Extra Övningar i PDE

1. Consider the Dirichlet problem

$$-\nabla \cdot (a(x)\nabla u) = f(x), \quad x \in \Omega \subset \mathbb{R}^2, \qquad u = 0, \text{ for } x \in \partial\Omega.$$

   Assume that c_0 and c_1 are constants such that $c_0 \leq a(x) \leq c_1$, $\forall x \in \Omega$ and let $U = \sum_{j=1}^{N} \alpha_j w_j(x)$ be a Galerkin approximation of u in a finite dimensional subspace M of $H_0^1(\Omega)$. Prove the a priori error estimate

$$||u - U||_{H_0^1(\Omega)} \leq C \inf_{\chi \in M} ||u - \chi||_{H_0^1(\Omega)}.$$

2. Prove an a priori and an a posteriori error estimate (in the energy norm: $\|u\|_E^2 := \|u'\|^2 + \alpha\|u\|^2$) for a finite element method for the problem

$$\begin{cases} -u'' + \alpha u = f, & 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases}$$

   where $\alpha = \alpha(x)$ is a bounded positive coefficient on $[0,1]$.

3. Consider the problem

$$\begin{cases} -\varepsilon u'' + xu' + \delta u = f, & \text{in } I = (0,1), \\ u(0) = u'(1) = 0. \end{cases}$$

   where ε and δ are positive constants, and $f \in L_2(I)$. Show that

$$\|\varepsilon u''\| \leq \|f\|, \qquad \sqrt{\varepsilon\delta}\|u'\| \leq \|f\|,$$

   where $\|\cdot\|$ is the $L_2(I)$-norm.

4. Consider the initial-boundary value problem

$$\begin{cases} \dot{u} - \Delta u = 0, & x \in \Omega, & t > 0, \\ u = 0, & x \in \partial\Omega, & t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

   where $\Omega \subset \mathbb{R}^2$ is a bounded smooth domain. Show the stability estimates:

$$||u(t)||^2 + \int_0^t ||\nabla u(s)||^2\, ds \leq ||u_0||^2 + C\int_0^t ||f(s)||^2\, ds,$$

$$||\nabla u(t)||^2 + \int_0^t ||\Delta u(s)||^2\, ds \leq ||\nabla u_0||^2 + C\int_0^t ||f(s)||^2\, ds.$$

5. Consider the initial value problem

$$\dot{u}(t) + au(t) = 0, \quad t > 0, \qquad u(0) = u_0, \quad a > 0, \quad (a = \text{constant}).$$

Assume a constant time step k and verify the iterative formulas for $dG(0)$ and $cG(1)$ approximations U and $\tilde{U}$ respectively: i.e.

$$U_n = \Big(\frac{1}{1+ak}\Big)^n u_0, \qquad \tilde{U}_n = \Big(\frac{1-ak/2}{1+ak/2}\Big)^n u_0.$$

6. Consider the heat equation

$$\begin{cases} \dot{u} - u'' = f(x), & 0 < x < 1, \quad t > 0, \\ u(0,t) = u'(1,t) = 0, & t > 0, \\ u(x,0) = u_0(x), & 0 < x < 1. \end{cases}$$

a) Show, for the homogeneous case: i.e., $f \equiv 0$ (with $\|u\| = (\int_0^1 u^2(x)\,dx)^{1/2}$), the estimates

$$(E1) \quad \frac{d}{dt}\|u\|^2 + 2\|u'\|^2 = 0, \qquad (E2) \quad \|u(.,t)\| \le e^{-t}\|u_0\|.$$

b) Let $u_s = u_s(x)$ be the solution of the corresponding stationary problem:

$$-u_s'' = f, \quad 0 < x < 1, \qquad u(0) = u'(1) = 0,$$

show that $\quad \|u - u_s\| \to 0, \quad$ as $t \to \infty$.

7. Prove an a priori and an a posteriori error estimate (in the H^1-norm: $\|u\|_{H^1}^2 = \|u'\|^2 + \|u\|^2$) for a finite element method for the problem

$$\begin{cases} -u'' + 2xu' + 3u = f, & 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases}$$

8. Determine the stiffness matrix and load vector if the $cG(1)$ finite element method applied to the Poisson's equation on a triangulation with triangles of side length 1/2 in both x_1- and x_2-directions:

$$\begin{cases} -\Delta u = 1, & \text{in} \quad \Omega = \{(x_1,x_2) : \ 0 < x_1 < 2, \ 0 < x_2 < 1\}, \\ u = 0, & \text{on} \quad \Gamma_1 = \{(0,x_2)\} \cup \{(x_1,0)\} \cup \{(x_1,1)\}, \\ \frac{\partial u}{\partial n} = 0, & \text{on} \quad \Gamma_2 = \{(2,x_2) : 0 \le x_2 \le 1\}. \end{cases}$$

9. a) Formulate a $cG(1)$ finite element method for the following system

$$\begin{cases} u(x) + v''(x) = f(x), & v(0) = v(1) = 0, \quad < x < 1, \\ u''(x) - v(x) = 0, & u(0) = u(1) = 0, \end{cases}$$

and show how the approximate solution (U, V) can be computed from the load vector F, using mass- and stiffness matrises.

b) Derive stability estimates for u and v, in terms of f, (e.g., through multiplying the first equation by u and the second by $-v$).

10. Consider the following *Schrödinger* equation

$$\dot{u} + i\Delta u = 0, \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega,$$

where $i = \sqrt{-1}$ and $u = u_1 + iu_2$. a) Show that the the L_2 norm of the solution, i.e., $\int_\Omega |u|^2$ is time independent.

Hint: Multiply the equation by $\bar{u} = u_1 - iu_2$, integrate over Ω and consider the real part.

b) Consider the corresponding *eigenvalue problem*, of finding $(\lambda, u \neq 0)$, such that

$$-\Delta u = \lambda u \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega.$$

Show that $\lambda > 0$, and give the relation between $||u||$ and $||\nabla u||$ for the corresponding eigenfunction u.

c) What is the optimal constant C (expressed in terms of smallest eigenvalue λ_1), for which the inequality $||u|| \leq C||\nabla u||$ can fullfil for all functions u, such that $u = 0$ on $\partial\Omega$?

11. Let Ω be a bounded convex polygonal domain in $\mathbb{R}^2$. Consider the problem

$$\beta \cdot \nabla u + u = f \quad \text{in} \quad \Omega, \qquad u = g \quad \text{on} \quad \Gamma^-,$$

where $\Gamma^- = \{x \in \Gamma = \partial\Omega : n(x) \cdot \beta < 0\}$, is the inflow boundary with $n(x)$ being outward unit normal to Γ at $x \in \Gamma$ and $\beta = (\beta_1, \beta_2)$ is a constant vector with $|\beta| = 1$, $\Gamma^+ = \Gamma \setminus \Gamma^-$.

a) Show that the folloeing general standard Galerkin method:

$$(\beta \cdot \nabla u_h + u^h, v) - < u^h, v >_- = (f, v) - < g, v >_-, \quad \forall\, v \in V_h,$$

is stable, here $(v, w) = \int_\Omega vw\, dx$ and $< v, w >_\pm = \int_{\Gamma^\pm} vw\, (n \cdot \beta)\, ds$.

b) Prove the error estimate

$$||u - u^h|| + |u - u^h| \leq Ch^r ||u||_{H^{r+1}(\Omega)},$$

where $||\cdot||$ is the L^2-norm and $|v|^2 = \int_\Gamma v^2 |n \cdot \beta|\, ds$.

12. Consider the initial-boundary value problem

$$\begin{cases} \ddot{u} - \Delta u + u = 0, & x \in \Omega, \qquad\qquad t > 0, \\ u = 0, & x \in \partial\Omega, \qquad\quad\;\; t > 0, \\ u(x,0) = u_0(x), & \dot{u}(x,0) = u_1(x), \quad x \in \Omega. \end{cases}$$

a) Define a suitable energy for this problem and prove that the energy is conserved.

b) Reformulate the problem as a system of 2 equations with time derivatives of at most order one. Why such a reformulation is necessary?

13. (a) Derive the stiffness matrix and load vector in piecewise polynomial (of degree q) approximation for the following ODE in population dynamics,

$$\begin{cases} \dot{u}(t) = \lambda u(t), & \text{for } 0 < t \leq 1, \\ u(0) = u_0. \end{cases}$$

(b) Let $\lambda = 1$ and $u_0 = 2$ and determine the approximate solution $U(t)$, for $q = 1$ and $q = 2$.

14. Consider the problem

$$-\Delta u = f \quad \text{in } \Omega, \qquad n \cdot \nabla u = -u \quad \text{on } \Gamma,$$

where Ω is a two-dimensional domain with the boundary Γ, and n is the outward unit normal to the boundary. Show that

$$||D^2 u|| \leq ||f||,$$

where $D^2 u = (u_{xx}^2 + u_{xy}^2 + u_{yx}^2 + u_{yy}^2)$ is a certain measure for the second derivative of u and $u_{xy} = u_{yx}$.

15. Consider the abstract parabolic problem

$$\begin{cases} \dot{u}(t) + Au(t) = 0, & \text{for } t > 0, \\ u(0) = u_0, \end{cases}$$

where A is a positive semi-definite symmetric linear operator on $\mathbb{R}^d$, $d \geq 1$. Show the following stability estimates for $T > 0$,

$$||u(T)||^2 + 2\int_0^T \Big(Au(t), u(t)\Big)\, dt \leq ||u_0||^2,$$

$$\int_0^T t||Au(t)||^2\, dt \leq \frac{1}{4}||u_0||^2,$$

$$||Au(T)||^2 \leq \frac{1}{\sqrt{2}T}||u_0||^2.$$