic derivation of the P_1 or SP_1 and SP_2 equations are derived. This part is based on an asymptotic expansion of the Boltzmann transport equation using the asymptotic scaling defined in Section 1.

1. Asymptotic expansion of angular flux

We consider the linear Boltazmann transport equation in three dimension given by
(1.1)

$$\Omega \cdot \bigtriangledown_x \psi(x,\Omega) + \sigma_T \psi(x,\Omega) = \int_{s^2} \sigma_s(\Omega \cdot \Omega^{'})\psi(x,\Omega^{'})d\Omega^{'} + \frac{Q(x)}{4\pi}, \quad x \in V.$$

$$\psi(x,\Omega) = \psi^b(x,\Omega), \quad x \in \partial V, \quad \Omega\cdot\mathbf{n}<0,$$

where σ_T is the total cross section and σ_s is the scattering cross section. Throughout this study, we shall use the following notation:

$\psi(x,\Omega) :=$ angular flux (intensity).

$\Omega := (\sqrt{1-\mu^2}\cos\gamma, \sqrt{1-\mu^2}\sin\gamma, \mu) = (\Omega_1, \Omega_2, \Omega_3), \quad \Omega \in \mathrm{S}^2,$

for $\quad -1 \leq \mu \leq 1 \;, \quad 0 \leq \gamma \leq 2\pi \quad |\Omega| = 1.$

$x := (x_1, x_2, x_3)$ position.

$\sigma_t := \sigma_t(x,\Omega)$ transport cross section.

$\sigma_T := \sigma_T(x,\Omega)$ the total cross section.

$\sigma_a := \sigma_a(x,\Omega)$ absorption cross section.

$\sigma_s := \sigma_s(x, \Omega)$ scattering cross section.

$Q(x) :=$ Isotropic internal source.

The right hand side of equation (1.1) is called the scattering term. Using the Legender polynomials, we may expand σ_s as

$$\sigma_s(\Omega \cdot \Omega^{'}) = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} \sigma_{s,l} P_l(\Omega \cdot \Omega^{'}). \tag{1.2}$$

Here, by addition formula for Legender polynomials for the expansion of the surface harmonic components, the l^{th} Legender polynomial on the interval $I = (-1, 1)$ is defined as:

$$\mathrm{P}_l(\mu) = \tfrac{1}{2^l l!} (\tfrac{d}{d\mu})^l \Big[(\mu^2 - 1)^l\Big], \quad \text{for } l \geq 0.$$

The Legender polynomials $\{P_l\}$ are orthogonal in the sence that:

$$\int_{-1}^{1} P_l(\mu) P_k(\mu) d\mu = \frac{2}{2l+1} \delta_{lk}.$$

Where δ_{lk} is the Dirac delta function:

$$\delta_{lk} = 0, \text{ for } l \neq k \text{ and } \delta_{kk} = 1.$$

A square integrable function $f(\mu)$, defined on the interval $I = [-1, 1]$, can be expanded in the Legender polynomial as

$$\mathrm{f}(\mu) = \sum_{l=0}^{\infty} \frac{2l+1}{2} f_l P_l(\mu), \quad \text{for } (-1 \leq \mu \leq 1),\ \mu \in I.$$

Where

$$f_l = \int_{-1}^{1} P_l(\mu) f(\mu_0) d(\mu),$$

as the Legender coefficient of f.

To proced, we define, for $l \geq 0, 0 \leq k \leq l$, the weighted derivatives of the P_l given by,

$$P_{l,k}(\mu) = (1-\mu^2)^{k/2} \frac{d^k P_l(\mu)}{d\mu^k} = \frac{(1-\mu^2)^{k/2}}{2^l l!} \quad \frac{d^{l+k}}{d\mu^{l+k}} \cdot \Big((\mu^2 - 1)^l\Big).$$

Next we recall the spherical harmonic functions given by;

$$\mathrm{Y}_{l,k}(\Omega) = a_{lk} \cdot P_{l,k}(\mu) e^{ik\gamma}, \quad \text{for } l \geq 0, \quad 0 \leq k \leq l.$$

Where

$$a_{l,k} = (-1)^{\frac{k+|k|}{2}} \Big[\frac{2l+1}{4\pi}\frac{(l-|k|)!}{(l+|k|)!}\Big]^{1/2}.$$

With, the spherical harmonic function of order 0 given by

$$Y_{0,0}(\Omega) = a_{0,0} = (\frac{1}{4\pi})^{1/2}.$$

In this way we can find the spherical harmonic function of order $1, 2, 3, \cdots$ We shall also need the complex conjugate of spherical harmonic functions denoted by Y^*_{nm}. It is known that $Y_{lk}(\Omega)$ and $Y^*_{nm}(\Omega)$ are orthonormal and satisfy,

$$\begin{cases} \displaystyle\int_{S^2} Y_{lk}(\Omega)Y^*_{nm}(\Omega)d\Omega = \delta_{ln}\delta_{km}. \\ \\ \delta_{ln}\cdot\delta_{mk} = 0 \text{ if either } l \neq n \text{ or } m \neq k, \text{ or} \\ \delta_{ln}\cdot\delta_{mk} = 0 \text{ if both } l \neq n \text{ and } m \neq k. \end{cases}$$

Now, we define,

$$\bar{\sigma}_{s,l}(x,\Omega) = (\hat{\sigma}_{s,l}) + \zeta_l\hat{\sigma}_a, \quad for \quad l \geq 0,$$

and consider the operator κ_0 represanting the difference between collision and scattering that is defined as

$$\kappa_0\psi(\Omega) = \bar{\sigma}_{s,0}\psi(\Omega) - \sum_{l=0}^{\infty}\frac{2l+1}{4\pi}\bar{\sigma}_{s,l}\int_{s^2} P_l(\Omega.\Omega')\psi(\Omega')d\Omega'. \tag{1.3}$$

Using the addition theorem and orthogonality of the spherical harmonic functions, as we have defined above, we obtain

$$\kappa_0 Y_{l,k}(\Omega) = (\bar{\sigma}_{s,0} - \bar{\sigma}_{s,l})Y_{l,k}(\Omega). \tag{1.4}$$

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So, the spherical harmonic fnctions $Y_{l,k}(\Omega)$ are the eigenfunctions of κ_0, with the eigenvalues $(\bar{\sigma}_{s,0} - \bar{\sigma}_{s,l})$.

Now we assume that the physical system satisfies the following conditions:
C1) The physically system is optically thick.
C2) The typical length scale for the solution is 0(1), and typically length scale for the solution is a distance in which ψ varies by amount 0(1) .

The physical system has mean free path of magnitude σ_T^{-1} and total cross section is equal to the sum of the absorpation and scattering cross sections

$$\sigma_T(x,\Omega) = \sigma_a(x,\Omega) + \sigma_{s,0}(x,\Omega). \tag{1.5}$$

The angular flux, cross sections and internal source are continous and vary spatially by at most, a small amount over the distance of mean free path. We assume $C1$ and $C2$, further we assume anisotropic scattering: i.e, the scattering has different properties in different direction that's why it is not forward peaked, and we have the following relation between the Legendre coefficients of σ_s:s expansion,

$$\sigma_{s,l} \leq \sigma_{s,0} < 1.$$

Here, consider the asymptotic scaling for the angular flux defined as [6],

$$\begin{cases} \sigma_T(x,\Omega) = \frac{\hat{\sigma}_T(x,\Omega)}{\epsilon}. \\ \sigma_a(x,\Omega) = \epsilon\hat{\sigma}_a(x,\Omega). \\ \sigma_{s,l}(x,\Omega) = \frac{\hat{\sigma}_{s,l}(x,\Omega)}{\epsilon}, \quad \text{for } l \geq 0. \\ \text{Q(x)}=\epsilon\hat{Q}(x). \end{cases} \tag{1.6}$$

$\sigma_T(x,\Omega)$, $\sigma_a(x,\Omega)$, $Q(x)$ are $0(1)$ and ϵ is a small and dimensionless parameter defined by

$$\epsilon = \frac{1}{K}L.$$

Where L= mean free path , and K= Typical length scale for the flux $\psi(x,\Omega)$. The collision rate is grater than absorpation. So, the infinite medium solution using asymptotic scaling is defined by, see [1];

$$\psi_\infty(x,\Omega) = \frac{1}{4\pi}(\sigma_T - \sigma_{s,0})^{-1}Q(x).$$

$$\sigma_{s,0}(x,\Omega) = \frac{\hat{\sigma}_{s,0}(x,\Omega)}{\epsilon}, \quad \text{for } l \geq 0.$$

$$\sigma_T(x,\Omega) = \frac{\hat{\sigma}_T(x,\Omega)}{\epsilon}.$$

$$\psi_\infty(x,\Omega) = \frac{1}{4\pi}[\frac{1}{\epsilon}(\hat{\sigma}_T - \hat{\sigma}_{s,0})]^{-1}(\epsilon\hat{Q}(x)).$$

$$\sigma_a = \sigma_T - \sigma_{s,0}.$$

If we use asymptotic scaling, then we get

$$\epsilon\hat{\sigma}_a = (\frac{1}{\epsilon}\hat{\sigma}_T) - (\frac{1}{\epsilon})\hat{\sigma}_{s,0}, \quad \text{for } l \geq 0.$$

$$\psi_\infty(x,\Omega) = \frac{1}{4\pi}\epsilon\hat{\sigma}_a^{-1}(\epsilon\hat{Q}(x)).$$

$$\sigma_a = \epsilon\hat{\sigma_a}, \qquad Q(x) = \epsilon\hat{Q}(x).$$

Where as before the asymptotic scaling, we get the infinity medium solution

$$\psi_\infty(x,\Omega) = \tfrac{Q(x)}{4\pi\sigma_a}. \tag{1.7}$$

The physical system is optically thick, i.e, $\sigma_T = 0(\frac{1}{\epsilon})$, $\sigma_a = 0(\epsilon)$, $Q(x) = 0(\epsilon)$ and infinite medium solution is $0(1)$. We insert the aymptotic scaling (1.6) and equation(1.2) into equation (1.1), to obtain,

$$\begin{cases} \Omega\cdot\nabla_x\psi(x,\Omega) + \epsilon^{-1}\hat{\sigma}_T\psi(x,\Omega) = \\ \displaystyle\sum_{l=0}^{\infty}\frac{2l+1}{4\pi}\epsilon^{-1}\hat{\sigma}_{s,l}\int_{S^2}P_l(\Omega\cdot\Omega^{'})\psi(x,\Omega^{'})d\Omega^{'} + \epsilon^{-1}\hat{Q}(x). \end{cases} \tag{1.8}$$

Using asymptotic scaling, we can rewrite the equation (1.5) as,

$$\sigma_{s,0}(x,\Omega) = \tfrac{\hat{\sigma}_T(x,\Omega)}{\epsilon} - \epsilon\hat{\sigma}_a(x,\Omega), \tag{1.9}$$

which is of order $0(\frac{1}{\epsilon})$ and is consistant with the scaling of $\sigma_{s,l}$, for $l \geq 1$ as defined in equation (1.6). We may rewrite equation (1.9) as

$$\sigma_{s,0}(x,\Omega) = \tfrac{\hat{\sigma}_{s,0}(x,\Omega)}{\epsilon}. \tag{1.10}$$

Hence using also equations (1.5) and (1.10) we get

$$\sigma_T(x,\Omega) = \tfrac{\hat{\sigma}_{s,0}(x,\Omega)}{\epsilon} + \epsilon\hat{\sigma}_a(x,\Omega). \tag{1.11}$$

By using these asymptotic scalings and the above results we can describ the general asymptotic scaling as,

$$\begin{cases} \sigma_{s,l}(x,\Omega) = \tfrac{\hat{\sigma}_{s,l}(x,\Omega)+\zeta_l\hat{\sigma}_a(x,\Omega)}{\epsilon} - \epsilon\zeta_l\hat{\sigma}_a(x,\Omega)\ , \quad \text{for } l \geq 0. \\ \\ \sigma_T(x,\Omega) = \tfrac{\hat{\sigma}_{s,0}(x,\Omega)+\zeta_0\hat{\sigma}_a(x,\Omega)}{\epsilon} + \epsilon(1-\zeta_0)\hat{\sigma}_a(x,\Omega)\ . \end{cases} \tag{1.12}$$

Here, $\sigma_{s,l}(x,\Omega)$ and $\sigma_T(x,\Omega)$ are $0(1)$ and ζ_l are arbitrary constants.
The (1.12) are special cases of asymptotic scaling as defined in (1.6). Here $\sigma_a(x,\Omega)$, $Q(x)$ has the same scaling as in (1.6). We shall use this general asymptotic scaling and (1.6) in the standard notation of Boltazmann transport equation to get the modified diffusion equation. The arbitrary constants of this general asymptotic scaling defined as follow:
If: $\zeta_0 = 1$ and $\zeta_l = 0, \quad for \quad l \geq 1$, then, according to general asmptotic scaling, we get the standard scalings defined previously in equations (1.6) and (1.10). If: $\zeta_l = 0, \quad for \quad l \geq 0$. then, by the general asmptotic scaling, we get equations (1.6),(1.10) and (1.11) and we have all arbitary constants of order $0(1)$. Now, here we have two cases for optically thick physical system: one is subcritical case and the other one characterises near critical case [6,7]. In subcritical case the largest eigenvalue is less then 1 and in the near

critical case $\lambda_0(x)$ is equal to 1, thus $\phi_0 = \phi_0(\Omega)$, Now

$$\kappa_0\phi_0(\Omega) = \phi_0(\Omega), \tag{1.13}$$

where $\phi_0(\Omega)$ is the eigenfunction of κ_0 with eigenvalue is 1. As we defined before through the equations (1.3) and (1.4), zero eigenvalues are leading to,

$$\kappa_0\phi_0(\Omega) = 0.$$

When subcritical and near critical condition are satisfied, we can write ψ in powers series expansion in ϵ as

$$\psi(x,\Omega) = \sum_{m=0}^{\infty} \epsilon^m \psi(x,\Omega). \tag{1.14}$$

Now inserting the (1.6) and (1.12) into the equation (1.1), we get

$$\begin{cases} \Omega \bigtriangledown_x \psi(x,\Omega) + \Big(\dfrac{\bar{\sigma}_{s,0}(x,\Omega)}{\epsilon} + (1-\zeta_0)\epsilon\hat{\sigma}_a\Big)\psi(x,\Omega) = \\ \displaystyle\sum_{l=0}^{\infty}\frac{2l+1}{4\pi}\Big(\frac{\bar{\sigma}_{s,l}(x,\Omega)}{\epsilon} - (\zeta_l\epsilon\hat{\sigma}_a)\Big)\int_{S^2} P_l(\Omega\cdot\Omega')\psi(x,\Omega')d\Omega' \\ +\frac{\epsilon\hat{Q}(x)}{4\pi}. \end{cases} \tag{1.15}$$

where

$$\bar{\sigma}_{s,l}(x,\Omega) = \hat{\sigma}_{s,l} + \zeta_l\hat{\sigma}_a, \quad for \quad l \geq 0.$$

Inserting the (1.14) into (1.15) and identify the coefficients of equal powers of ϵ, Then we this obtain for $m \geq 0$,

$$\begin{cases} \bar{\sigma}_{s,0}\psi_m(x,\Omega) - \displaystyle\sum_{l=0}^{\infty}\frac{2l+1}{4\pi}\bar{\sigma}_{s,l}\int_{S^2} P_l(\Omega,\Omega')\psi_m(x,\Omega')d\Omega' = \\ \qquad -\Omega\bigtriangledown_x\psi_{m-1}(x,\Omega) - (1-\zeta_0)\hat{\sigma}_a\psi_{m-2}(x,\Omega) \\ -\hat{\sigma}_a\displaystyle\sum_{l=0}^{\infty}\frac{2l+1}{4\pi}\zeta_l\int_{S^2} P_l(\Omega.\Omega')\psi_{m-2}(x,\Omega')d\Omega' + \delta_{m,2}\frac{\hat{Q}(x)}{4\pi}. \end{cases} \tag{1.16}$$

If $m = 0$, then

$$\bar{\sigma}_{s,0}\psi_0(x,\Omega) - \sum_{l=0}^{\infty}\frac{2l+1}{4\pi}\bar{\sigma}_{s,l}\int_{S^2} P_l(\Omega,\Omega')\psi_0(x,\Omega')d\Omega' = 0, \tag{1.17}$$

where we have used $\psi_{-2} = \psi_{-1} = 0$. Equation (1.17) has the general solution by the spherical harmonic functions properties and equations (1.3) and

(1.4) are defined previously. If for an equation of the form

$$\kappa_0 \psi = 0. \tag{1.18}$$

Then, we have only the general solution

$$\psi(\Omega) = AY_{0,0}(\Omega) = A \cdot \tfrac{1}{\sqrt{4\pi}} = A\sqrt{4\pi} \cdot \tfrac{1}{4\pi} = C \cdot \tfrac{1}{4\pi}. \tag{1.19}$$

where C is a constant. So by (1.18) and (1.19), we get the general solution of (1.17) as:

$$\psi_0(x,\Omega) = \tfrac{\phi_0(x)}{4\pi}. \tag{1.20}$$

where $\phi_0(x)$ is arbitrary. For $m = 1$ using (1.16) and (1.20), we get:

$$\begin{cases} \bar{\sigma}_{s,0}(x,\Omega)\psi_1(x,\Omega) - \displaystyle\sum_{l=0}^{\infty} \frac{2l+1}{4\pi} \bar{\sigma}_{s,l} \int_{S^2} P_l(\Omega,\Omega^{'})\psi_1(x,\Omega^{'})d\Omega^{'} = \\ \qquad -\Omega \bigtriangledown_x \tfrac{\phi_0(x)}{4\pi}. \end{cases} \tag{1.21}$$

Using the equation (1.13), we have the general solution in the form

$$\psi(\Omega) = \tfrac{C}{4\pi} + (\tfrac{1}{\hat{\sigma}_t})(\Omega). \tag{1.22}$$

Where $\hat{\sigma}_t = \bar{\sigma}_{s,0} - \bar{\sigma}_{s,1}$. Then, the general solution of the equation (1.21) will be of the form

$$\psi_1(x,\Omega) = \tfrac{\phi_1}{4\pi} - (\tfrac{1}{3\hat{\sigma}_t})\Omega \cdot \bigtriangledown_x \tfrac{\phi_0(x)}{4\pi}. \tag{1.23}$$

Where ϕ_1 is arbitary. If we put $m = 2$ in equation (1.16) and also use the results for $m = 0$ and $m = 1$, then by (1.20) and (1.23), we get

$$\begin{cases} \bar{\sigma}_{s,0}\psi_2(x,\Omega) - \displaystyle\sum_{l=0}^{\infty} \frac{2l+1}{4\pi} \bar{\sigma}_{s,l} \int_{S^2} P_l(\Omega,\Omega^{'})\psi_2(x,\Omega^{'})d\Omega^{'} = \\ -\Omega \cdot \bigtriangledown_x \Big[\dfrac{\phi_1}{4\pi} - \Big(\dfrac{1}{3\hat{\sigma}_t}\Omega\Big) \bigtriangledown_x (\dfrac{\phi_0}{4\pi})\Big] - \hat{\sigma}_a \dfrac{\phi_0(x)}{4\pi} + \dfrac{\hat{Q}(x)}{4\pi}. \end{cases} \tag{1.24}$$

We denote $-\Omega \bigtriangledown_x [\frac{\phi_1}{4\pi} - \frac{1}{3\hat{\sigma}_t}\Omega \bigtriangledown_x (\frac{\phi_0}{4\pi})]$, by J .
The resultant equation (1.24) , has a solvability condition, because the right hand side of this equation vanishes after taking the integration over the unit sphere (S^2). Also using the identities from J, as those introduced in equation (1.22), we get

$$\begin{cases} \int_{S^2} \Omega \cdot \nabla_x(\phi_1(x))d\Omega = 0. \\ \int_{S^2} \Omega \cdot \nabla_x \frac{1}{3\hat{\sigma}_t}\Omega \cdot \nabla_x(\phi_1(x))d\Omega = 4\pi \nabla_x \left(\frac{1}{3\hat{\sigma}_t}\right) \nabla \phi_0(x). \end{cases} \tag{1.25}$$

Then (1.25) and (1.24), yield

$$0{=}\nabla_x\left(\frac{1}{3\hat{\sigma}_t} \nabla_x \right)\phi_0(x) - \hat{\sigma}_a\phi_0(x) + \hat{Q}(x). \tag{1.26}$$

Multiplying (1.26) by ϵ, also using general aymptotic scaling defined in (1.13), and rearrange the terms we get the equation

$$- \nabla_x \left(\frac{\epsilon}{3\hat{\sigma}_t} \nabla_x \right)\phi_0(x) - \epsilon\hat{\sigma}_a\phi_0(x) = \epsilon\hat{Q}(x). \tag{1.27}$$

Now using the scaling (1.6) and (1.12) and then in terms of non-asymptotic scaling this is written as:

$$- \nabla_x \left(\frac{1}{3(\sigma_t + \eta_d\sigma_a)} \nabla_x \right)\phi_0(x) - \sigma_a\phi_0(x) = Q(x). \tag{1.28}$$

Where η_d is a diffusion parameter and if it is equal to zero in (1.26) then, we get the convential diffusion equation or standard diffusion equation.

$$- \nabla_x (D_d\nabla_x)\phi_0(x) - \sigma_a\phi_0(x) = Q(x). \tag{1.29}$$

Where D_d is a diffusion coeffients of order $0(\epsilon^3)$. The equation (1.28) is known as the modified diffusion equation, which is also called as P_1 equation.

2. **Asymptotic Expansion of Boltazmann Equation**

We can find simplified P_N equations by expanding the linear Boltazmann transport equation, instead of its solution, see in [14],[18]. For this approach, we take the standard notation of linear Boltazmann transport equation (1.1) togather with (1.2):

$$\begin{aligned} \Omega \cdot \nabla_x\psi(x,\Omega) + \sigma_T\psi(x,\Omega) = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi}\sigma_{s,l} \int_{s^2} P_l(\Omega \cdot \Omega^{'})\times \\ \psi(x,\Omega^{'})d\Omega^{'} + \tfrac{Q(x)}{4\pi}, \quad \text{x} \in V. \end{aligned} \tag{2.1}$$

We can also write the equation in asymptotic scaling forms as defined previously in equation (1.4) and get (2.1) reformulated as

$$\begin{aligned}\Omega \cdot \bigtriangledown_x \psi(x,\Omega) + \tfrac{\hat{\sigma}_T(x,\Omega)}{\epsilon}\psi(x,\Omega) = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} \cdot \frac{\hat{\sigma}_{s,l}(x,\Omega)}{\epsilon} \times \\ \int_{s^2} P_l(\Omega \cdot \Omega^{'})\psi(x,\Omega^{'})d\Omega^{'} + \frac{\epsilon \hat{Q}(x)}{4\pi}.\end{aligned} \tag{2.2}$$

Now, we define the scalr flux and current as

$$\phi_0(x) = \int_{s^2} \psi(x,\Omega)d\Omega, \tag{2.3}$$

and

$$\phi_1(x) = \int_{s^2} \Omega\psi(x,\Omega)d\Omega, \tag{2.4}$$

respectively. Let now

$$\mathrm{P}\psi(x,\Omega) = \tfrac{1}{4\pi}\int_{s^2} \psi(x,\Omega^{'})d\Omega'. \tag{2.5}$$

than, applying the operators P and $I-P$ on equation (2.2), we get the *balance equation*:

$$\bigtriangledown_x \phi_1(x) + \frac{1}{\epsilon}\Big[\hat{\sigma}_T(x,\Omega) - \hat{\sigma}_{s,0}(x,\Omega)\Big]\phi_0(x) = \epsilon\hat{Q}(x) \tag{2.6}$$

and

$$\begin{aligned}(I-P)\Omega \bigtriangledown_x \psi(x,\Omega) + \frac{\hat{\sigma}_T(x,\Omega)}{\epsilon}\Big(\psi(x,\Omega) - \frac{1}{4\pi}\phi_0(x)\Big) = \\ \frac{1}{\epsilon}\int_{s^2}\Big(\sum_{l=1}^{\infty}\frac{2l+1}{4\pi}\hat{\sigma}_{s,l}P_l(\Omega\cdot\Omega^{'})\Big)\psi(x,\Omega)d\Omega^{'}.\end{aligned} \tag{2.7}$$

If we define the Laplace operator L by

$$L\psi(x,\Omega) = \hat{\sigma}_T\psi(x,\Omega) - \int_{s^2}\sum_{l=1}^{\infty}\frac{2l+1}{4\pi}\hat{\sigma}_{s,l}P_l(\Omega\cdot\Omega^{'})\psi(x,\Omega^{'})d\Omega^{'} \tag{2.8}$$

Then inserting (2.8) in (2.7), we get

$$\mathrm{L}\psi(x,\Omega) + \epsilon\cdot(I-P)\Omega\bigtriangledown_x\psi(x,\Omega) = \tfrac{\sigma_T}{4\pi}\phi_0(x). \tag{2.9}$$

Here, L is the same as the collision operator that have $O(\frac{1}{\epsilon})$ terms in equation (2.2), where the summ over L does not include $l=0$ term of the scattering operator.

Thus, if scattering is isotropic L reduces to a simple multiplicative operator.

Here, L^{-1} exists and is of order 0(1). Thus equation (2.9) can be rewriten as:

$$[\mathrm{I}+\epsilon L^{-1}(I-P)\Omega\cdot\bigtriangledown_x]\psi(x,\Omega)=\tfrac{1}{4\pi}\phi_0(x),$$

i.e.

$$\psi(x,\Omega)+\bigtriangledown_x\psi(x,\Omega)\Big(\epsilon L^{-1}(I-P)\Omega\Big)=\frac{1}{4\pi}\phi_0(x),$$

and hence

$$\psi(x,\Omega)=[I+\epsilon L^{-1}(I-P)\Omega\cdot\bigtriangledown_x]^{-1}\frac{\phi_0(x)}{4\pi}. \tag{2.10}$$

This equation (2.10) inserted into te current equation (2.4), implies that

$$\phi_1(x)=\frac{1}{4\pi}\int_{s^2}\Omega[I+\epsilon L^{-1}(I-P)\Omega\cdot\bigtriangledown_x]^{-1}d\Omega\,\phi_0(x). \tag{2.11}$$

Hence, using equation (2.11) in the Balance equation (2.6), we end up with

$$\begin{cases} \left(\frac{1}{4\pi}\right)\cdot\int_{s^2}\Omega\cdot\bigtriangledown_x[I+\epsilon L^{-1}(I-P)\Omega\cdot\bigtriangledown_x]^{-1}d\Omega\,\phi_0(x) \\ \qquad\qquad +\frac{1}{\epsilon}(\hat\sigma_T-\hat\sigma_{s,0})\phi_0=\epsilon\hat Q(x). \end{cases} \tag{2.12}$$

Equations (2.6) and (2.11) make an exact system of equations for the current ϕ_1 and the scalar flux ϕ_0. However, equation (2.12) can be more simplified for immediate use, so we approximate it by expanding for $\epsilon << 1$ and the result is then

$$\phi_1\approx\sum_{l=0}^{\infty}\epsilon^l L_l\phi(x),$$

where the oprator L is defied by

$$\mathrm{L}\phi(x)=\tfrac{1}{4\pi}\int_{s^2}\Omega\cdot\bigtriangledown_x[I+\epsilon L^{-1}(I-P)\Omega\cdot\bigtriangledown_x]^{-1}\phi_0(x)d\Omega.$$

$$\mathrm{L}\phi(x)=\sum_{l=0}^{\infty}\frac{(-1)^l}{4\pi}\int_{s^2}\Omega\cdot\bigtriangledown_x[L^{-1}(I-P)\Omega\cdot\bigtriangledown_x]^l\phi_0(x)d\Omega, \tag{2.13}$$

and

$$\mathrm{L}\phi(x)=\sum_{l=0}^{\infty}(-1)^l L_l\phi_0(x), \tag{2.14}$$

where

$$L_l=\frac{1}{4\pi}\int_{s^2}\Omega\cdot\bigtriangledown_x[L^{-1}(I-P)\Omega\cdot\bigtriangledown_x]^l d\Omega.$$

The first few of the operators L_l can be easily evaluated using the equation (2.14).

Now, we define some spherical harmonic functions which we shall need to proceed our expansion:

A constant is proportional to the spherical harmonic function of order 0.

For $1 \leq i \leq 3$, each of the 3 funtions,

$$\omega_i \equiv \Omega_i, \tag{2.15}$$

is a linear combination of spherical harmonic funtions of order 1.

For $1 \leq i, j \leq 3$, each of these 9 functions

$$\omega_{i,j} \equiv \Omega_i \Omega_j - \tfrac{1}{3}\delta_{i,j}. \tag{2.16}$$

is a linear combination of spherical harmonic functions of order 2.

For $1 \leq i, j, k \leq 3$, each of the 27 functions below

$$\omega_{i,j,k} \equiv \Omega_i \Omega_j \Omega_k - \tfrac{1}{5}(\Omega_i \delta_{j,k} + \Omega_j \delta_{ki} + \Omega_k \delta_{i,j}), \tag{2.17}$$

is a linear combination of sphrical harmonic fuctions of order 3.
Therefore, with L defined by the equation (2.9) L^{-1} can be written as

$$\begin{cases} \mathrm{L}^{-1}(I-P)\omega_i = (\hat{\sigma}_T - \hat{\sigma}_{s,1})^{-1}\omega_i. \\ \mathrm{L}^{-1}(I-P)\omega_{i,j} = (\hat{\sigma}_T - \hat{\sigma}_{s,2})^{-1}\omega_{i,j}. \\ \mathrm{L}^{-1}(I-P)\omega_{i,j,k} = (\hat{\sigma}_T - \hat{\sigma}_{s,3})^{-1}\omega_{i,j,k}. \end{cases} \tag{2.18}$$

Equation's ((2.15)-(2.18)) are used to derive the following results.
Let

$$\begin{cases} \Omega \cdot \bigtriangledown = \displaystyle\sum_{i=1}^{3} \Omega_i \bigtriangledown_i. \\ \Omega \cdot \bigtriangledown = \displaystyle\sum_{i=1}^{3} \omega_i \bigtriangledown_i. \quad (\omega_i \equiv \Omega_i) \end{cases} \tag{2.19}$$

Then,

$$L^{-1}(I-P)\Omega \cdot \bigtriangledown = \sum_{i=1}^{3} (\hat{\sigma}_T - \hat{\sigma}_{s,1})^{-1}\omega_{ii}. \tag{2.20}$$

$$\begin{cases} \Omega\cdot\bigtriangledown(L^{-1}(I-P)\Omega\cdot\bigtriangledown) = \sum_{i,j=1}^{3}(\Omega_j\bigtriangledown_j)((\hat{\sigma}_T-\hat{\sigma}_{s,1})^{-1}\omega_i\bigtriangledown_i). \\ \Omega\cdot\bigtriangledown(L^{-1}(I-P)\Omega\cdot\bigtriangledown) = \sum_{i,j=1}^{3}(\Omega_j\omega_i)(\bigtriangledown_j(\hat{\sigma}_T-\hat{\sigma}_{s,1})^{-1}\bigtriangledown_i). \\ \Omega\cdot\bigtriangledown(L^{-1}(I-P)\Omega\cdot\bigtriangledown) = \sum_{i=j=1}^{3}(\omega_{i,j}+\frac{1}{3}\delta_{i,j})(\bigtriangledown_j(\hat{\sigma}_T-\hat{\sigma}_{s,1})^{-1}\bigtriangledown_i). \end{cases}$$

$$\begin{cases} \Omega\cdot\bigtriangledown(L^{-1}(I-P)\Omega\cdot\bigtriangledown) = \sum_{i=j=1}^{3}\Omega_{i,j}(\bigtriangledown_j(\hat{\sigma}_T-\hat{\sigma}_{s,1})^{-1}\bigtriangledown_i)+ \\ (\bigtriangledown\cdot\tfrac{1}{3}(\hat{\sigma}_T-\hat{\sigma}_{s,1})^{-1}\bigtriangledown). \end{cases} \tag{2.21}$$

Using equations (2.19)-(2.21), in equation (2.13) , we get,

$$L_0 = \tfrac{1}{4\pi}\sum_{l=0}^{\infty}\int_{s^2}\Omega\cdot\bigtriangledown d\Omega = 0. \tag{2.22}$$

$$L_1 = \frac{1}{4\pi}\int_{s^2}\Omega\cdot\bigtriangledown(L^{-1}(I-P)\Omega\cdot\bigtriangledown)d\Omega. \tag{2.23}$$

$$L_1 = \bigtriangledown\cdot\frac{1}{3}(\hat{\sigma}_T-\hat{\sigma}_{s,1})^{-1}\bigtriangledown. \tag{2.24}$$

Here, we may say that $\sigma_T - \sigma_{s,1}$ is equal to the transport cross section and defined by

$$\sigma_t = \sigma_T - \sigma_{s,1}.$$

$$\frac{\hat{\sigma}_t}{\epsilon} = \frac{\hat{\sigma}_T}{\epsilon} - \frac{\hat{\sigma}_{s,1}}{\epsilon}.$$

Then multiplying by ϵ, we get equation (2.24) in the following form:

$$L_1 = \bigtriangledown\cdot\frac{1}{3\sigma_t}\bigtriangledown. \tag{2.25}$$

Equations (2.22) and (2.24) are exact for homogeneous and heterogeneous media. These equation introduced in equation (2.13), togather with asymptotic scalings defined in (1.12) and (1.6), and inserted in equation (2.12), gives:

$$-\bigtriangledown\cdot\Big(\frac{1}{3(\sigma_t+\eta_d\sigma_a)}\bigtriangledown\Big)\phi_0(x) + \frac{1}{\epsilon}(\hat{\sigma}_T-\hat{\sigma}_{s,0})\phi_0(x) = \epsilon\hat{Q}(x).$$

$$-\bigtriangledown\cdot\Big(\frac{1}{3(\sigma_t+\eta_d\sigma_a)}\bigtriangledown\Big)\phi_0(x)+(\epsilon\hat{\sigma}_a)\phi_0(x)=\epsilon\hat{Q}(x)+0(\epsilon^3). \tag{2.26}$$

Without aymptotic scaling this equation would be in the form

$$-\bigtriangledown\cdot\Big(\frac{1}{3(\sigma_t+\eta_d\sigma_a)}\bigtriangledown\Big)\phi_0(x)+\sigma_a\phi_0(x)=Q(x)+0(\epsilon^3). \tag{2.27}$$

Omitting $0(\epsilon^3)$ term, the equation (2.27) is the same as the modified diffusion equation approximation that is find in Section 1. But here we get this equation by the asymptotic expansion of the Boltazmann transport equation. The standard diffusion approximation is obtained if diffusion parameter $\eta_d = 0$:

$$-\bigtriangledown(D_d\bigtriangledown)\phi_0(x)+\sigma_a\phi_0(x)=Q(x), \tag{2.28}$$

where $D_d = \frac{1}{3\sigma_t}$ is a diffusion coefficient in standard diffusion equation.
Equation (2.28) is the same as the modified diffusion result which we previously drived in Section 1.
In next Section, we shell disscuse about exponential decay rate, in which calculating the exponential decay rate of the solution we try to find exact information about the solution and also that the standard diffusion equation has a different solution than that of the transport equation.

3. **Exponential Decay Rate**

We consider the planar geometry tranport eqaution and if we also assume linear anisotropic scattering, then the transport equation given in the plane geometry is defined by

$$\begin{cases} \mu\bigtriangledown_x\cdot\psi(x,\mu)+\sigma_T\psi(x,\mu)=\dfrac{\sigma_{s,0}}{2}\displaystyle\int_{-1}^{1}\psi(x,\mu^{'})d\mu^{'}+ \\ \qquad\frac{3\sigma_{s,1}}{2}\mu\displaystyle\int_{-1}^{1}\mu^{'}\psi(x,\mu')d\mu'. \end{cases} \tag{3.1}$$

Equation (3.1) has a solution that osccilates exponentially in space [19 , 20] and is given by

$$\psi(x,\mu)=Cb(\mu)e^{-\sigma_T N_{tr}x}. \tag{3.2}$$

Where C and N_{tr} are constants, $b(\mu)$ is unknown and normalized as:

$$\mathrm{b}_0=\int_{-1}^{1}b(\mu^{'})d\mu^{'}=1. \tag{3.3}$$

Inserting (3.2) into the equation (3.1), we get

$$\begin{cases} -\mu C b(\mu) N_{tr}\sigma_T e^{-\sigma_T N_{tr} x} + \sigma_T C b(\mu) e^{-\sigma_T N_{tr} x} = \frac{\sigma_{s,0}}{2}\int_{-1}^{1} C b(\mu^{'}) e^{-\sigma_T N_{tr} x} d\mu^{'} + \\ \frac{3\sigma_{s,1}}{2}\mu\int_{-1}^{1} C\mu^{'} b(\mu^{'}) e^{-\sigma_T N_{tr} x} d\mu^{'}. \end{cases}$$

$$-\mu b(\mu) N_{tr} + b(\mu) = \tfrac{1}{2}\tfrac{\sigma_{s,0}}{\sigma_T}\int_{-1}^{1} b(\mu^{'}) d\mu^{'} + \frac{3}{2}\frac{\sigma_{s,1}}{\sigma_T}\mu\int_{-1}^{1}\mu^{'} b(\mu^{'}) d\mu^{'}. \tag{3.4}$$

Here, we defined some quantities, which we have use in what follows:

$$\mathrm{b}_1 = \int_{-1}^{1}\mu^{'} b(\mu^{'}) d\mu^{'}. \tag{3.5}$$

$$s = \frac{\sigma_{s,0}}{\sigma_T} = scattering \quad ratio. \tag{3.6}$$

$$\bar{\mu}_0 = \frac{\sigma_{s,1}}{\sigma_T} = mean \quad scattering \quad ratio. \tag{3.7}$$

Inserting (3.6) and (3.7) in the equation (3.4), we get

$$(\text{-}\mu N_{tr} + 1) b(\mu) = \tfrac{1}{2} s\int_{-1}^{1} b(\mu^{'}) d\mu^{'} + \frac{3}{2}\bar{\mu}_0\mu\int_{-1}^{1}\mu^{'} b(\mu^{'}) d\mu^{'}.$$

Then, using (3.3) and (3.5) yields

$$(-\mu N_{tr} + 1) b(\mu) = \frac{1}{2}s + \frac{3}{2}\bar{\mu}_0\mu b_1. \tag{3.8}$$

We integrate (3.8) over $\mu \in (-1, 1)$, to get

$$\int_{-1}^{1} b(\mu)(-\mu N_{tr} + 1) d\mu = \frac{1}{2}\int_{-1}^{1}(s + 3\bar{\mu}_0\mu b_1) d\mu.$$

Performing this integration we have

$$\text{-N}_{tr} b_1 + 1 = s., \quad b_1 = \tfrac{1-s}{N_{tr}}. \tag{3.9}$$

Now, we insert the value of b_1 from (3.9) into the equation (3.8), to obtain:

$$b(\mu)(1 - \mu N_{tr}) = \frac{1}{2}\Big(s + 3\bar{\mu}_0(\frac{1-s}{N_{tr}})\mu\Big).$$

This gives

$$b(\mu) = \frac{1}{2}\cdot\frac{(s + 3\bar{\mu}_0(\frac{1-s}{N_{tr}})\mu)}{(1 - \mu N_{tr})}. \tag{3.10}$$

Here, we may state that equation (3.10) defines $b(\mu)$ in term of N_{tr}. By the normalization relation (3.3), we have that,

$$1 = \frac{1}{2}\int_{-1}^{-1} \frac{(s + 3\bar{\mu}_0(\frac{1-s}{N_{tr}})\mu)}{(1-\mu N_{tr})} \times \frac{(1+\mu N_{tr})}{(1+\mu N_{tr})} d\mu.$$

In other words,

$$1{=}\int_{0}^{1} \frac{(s + 3\bar{\mu}_0(1-s)\mu^2)}{(1-\mu^2 N_{tr}^2)} d\mu. \tag{3.11}$$

This is as dispersion law. Thus, the modified diffusion equation corresponding to the equation (3.1) is given by:

$$\tfrac{1}{3(\sigma_t)} \cdot \bigtriangledown_x^2 \phi(x) + \sigma_a \phi(x) = 0. \tag{3.12}$$

Where the transport cross section $\sigma_t = \sigma_{a,1} + \eta_d \sigma_a$. if $\eta_d = 0$ then we have a diffusion equation.
Equation (3.12) has the exponential solution defined as:

$$\phi(x) = Ce^{-\sigma_T N_d x}. \tag{3.13}$$

Where using (3.13) N_d satisfies

$$\sigma_a - \tfrac{\sigma_T^2 N_d^2}{3\sigma_t} = 0. \tag{3.14}$$

Using equations (3.5)-(3.7) into (3.14) for solving N_d^2 value, we obtained:

$$N_d^2 = 3\Big(1 - \frac{\sigma_{s,1}}{\sigma_T}\Big)\Big(1 - \frac{\sigma_{s,0}}{\sigma_T}\Big) + 3\Big(\eta_d + \frac{\sigma_{s,1}}{\sigma_T}\Big)\Big(1 - \frac{\sigma_{s,0}}{\sigma_T}\Big)^2,$$

i.e.

$$\mathrm{N}_d^2 = 3(1-\bar{\mu}_0)(1-s) + 3(\eta_d + \bar{\mu}_0)(1-s)^2. \tag{3.15}$$

By the standard diffusion theory $\eta_d = 0$ and N_d does not satisfy (3.11) thus $N_{tr} \neq N_d$.

So by the above result i.e. $N_{tr} \neq N_d$, we may say that the standard diffuion equation has different exponential solution as compared to the transport eqaution.

This is the main reason why the solutions of standard diffusion eqaution for deep panetration shielding problems can be highly inaccurate, which is a strong reason why it is not used in these types of applications.

The modified SP_3 equations corresponding to equation (3.1), are given by

$$\left\{\begin{array}{l} \frac{1}{3\sigma_t}\cdot\nabla_x^2(\phi(x)+2\phi_2(x))+\sigma_a\phi(x)=0. \\ \\ \dfrac{9}{35(\sigma_T+\eta_{sp_3}\sigma_a)}\cdot\nabla_x^2\phi_2(x)+\sigma_T\phi_2(x)=\dfrac{4}{5}\sigma_a(x). \end{array}\right. \tag{3.16}$$

The equations (3.16) have exponentially varying solutions respectively given below:

$$\left\{\begin{array}{c} \phi(x)=Ce^{-\sigma_T N_{sp_3}x}. \\ \phi_2(x)=C\beta e^{-\sigma_T N_{sp_3}x}. \end{array}\right. \tag{3.17}$$

Where C is a given constant. The exponentially varing solutions inserted into the modified SP_3 equations (3.16), and resultant equations, yields the system of equations

$$\left\{\begin{array}{l} \Big(\dfrac{1}{3(1-s\bar{\mu}_0)}\Big)(1+2\beta)+(1-c)=0. \\ \\ \Big(\dfrac{9N_{sp_3}^2}{35[1+\eta_{sp_3}(1-s)]}+1\Big)\beta=\dfrac{2}{5}(1-s). \end{array}\right. \tag{3.18}$$

From the equation (3.18), we can determin the valuse of $N_{sp_3}^2$,
where we put $\eta_{sp_3}=0$, then we get the value of N_{sp_3} does not satisfy the (3.11) and (3.15) thus $N_{tr}\neq N_{sp_3}$. So, this is the reason, why it does not contain the exponential decay rate of solutions for the standard diffusion and SP_3 equations of Boltazmann transport equation and standard SP_3 equations are not acurate for the deep penetration problems that are not approximate Boltazmann equations.
The term N_{tr} is more important. If we take $N_d=N_{tr}$, $N_{sp_3}=N_{tr}$ in modified diffusion equation (3.12) and modified SP_3 equations (3.16), and solve these equations for η_d, η_{sp_3} then we get the following resultant values:

$$\sigma_T^2N_{tr}^2=3\sigma_{a,1}\sigma_a+3\eta_d\sigma_a^2.$$

This implies that

$$\eta_d=\frac{\sigma_T^2N_{tr}^2}{3\sigma_a^2}-\frac{\sigma_{a,1}\sigma_a}{\sigma_a^2}.$$

$$\eta_d=\frac{1}{\sigma_a}\Big(\frac{\sigma_T^2N_{tr}^2}{3\sigma_a}-\sigma_{a,1}\Big). \tag{3.19}$$

Here, η_{sp_3} is given by:

$$\eta_{sp_3}=\frac{\sigma_T}{\sigma_a}\Big[\frac{9N_{tr}^2(\sigma_T^2N_{tr}^2-3\sigma_a\sigma_{a,1})}{35[\sigma_T(\sigma_T+\frac{4}{5}\sigma_a)N_{tr}^2-3\sigma_a\sigma_{a,1}]}-1\Big]. \tag{3.20}$$

Using scattering ration (3.6) and mean scattering ration (3.7) in the equations (3.19) and (3.20), we get:

$$\eta_d = \frac{1}{1-s}\Big(\frac{N_{tr}^2}{3(1-s)} - (1-\bar{\mu}_0 s)\Big). \tag{3.21}$$

$$\eta_{sp_3} = \frac{1}{1-s}\Big[\frac{9N_{tr}^2(N_{tr}^2 - 3(1-s)(1-s\bar{\mu}_0))}{35[(1+\frac{4}{5}(1-s))N_{tr}^2 - 3(1-s)(1-s\bar{\mu}_0)]} - 1\Big]. \tag{3.22}$$

Finally, we get the modified diffusion equation and modified SP_3 equations as:

$$-\bigtriangledown_x D_{d,exp} \bigtriangledown \phi(x) + \sigma_a \phi(x) = 0. \tag{3.23}$$

and

$$\begin{cases} \bigtriangledown_x \dfrac{1}{3\sigma_{a,1}} \bigtriangledown (\phi(x) + 2\phi_2(x)) + \sigma_a \phi(x) = 0. \\ \bigtriangledown D_{sp_3,exp} \bigtriangledown \phi_2(x) + \sigma_T \phi_2(x) = \dfrac{2}{5}\sigma_a \phi(x). \end{cases} \tag{3.24}$$

Where $D_{d,exp}$ and $D_{sp_3,exp}$ are diffusion and SP_3 coefficients.
The modified diffusion equation (2.23) contains the first nonzero spatial moment $< x^0\phi >$ and N_{tr}. It does not contain the second nonzero spatial moment $< x^2\phi >$. The modified SP_3 (2.24) contains first three spatial moments which are $< x^0\phi >$, $< x^2\phi >$ and $< x^4\phi >$, as well as N_{tr}, and does not contain fourth nonzero spatial moment $< x^6\phi >$. We can also confirm theorically that for the modfied diffusion and modified SP_3 equations, the parameters η_d and η_{sp_3} are of order one $0(1)$ as $s \to 1$ (E. W. Larsen, 2011). Thus in this part, we have derived exponentially decay rate for modified diffusion and modified SP_3 equations that are defined in (3.23) and (3.24), these are the special cases of modified asymptotic diffusion and SP_3 equations which we have obtained previously.

4. Asymptotically Derived SP_1 and SP_2 Approximation

In this section, we shall derive the SP_1 or P_1 and SP_2 approximation by using the standard notation of Boltazmann transport equation and asymptotic scaling that were defined in Section 1, equation (1.6), see [10], [13]. Consider:

$$\Omega \cdot \bigtriangledown_x \psi(x,\Omega) + \tfrac{\hat{\sigma}_T}{\epsilon}\psi(x,\Omega) = \frac{1}{4\pi}\Big(\frac{\hat{\sigma}_{s,0}}{\epsilon}\Big)\int_{S^2}\psi(x,\Omega')d^2\Omega' + \epsilon\frac{\hat{Q}(x)}{4\pi},\ x \in V. \tag{4.1}$$

Since

$$\sigma_{s,0} = \sigma_T - \sigma_a,$$

in asymptotic scaling

$$\frac{\hat{\sigma}_{s,0}}{\epsilon} = \frac{\hat{\sigma}_T}{\epsilon} - \epsilon\hat{\sigma}_a. \tag{4.2}$$

Inserting (4.1) in (4.2), we get

$$\begin{cases} \Omega \cdot \bigtriangledown_x \psi(x,\Omega) + \dfrac{\hat{\sigma}_T}{\epsilon}\psi(x,\Omega) = \dfrac{1}{4\pi}\Big(\dfrac{\hat{\sigma}_T}{\epsilon} - \epsilon\hat{\sigma}_a\Big)\displaystyle\int_{S^2}\psi(x,\Omega^{'})d^2\Omega^{'}. \\ \qquad\qquad\qquad\qquad + \dfrac{\epsilon\hat{Q}(x)}{4\pi}. \end{cases} \tag{4.3}$$

Multiplying equation (4.3) by $\frac{\epsilon}{\hat{\sigma}_T}$, we obtain

$$\tfrac{\epsilon}{\hat{\sigma}_T}\Omega \cdot \bigtriangledown_x \psi(x,\Omega) + \psi(x,\Omega) = \tfrac{1}{4\pi}\Big(I - \epsilon^2\tfrac{\hat{\sigma}_a}{\hat{\sigma}_T}\Big)\int_{S^2}\psi(x,\Omega^{'})d^2\Omega^{'} + \frac{1}{4\pi}\epsilon^2\frac{\hat{Q}(x)}{\hat{\sigma}_T}.$$

We may simplify the above relation as:

$$\Big(I + \frac{\epsilon}{\hat{\sigma}_T}\Omega \cdot \bigtriangledown_x\Big)\psi(x,\Omega) = \frac{1}{4\pi}\Big[\Big(1 - \epsilon^2\frac{\hat{\sigma}_a}{\hat{\sigma}_T}\Big)\phi_0 + \epsilon^2\frac{\hat{Q}(x)}{\hat{\sigma}_T}\Big]. \tag{4.4}$$

Here, ϕ_0 is the scalar flux which was defined in Section 2.

$$\psi(x,\Omega) = \Big(I + \frac{\epsilon}{\hat{\sigma}_T}\Omega \cdot \bigtriangledown_x\Big)^{-1}\frac{1}{4\pi}\Big[\Big(1 - \epsilon^2\frac{\hat{\sigma}_a}{\hat{\sigma}_T}\Big)\phi_0 + \epsilon^2\frac{\hat{Q}(x)}{\hat{\sigma}_T}\Big]. \tag{4.5}$$

Integrating (4.5) over Ω and we get Peierls integral equation for the scalar flux:

$$\int_{S^2}\psi(x,\Omega)d^2\Omega = \Big[\frac{1}{4\pi}\int_{S^2}\Big(I + \Omega \cdot \bigtriangledown_x\frac{\epsilon}{\hat{\sigma}_T}\Big)^{-1}d^2\Omega\Big]\Big[\Big(1 - \epsilon^2\frac{\hat{\sigma}_a}{\hat{\sigma}_T}\Big)\phi_0 + \epsilon^2\frac{\hat{Q}(x)}{\hat{\sigma}_T}\Big].$$

This yields

$$\phi_0 = \Big[\tfrac{1}{4\pi}\int_{S^2}\Big(I + \Omega \cdot \bigtriangledown_x\frac{\epsilon}{\hat{\sigma}_T}\Big)^{-1}d^2\Omega\Big]\Big[\Big(1 - \epsilon^2\frac{\hat{\sigma}_a}{\hat{\sigma}_T}\Big)\phi_0 + \epsilon^2\frac{\hat{Q}(x)}{\hat{\sigma}_T}\Big]. \tag{4.6}$$

In equation (4.6) the integral operator expanding in powers of dimensionaless parameter ϵ , can be writen as:

$$\phi_0 = \Big[\sum_{l=0}^{\infty}\epsilon^{2l}L_{2l}\Big]\Big[(1 - \epsilon^2\frac{\hat{\sigma}_a}{\hat{\sigma}_T})\phi_0 + \epsilon^2\frac{\hat{Q}(x)}{\hat{\sigma}_T}\Big], \tag{4.7}$$

where, the laplace operator L_{2l} is defined as

$$\mathrm{L}_{2l} = \tfrac{1}{4\pi}\int_{S^2}\Big(\frac{1}{\hat{\sigma}_T}(\Omega\cdot\bigtriangledown)\Big)^{2l} d^2\Omega. \tag{4.8}$$

Next, we shell find some equations, which are needed to close our derivation. Inserting $l = 0, 1, 2$ in (4.8), we get the values of L_0, L_2, L_4 respectively as follow:

$$\mathrm{L}_0 = \tfrac{1}{4\pi}\int_{S^2} d^2\Omega.$$

$$\Rightarrow$$
$$\mathrm{L}_0 = \tfrac{1}{4\pi}4\pi = I. \tag{4.9}$$

$$L_2 = \frac{1}{4\pi}\int_{S^2}\Big(\frac{1}{\hat{\sigma}_T}(\Omega\cdot\bigtriangledown)\Big)^2 d^2\Omega.$$

Since

$$\int_{S^2}\Omega_i\Omega_j d^2\Omega = \frac{4\pi}{3}\delta_{ij}.$$

$$L_2 = \frac{1}{3}\sum_{i,j=1}^{3}\delta_{ij}\Big(\frac{1}{\hat{\sigma}_T}\frac{\partial}{\partial x_i}\frac{1}{\hat{\sigma}_T}\frac{\partial}{\partial x_j}\Big).$$

$$\Rightarrow$$
$$L_2 = \frac{1}{\hat{\sigma}_T}\cdot\bigtriangledown\Big(\frac{1}{3\hat{\sigma}_T}\bigtriangledown\Big). \tag{4.10}$$

$$L_4 = \frac{1}{4\pi}\int_{S^2}\Big(\frac{1}{\hat{\sigma}_T}(\Omega\cdot\bigtriangledown)\Big)^4 d^2\Omega.$$

Since

$$\int_{S^2}\Omega_i\Omega_j\Omega_k\Omega_l d^2\Omega = \frac{4\pi}{15}[\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{ik}\delta_{jk}].$$

$$\frac{1}{15}\sum_{i,j,k=1}^{3}[\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{ik}\delta_{jk}]\Big(\frac{1}{\hat{\sigma}_T}\frac{\partial}{\partial x_i}\frac{1}{\hat{\sigma}_T}\frac{\partial}{\partial x_j}\frac{1}{\hat{\sigma}_T}\frac{\partial}{\partial x_k}\frac{1}{\hat{\sigma}_T}\frac{\partial}{\partial x_l}\Big).$$

$$L_4 = \frac{1}{5}\Big(\frac{1}{\hat{\sigma}_T}\cdot\bigtriangledown\Big)\Big(\frac{1}{\hat{\sigma}_T}\cdot\bigtriangledown\Big)\Big(\frac{1}{\hat{\sigma}_T}\cdot\bigtriangledown\Big)\Big(\frac{1}{\hat{\sigma}_T}\cdot\bigtriangledown\Big).$$

$$L_4 = \frac{9}{5}\Big(\frac{1}{\hat{\sigma}_T}\cdot\bigtriangledown\Big)\Big(\frac{1}{3\hat{\sigma}_T}\cdot\bigtriangledown\Big)\Big(\frac{1}{\hat{\sigma}_T}\cdot\bigtriangledown\Big)\Big(\frac{1}{3\hat{\sigma}_T}\cdot\bigtriangledown\Big).$$

Since by using (4.10) in above equation, we get

(4.11) $$L_4 = \frac{9}{5}L_2^2.$$

Using (4.9)-(4.11) in (4.7), we need just first three even terms in laplace operator series expansion explicitely and we can rewrite (4.7) as

(4.12) $$\phi_0 = [I + \epsilon^2 L_2 + \epsilon^4 L_4]\Big[(1 - \epsilon^2\frac{\hat{\sigma}_a}{\hat{\sigma}_T})\phi_0 + \epsilon^2\frac{\hat{Q}(x)}{\hat{\sigma}_T}\Big] + 0(\epsilon^6).$$

Inverting the operator on the right hand side of (4.12), we get

$$\Rightarrow [I - \epsilon^2 L_2 - \epsilon^4 L_4]\phi_0 = \Big[\Big(1 - \epsilon^2\frac{\hat{\sigma}_a}{\hat{\sigma}_T})\phi_0 + \epsilon^2\frac{\hat{Q}(x)}{\hat{\sigma}_T}\Big] + 0(\epsilon^6).$$

Thus we obtained the successive relations

$$\Rightarrow \phi_0 - \epsilon^2 L_2\phi_0 - \epsilon^4 L_4\phi_0 = \phi_0 - \epsilon^2\frac{\hat{\sigma}_a}{\hat{\sigma}_T}\phi_0 + \epsilon^2\frac{\hat{Q}(x)}{\hat{\sigma}_T} + 0(\epsilon^6).$$

$$\Rightarrow -\hat{\sigma}_T\epsilon^2 L_2\phi_0 - \hat{\sigma}_T\epsilon^4 L_4\phi_0 = -\epsilon^2\hat{\sigma}_a\phi_0 + \epsilon^2\hat{Q}(x) + 0(\epsilon^6).$$

$$\Rightarrow \epsilon^2[-\hat{\sigma}_T L_2\phi_0 - \hat{\sigma}_T\epsilon^2 L_4\phi_0] = \epsilon^2[-\hat{\sigma}_a\phi_0 + \hat{Q}(x) + 0(\epsilon^4)].$$

(4.13) $$-\hat{\sigma}_T L_2\phi_0 + \hat{\sigma}_T\epsilon^2 L_4\phi_0 + \hat{\sigma}_a\phi_0 = \hat{Q}(x) + 0(\epsilon^4).$$

Using (4.13), we can drive P_1 or SP_1 equation since P_1, SP_1 are similar and by neglecting the $0(\epsilon^2)$ in (4.13) then multiplying by ϵ, and using (4.10) and the asymptotic scaling (1.6), we get SP_1 or P_1 equation as:

(4.14) $$-\bigtriangledown\cdot\Big(\frac{1}{3\sigma_t}\bigtriangledown\Big)\phi_0 + \sigma_a\phi_0 = Q(x).$$

For SP_2, we neglect the $0(\epsilon^4)$ in (4.13), to get

$$-\hat{\sigma}_T L_2\phi_0 + \hat{\sigma}_T\epsilon^2 L_4\phi_0 + \hat{\sigma}_a\phi_0 = \hat{Q}(x).$$

$$-L_2\phi_0 + \epsilon^2 L_4\phi_0 = \frac{\hat{Q}(x) - \hat{\sigma}_a\phi_0}{\hat{\sigma}_T}.$$

Now using (4.11), we get

$$-L_2\phi_0 + \epsilon^2\Big(\Big(\frac{9}{5}\Big)L_2^2\Big)\phi_0 = \frac{\hat{Q}(x) - \hat{\sigma}_a\phi_0}{\hat{\sigma}_T}.$$

i.e.

$$-L_2\Big(I + \epsilon^2\frac{9}{5}L_2\Big)\phi_0 = \frac{\hat{Q}(x) - \hat{\sigma}_a\phi_0}{\hat{\sigma}_T}. \tag{4.15}$$

Operating on equation (4.15) by $\Big(I - \epsilon^2\frac{9}{5}L_2\Big)$ and neglecting $0(\epsilon^4)$, we get resulatant equation as

$$-L_2\phi_0 = \Big(I - \epsilon^2\frac{9}{5}L_2\Big)\frac{\hat{Q}(x) - \hat{\sigma}_a\phi_0}{\hat{\sigma}_T}.$$

$$-L_2\phi_0 = \frac{\hat{Q}(x) - \hat{\sigma}_a\phi_0}{\hat{\sigma}_T} + \Big(\epsilon^2\frac{9}{5}L_2\Big)\frac{\hat{\sigma}_a\phi_0 - \hat{Q}(x)}{\hat{\sigma}_T}.$$

$$-L_2\Big[\phi_0 + \epsilon^2\frac{9}{5}\Big(\frac{\hat{\sigma}_a\phi_0 - \hat{Q}(x)}{\hat{\sigma}_T}\Big)\Big] = \frac{\hat{Q}(x) - \hat{\sigma}_a\phi_0}{\hat{\sigma}_T}.$$

$$-\hat{\sigma}_T L_2\Big[\phi_0 + \epsilon^2\frac{9}{5}\Big(\frac{\hat{\sigma}_a\phi_0 - \hat{Q}(x)}{\hat{\sigma}_T}\Big)\Big] + \hat{\sigma}_a\phi_0 = \hat{Q}(x).$$

Using (4.11), we obtained

$$-\bigtriangledown\Big(\frac{1}{3\hat{\sigma}_T}\bigtriangledown\Big)\Big[\phi_0 + \epsilon^2\frac{9}{5}\Big(\frac{\hat{\sigma}_a\phi_0 - \hat{Q}(x)}{\hat{\sigma}_T}\Big)\Big] + \hat{\sigma}_a\phi_0 = \hat{Q}(x). \tag{4.16}$$

Equation (4.16), multiplied by ϵ and by the asymptotic scaling in (1.6), yields

$$-\bigtriangledown\Big(\frac{1}{3\sigma_t}\bigtriangledown\Big)\Big[\phi_0 + \frac{9}{5}\Big(\frac{\sigma_a\phi_0 - Q(x)}{\sigma_t}\Big)\Big] + \sigma_a\phi_0 = Q(x). \tag{4.17}$$

Equation (4.17) is known as the SP_2 approximation equation.

5. Conclusion

In this study, expansion the theoretical foundation and extensibility of application of the diffusion and $SP_N(SP_1, SP_2)$ approximations for the steady-state linear transport equation. These diffusion-based methods are well-defined and have been used from last 50 years; to reproduce with the equation for optically thick system with the weak absorption. These are classical methods and are not precise for the strong absorbing deep penetration systems or regions, which happen in the shielding problems. In this analysis it has been found that the classical diffusion-based methods are the more accurate for the these type of problems by the modifications of the methods, and the resulted approximations are similar to the classical approximate equations. So, for conclusive low-order spatial moments are preserving for the large fissile regions, it is more adequate to use the standard diffusion and SP_N approximations for the transport solution. The transport exponential decay rate and less one moment as we have done in Section 4 are preserving for the deep penetration regions and is best used of the modified diffusion or SP_N equations for the transport solution. When ϵ is small it means that the scattering dominates absorption and the physical system is optically thick, then the asymptotic approximations are the most accurate ones.

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