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L_2-ERROR ESTIMATES FOR THE DISCRETE ORDINATES METHOD FOR THREE-DIMENSIONAL NEUTRON TRANSPORT

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Abstract. We prove L_2-error estimates for the discrete ordinates method for the angular discretization of the three-dimensional neutron transport equation. The analysis is for monoenergetic three-dimensional transport of neutrons in a homogeneous uniform media and isotropic scattering is assumed. A special quadrature rule with relatively uniformly distributed discrete directions is considered.

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0. Introduction. Recall the stationary one-velocity process of neutron transport in a substance surrounded by vacuum: Given a source f and the coefficients α and σ find the angular flux $u = u(x,\mu)$ such that

$$\text{(0.1a)}\quad \mu\cdot\nabla u(x,\mu) + \alpha(x)u(x,\mu) = \int_{S^2} \sigma(x,\mu,\mu')u(x,\mu')d\mu' + f(x,\mu),\quad (x,\mu)\in\Omega\times S^2$$

$$\text{(0.1b)}\quad u(x,\mu) = 0,\ x\in\Gamma_\mu^- = \{x\in\Gamma:\ \mu\cdot n(x) < 0\},\ \mu = (\mu_1,\mu_2,\mu_3),$$

where Ω is a domain in R^3 with boundary Γ, $S^2 = \{\mu\in R^3:\ |\mu| = 1\}$, α is the total cross-section, σ is the transfer kernel, $n(x)$ is the outward unit normal to Γ at $x\in\Gamma$ and

$$\mu\cdot\nabla = \sum_{i=1}^{3} \mu_i \frac{\partial}{\partial x_i}.$$

The purpose of this note is to prove L_2-error estimates for the discrete ordinates method for the three dimensional model problem, in a convex bounded polygonal domain Ω, obtained from (0.1) by setting $\alpha\equiv 1$ and $\sigma\equiv$ cte.

Previous convergence results have been obtained in supremum norm for the discrete ordinates method for neutron transport, see e.g. [8], [9], [11] and [12] in the slab case and [10] and [17], in two and three-dimensional cases, respectively (these results give no rate of convergence). In the case of slab geometry L_p, $1\le p\le\infty$, and eigenvalue error estimates for the discrete ordinates method are given by Pitkäranta and Scott [15], where also discretizations in space variable using finite element approximations are considered. L_2 error estimates for a two- dimensional model problem are given in Johnson and Pitkäranta [5] and for infinite cylindrical domains by this author [2], where cylindrical symmetry is assumed. L_p, $1\le p\le\infty$, and eigenvalue error estimates for the discrete ordinates method for two-dimensional neutron transport are analyzed in [3]. In a recent paper by Pitkäranta [14], for the case of slab geometry, a family of projection schemes are studied. This family covers the discontinuous Galerkin method, the balance equations approach and the finite moments method for discretization of the space variable, where a discretization method for the angular variable is also analyzed. Finally the present work is focused on extending the angular discretization studied in [5] to a three-dimensional case.

An outline of this paper is as follows: In Section 1 we present our model problem and show that this problem can also be formulated as a Fredholm integral equation of the second kind for the scalar flux. Notation, assumptions and a previous result, which are fundamental in the analysis, are also included in this section. Section 2 is devoted to a quadrature rule on the surface of the unit sphere in R^3. In the concluding Section 3 we study the stability of the discrete ordinates method and give error estimates.

1. A model problem. We consider the following model problem: Given a source density f and a parameter $\lambda > 0$ find $u(x,\mu)$, the density of particles at the point $x\in\Omega$ moving in the direction $\mu\in S^2$, such that

$$\text{(1.1a)}\quad \mu\cdot\nabla u(x,\mu) + u(x,\mu) = \lambda\int_{S^2} u(x,\mu')d\mu' + f(x),\ (x,\mu)\in\Omega\times S^2$$

$$\text{(1.1b)}\quad u(x,\mu) = 0 \qquad x\in\Gamma_\mu^- = \{x\in\Gamma: n(x)\cdot\mu<0\},$$

where Ω is a bounded convex polygonal domain in R^3 with boundary Γ and $n(x)$ is the outward unit normal to Γ at $x \in \Gamma$.

For $\mu \in S^2$ let T_μ be the solution operator for the following problem: Given $g \in L_2(\Omega)$ find u such that

$$\begin{cases} \mu \cdot \nabla u + u = g & \text{in } \Omega, \quad (1.2a) \\ u = 0 & \text{on } \Gamma_\mu^-, \quad (1.2b) \end{cases}$$

i.e., $u = T_\mu g$ if u satisfies (1.2). By a simple calculation we find that

$$T_\mu g(x) = \int_0^{d(x,\mu)} e^{-s} g(x-s\mu)\,ds, \tag{1.3}$$

where $d(x,\mu)$ is the distance from $x \in \Omega$ to Γ in the direction $-\mu$, i.e.,

$$d(x,\mu) = \inf\{s > 0 : (x-s\mu) \notin \Omega\}.$$

Introducing the scalar flux

$$U(x) = \int_{S^2} u(x,\mu)\,d\mu, \tag{1.4}$$

the problem (1.1) can now be formulated as

$$u(x,\mu) = T_\mu(\lambda U + f)(x), \quad (x,\mu) \in \Omega \times S^2. \tag{1.5}$$

Integrating over S^2 we obtain the following integral equation for the scalar flux U,

$$(I - \lambda T)U = Tf, \tag{1.6}$$

where

$$T = \int_{S^2} T_\mu d\mu.$$

Using (1.3) we have the following explicit formula for the integral operator T,

$$Tg(x) = \int_{S^2} T_\mu g(x)\,d\mu = \int_{S^2} \int_0^{d(x,\mu)} e^{-s} g(x-s\mu)\,ds\,d\mu$$
$$= \int_{S^2} \int_0^{d(x,\mu)} \frac{e^{-s}}{s^2} g(x-s\mu) s^2\,ds\,d\mu,$$

so that changing from polar to Cartesian coordinates

$$Tg(x) = \int_\Omega \frac{e^{-|x-y|}}{|x-y|^2} g(y)\,dy. \tag{1.7}$$

Thus T is an integral operator with weakly singular kernel and one can show (see e.g. [6]) that $T: L_2(\Omega) \to L_2(\Omega)$ is compact and consequently (1.6) is a Fredholm integral equation of the second kind.

Remark 1.1. The degree of regularity of the scalar flux U in problem (1.6) is limited even if f is smooth. The best one can hope for in general, using the Sobolev spaces $H^1(\Omega)$, is that $U \in H^{3/2-\epsilon}(\Omega)$ for $\epsilon > 0$ (this will be the case e.g. if f is smooth see [13]). □

Throughout this paper we shall use the following notation: $\|\cdot\|_1$ will denote the norm in Sobolev space $H^1(\Omega)$ and $\|\cdot\|$ denotes the $L_2(\Omega)$-norm. C will denote a constant not necessarily the same at each occurrence and independent of N.

We assume that λ^{-1} is not in the spectrum of T. Thus $(I-\lambda T)$ is invertible and $(I-\lambda T)^{-1}: L_2(\Omega) \to L_2(\Omega)$ is a continuous linear mapping. This implies that

(i) For a given $f \in L_2(\Omega)$, the problem $(I - \lambda T)U = Tf$ has a unique solution.

(ii) There exists a constant $C > 0$ such that

$$\|(I-\lambda T)v\| \geq C\|v\|, \quad \forall v \in L_2(\Omega). \tag{1.8}$$

Once U has been determined from (1.6) we can for a given $\mu \in S^2$ find the angular flux $u(\cdot,\mu) = T_\mu g(\cdot)$ with $g = \lambda U + f$.

We shall also use the following Proposition due to Anselone [1].

Proposition 1.1. Let $T: L_2(\Omega) \to L_2(\Omega)$ be a bounded linear operator such that for some positive constant C,

$$\|(I - \lambda T)v\| \ge C\|v\|, \quad \forall v \in L_2(\Omega),$$

and let $(T_N)_{N=1}^{\infty}$ be a uniformly bounded sequence of linear operators on $L_2(\Omega)$ such that for some positive integer m,

$$\epsilon_N := \|(T - T_N)T_N^m\| \to 0 \text{ as } N \to \infty. \tag{1.9}$$

Then there exists a positive constant C_1 such that for N large enough

$$\|(I - \lambda T_N)v\| \ge C_1\|v\| \quad \forall v \in L_2(\Omega). \quad \square$$

Finally we shall use the following stability estimate for (1.2):

$$\|\mu\cdot\nabla T_\mu g\| + \|T_\mu g\| + [\tfrac{1}{2}\int_\Gamma (T_\mu g)^2 |\mu\cdot n| d\sigma]^{1/2} \le C\|g\|. \tag{1.10}$$

To obtain (1.10) we multiply (1.2a) by $u = T_\mu g$ and integrate over Ω. Using Green's formula

$$\int_\Omega (\mu\cdot\nabla u)u dx = \int_\Gamma u^2(\mu\cdot n)d\sigma - \int_\Omega (\mu\cdot\nabla u)u\, dx,$$

we then find that

$$\frac{1}{2}\int_\Gamma (T_\mu g)^2(\mu\cdot n)d\sigma + \int_\Omega (T_\mu g)^2 dx = \int_\Omega g T_\mu g\, dx,$$

from which (1.10) follows by using (1.2a). $\square$

2. The quadrature rule. We shall introduce a semidiscrete analogue of (1.6) where we use the discrete ordinates method for the angular variable μ. Using the quadrature rule

$$\int_{S^2} u(x,\mu)d\mu \sim \sum_{\mu\in Q} u(x,\mu)\omega_\mu, \tag{2.1}$$

where $Q = Q_N = \{\mu^1, \ldots, \mu^N\}$ is a finite set of quadrature points $\mu^i \in S^2$ with positive weights ω_{μ^i}, we obtain the following semidiscrete analogue of (1.5): Find $u_N(x,\mu)$ such that

$$u_N(x,\mu) = T_\mu(\lambda U_N + f)(x), \quad (x,\mu) \in \Omega \times Q, \tag{2.2}$$

where

$$U_N(x) = \sum_{\mu\in Q} u_N(x,\mu)\omega_\mu.$$

Multiplying (2.2) by ω_μ and summing over $\mu \in Q$, we obtain the following integral equation: Find $U_N \in L_2(\Omega)$ such that

$$(I - \lambda T_N)U_N = T_N f, \tag{2.3}$$

where

$$T_N = \sum_{\mu\in Q} T_\mu \omega_\mu.$$

Remark 2.1. Proposition 1.1 together with the fact that λ^{-1} is not in the spectrum of T, imply that once (1.9) is established for some positive integer m, then for sufficiently large N, $(I - \lambda T_N)^{-1}$ exists and is a bounded linear operator on $L_2(\Omega)$. On the other hand from (1.6) and (2.3) we find that

$$U - U_N = \lambda T_N(U - U_N) + (T - T_N)(\lambda U + f),$$

so that for large N,

$$U - U_N = (I - \lambda T_N)^{-1}(T - T_N)(\lambda U + f) = (I - \lambda T_N)^{-1} e_N,$$

and hence

$$\|U - U_N\| \le C\|e_N\|.$$

Here the angular discretization error $e_N = (T - T_N)(\lambda U + f)$ is just the quadrature error in evaluating the scalar flux; i.e.,

$$e_N(x) = \int_{S^2} T_\mu(\lambda U+f)(x)\,d\mu - \sum_{\mu\in Q} \omega_\mu T_\mu(\lambda U+f)(x),$$

and may readily be estimated. □

In the following example we construct a special quadrature rule which is more suitable for detailed analysis. In this particular rule the points and weights are relatively uniformly distributed in compairing with the usual S_N-methods. The theoretical advantage of this particular rule is that using interpolation space arguments optimal error estimates may be obtained for the maximally regular scalar flux U, i.e. $U \in H^{3/2-\epsilon}(\Omega)$, however conventional quadrature rules often require higher regularity of the solution U than $U \in H^{3/2-\epsilon}(\Omega)$.

We shall use spherical coordinates for $\mu = (\mu_1, \mu_2, \mu_3)$,

$$\mu_1 = \cos\beta\cos\alpha, \qquad 0 \le \alpha \le 2\pi, \quad -\frac{\pi}{2} \le \beta \le \frac{\pi}{2},$$
$$\mu_2 = \cos\beta\sin\alpha,$$
$$\mu_3 = \sin\beta.$$

The Jacobian of this transformation is equal to $\cos\beta$.

<u>Example 2.1.</u> We construct a relatively uniform distribution of a set of quadrature points $Q \subset S^2$ and weights ω_μ with $\mu \in Q$. More precisely let M be an integer and set $A = S^2 \cap \{(x_1, x_2, x_3) \in R^3 : x_i \ge 0,\ i = 1,2,3\}$. To construct the quadrature points on A we divide first the interval $[0,\frac{\pi}{2}]$ into M equal subintervals $I_k = [\frac{k\pi}{2M}, \frac{(k+1)\pi}{2M}]$, and let β_k be the midpoint of I_k, $k = 0,1,\dots, M-1$, and then for each k we choose $[M\cos\beta_k + 1]$ equidistant points on the circular arc $A \cap \{x_3 = \sin\beta_k\}$. We obtain in this way $M_0 = \sum_{k=0}^{M-1} [M\cos\beta_k + 1]$ quadrature points on A. We choose the quadrature points on S^2 to be those on A together with their reflections with respect to all the planes defined by the coordinate axis. We shall write the weights ω_μ as $\omega_{\alpha_{\ell,k}}\omega_{\beta_k}$ where

$$\omega_{\alpha_{\ell,k}} = \frac{\pi}{2}\cdot\frac{1}{[M\cos\beta_k+1]}, \quad \ell = 1, 2, \dots, [M\cos\beta_k + 1],$$
$$\omega_{\beta_k} = \frac{\pi}{2}\cdot\frac{1}{M}\cos\beta_k, \qquad k = 0, 1, \dots, M-1.$$

Reflected points have the weights of the original ones. □

<u>Lemma 2.1.</u> Let $N = 8M_0$. Then the quadrature points and weights constructed in Example 2.1 satisfy the following conditions:

(2.4a) There is a constant C such that for u sufficiently smooth

$$\Big|\int_{S^2} u(x,\mu)\,d\mu - \sum_{\mu\in Q} u(x,\mu)\omega_\mu\Big| \le CN^{-1/2}\Big[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi} \Big|\frac{\partial}{\partial\alpha}u(x,\alpha,\beta)\Big|\,d\alpha\, d\beta$$
$$+ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sum_\ell \omega_{\alpha_\ell}(\beta)\Big(\Big|\frac{\partial}{\partial\beta}(u(x,\alpha_\ell(\beta),\beta))\cos\beta\Big| + |u(x,\alpha_\ell(\beta),\beta)\sin\beta|\Big)d\beta\Big]$$

where $\omega_{\alpha_\ell}(\beta) = \omega_{\alpha_{\ell,k}}$ for $\beta \in I_k$.

(2.4b) $\qquad \sigma(\epsilon,N) \to 0, \qquad$ as $\max(\epsilon, \frac{1}{N^{1/2}}) \to 0,$

where for $\epsilon > 0$,

$$\sigma(\epsilon,N) := \sum_{(\mu,\nu,\gamma)\in I''_\epsilon} \omega_\mu\omega_\nu\omega_\gamma,$$

and

$$Q^3 := I'_\epsilon \cup I''_\epsilon,$$

with

$$I'_\epsilon := \{(\mu,\nu,\gamma) \in Q^3 :\ \min[\delta(\mu,d_i),\ \delta(\nu,d_i),\ \delta(\gamma,d_i),$$
$$|\mu\cdot(\nu\times\gamma)|] \ge \epsilon,\ i = 1, \dots, P_0\}$$

and

$$I''_\epsilon := \{(\mu,\nu,\gamma) \in Q^3 : (\mu,\nu,\gamma) \notin I'_\epsilon\},$$

where $\delta(\phi,\psi) = \sin a(\phi,\psi)$ with $a(\phi,\psi)$ the smallest angle between ϕ and ψ. Further d_i are the sides of Ω and P_0 is the number of sides of Ω. Observe that the condition $|\mu\cdot(\nu\times\gamma)| \ge \epsilon$, in the definition of I'_ϵ implies that: If $(\mu,\nu,\gamma) \in I'_\epsilon$, then μ,ν and γ are not in the same plane.

Remark 2.2. The condition (2.4b) is a stability condition assuring that the quadrature points $\mu \in Q$ together with the associated weights $\omega_\mu = \omega_{\alpha_{\ell,k}} \omega_{\beta_k}$ are not too nonuniformly distributed. A totally uniform structure of the quadrature points on the surface of the sphere is not known. An almost uniform structure may be achieved via imbedding regular polygons in the sphere with vertices on the surface of the sphere, then triangulating the faces of these polygons and finally projecting the so obtained nodal points on the surface of the sphere. For other constructions see Stroud [16]. □

Proof of Lemma 2.1. To prove (2.4a) we note that

$$|\int_A u(x,\mu)\,d\mu - \sum_{\mu\in Q\cap A} u(x,\mu)\omega_\mu| \le$$

$$\le |\int_0^{\frac{\pi}{2}}\Big[\int_0^{\frac{\pi}{2}} u(x,\alpha,\beta)\,d\alpha - \sum_\ell \omega_{\alpha_\ell}(\beta)u(x,\alpha_\ell(\beta),\beta)\Big]\cos\beta\,d\beta| +$$

$$|\int_0^{\frac{\pi}{2}} \sum_\ell \omega_{\alpha_\ell}(\beta)u(x,\alpha_\ell(\beta),\beta)\cos\beta\,d\beta - \sum_k\sum_\ell \omega_{\alpha_{\ell,k}}\omega_{\beta_k} u(x,\alpha_{\ell,k},\beta_k)|$$

$$:= I + II.$$

Below we shall estimate I and II separately. Using the quadrature error approximation for a uniform division (see, Krylov [7], pp. 153–155) we obtain

$$|\int_0^{\frac{\pi}{2}} u(x,\alpha,\beta)\,d\alpha - \sum_\ell \omega_{\alpha_\ell}(\beta)u(x,\alpha_\ell(\beta),\beta)| \le$$

$$\le \frac{C}{[M\cos\beta_k + 1]}\int_0^{\frac{\pi}{2}} |\frac{\partial}{\partial\alpha}u(x,\alpha,\beta)|\,d\alpha,$$

so that

$$I = |\int_0^{\frac{\pi}{2}}\Big[\int_0^{\frac{\pi}{2}} u(x,\alpha,\beta)\,d\alpha - \sum_\ell \omega_{\alpha_\ell}(\beta)u(x,\alpha_\ell(\beta),\beta)\Big]\cos\beta\,d\beta| \tag{2.5}$$

$$\le C\sum_{k=0}^{M-1}\int_{I_k}\frac{\cos\beta}{[M\cos\beta_k + 1]}\int_0^{\frac{\pi}{2}} |\frac{\partial}{\partial\alpha}u(x,\alpha,\beta)|\,d\alpha d\beta$$

$$\le \frac{C}{M}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} |\frac{\partial}{\partial\alpha}u(x,\alpha,\beta)|\,d\alpha\,d\beta.$$

Moreover

$$II = |\int_0^{\frac{\pi}{2}} \sum_\ell \omega_{\alpha_\ell}(\beta)u(x,\alpha_\ell(\beta),\beta)\cos\beta d\beta - \sum_\ell\sum_k \omega_{\alpha_{\ell,k}}\omega_{\beta_k} u(x,\alpha_{\ell,k},\beta_k)| \tag{2.6}$$

$$= |\int_0^{\frac{\pi}{2}} \sum_\ell \omega_{\alpha_\ell}(\beta)u(x,\alpha_\ell(\beta),\beta)\cos\beta d\beta - \sum_k \frac{\omega_{\beta_k}}{\cos\beta_k}\sum_\ell \omega_{\alpha_{\ell,k}} u(x,\alpha_{\ell,k},\beta_k)\cos\beta_k|$$

$$\le \frac{C}{M}\sum_{k=0}^{M-1}\int_{I_k}\sum_\ell \omega_{\alpha_\ell}(\beta_k)|\frac{\partial}{\partial\beta}(u(x,\alpha_\ell(\beta),\beta)\cos\beta)|\,d\beta$$

$$\le \frac{C}{M}\sum_{k=0}^{M-1}\int_{I_k}\sum_\ell \omega_{\alpha_\ell}(\beta_k)\Big[|\frac{\partial}{\partial\beta}(u(x,\alpha_\ell(\beta),\beta))\cos\beta| + |u(x,\alpha_\ell(\beta),\beta)\sin\beta|\Big]\,d\beta.$$

Finally (2.5)–(2.6) together with the fact that $\omega_{\alpha_\ell}(\beta) = \omega_{\alpha_\ell}(\beta_k)$ for $\beta \in I_k$ give the proof of (2.4a).

The crucial part in the proof of (2.4b) is to show that (2.4b) is valid with I''_ϵ, replaced by the subset of Q^3 consisting of all (μ,ν,γ) for which the volume $|\mu\cdot\nu\times\gamma|$ is zero or less than ϵ. There are at most M^5 combinations of μ,ν and γ for which $|\mu\cdot(\nu\times\gamma)| = 0$. On the other hand

$$\Sigma_{\{(\mu,\nu,\gamma):\, 0<|\mu\cdot(\nu\times\gamma)|<\epsilon\}} \omega_\mu \omega_\nu \omega_\gamma = \Sigma_{(\nu,\gamma)} \omega_\nu \omega_\gamma \Sigma_{\{\mu:\, |\mu\cdot(\nu\times\gamma)|\le\epsilon\}} \omega_\mu .$$

In the last sum above μ lies in a strip of width $H := C\dfrac{\epsilon}{|\sin(\nu,\gamma)|}$, and thus the number of μ is $M\dfrac{H}{\frac{1}{M}} = HM^2$. Hence

$$\Sigma_{\{(\mu,\nu,\gamma):\, 0<|\mu\cdot(\nu\times\gamma)|<\epsilon\}} \omega_\mu\omega_\nu\omega_\gamma \le C \sum_\nu \omega_\nu \Sigma_\gamma \omega_\gamma \frac{\epsilon}{|\sin(\nu,\gamma)|}$$
$$\le C\,\epsilon \log M,$$

so that

$$\sigma(\epsilon,N) \le C[\Sigma_{\substack{(\mu,\nu,\gamma)\\ |\mu\cdot(\nu\times\gamma)|=0}} \omega_\mu\omega_\nu\omega_\gamma + \Sigma_{\substack{(\mu,\nu,\gamma)\\ 0<|\mu\cdot(\nu\times\gamma)|<\epsilon}} \omega_\mu\omega_\nu\omega_\gamma]$$

$$\le C(\tfrac{1}{M} + \epsilon \log M) \to 0 \quad \text{as } \max(\tfrac{1}{M}, \epsilon) \to 0$$

and this completes the proof of Lemma 2.1. □

3. The discrete ordinates method. The aim of this section is to prove stability of the semidiscrete problem (2.3) using Proposition 1.1. For this purpose we assume that λ^{-1} is not in the spectrum of T, i.e. (1.8) holds and we prove (1.9) with $m = 3$.

Theorem 3.1. If the quadrature rule (2.1) satisfies (2.4a,b), then for $\lambda^{-1} \notin \sigma(T)$ there is a constant C and an integer N_λ such that for $N \ge N_\lambda$,

$$\|(I - \lambda T_N)v\| \ge C\|v\|, \quad \forall v \in L_2(\Omega). \tag{3.1}$$

Note that since T_N is not compact, (3.1) does not directly imply the existence of a solution to (2.3). To prove that $(I-\lambda T_N)$ is onto we may argue as in [3] and we thus have the following result.

Proposition 3.1. If $\lambda^{-1} \notin \sigma(T)$, then there is an integer N_λ and a constant C such that for $N \ge N_\lambda$, $\|(I-\lambda T_N)^{-1}\| \le C$.

To prove Theorem 3.1 we need the following two lemmas:

Lemma 3.1. There exists a constant C such that if $(\mu,\nu,\gamma) \in I'_\epsilon$ then for $g \in L_2(\Omega)$,

$$\|T_\mu T_\nu T_\gamma g\|_1 \le C\epsilon^{-2}\|g\|.$$

Lemma 3.2. There exists a constant C such that for $g \in H^1(\Omega)$

$$\|(T-T_N)g\| \le CN^{-1/2}\|g\|_1.$$

Let us postpone the proofs of Lemmas 3.1 and 3.2 and first show that Theorem 3.1 follows from these two lemmas and Lemma 2.1.

Proof of Theorem 3.1. Using Lemmas 3.1 and 3.2, and (1.10) we have as in the proof of Lemma 4.1 in [2]

$$\|(T-T_N)T_N^3 g\| = \|(T-T_N) \Sigma_{(\mu,\nu,\gamma)\in Q^3} \omega_\mu\omega_\nu\omega_\gamma T_\mu T_\nu T_\gamma g\|$$
$$\le \Sigma'_\epsilon\, \omega_\mu\omega_\nu\omega_\gamma \|(T-T_N)T_\mu T_\nu T_\gamma g\|$$

$$+\Sigma''_\epsilon\,\omega_\mu\omega_\nu\omega_\gamma\|(T-T_N)T_\mu T_\nu T_\gamma g\|$$
$$\le(\Sigma'_\epsilon\,\omega_\mu\omega_\nu\omega_\gamma)\,CN^{-1/2}\|T_\mu T_\nu T_\gamma g\|_1$$
$$+\|T-T_N\|\,\Sigma''_\epsilon\,\omega_\mu\omega_\nu\omega_\gamma\|T_\mu T_\nu T_\gamma g\|$$
$$\le C[N^{-1/2}\epsilon^{-2}+\sigma(\epsilon,N)]\|g\|.$$

Now choosing e.g., $\epsilon=N^{-1/5}$ and using (2.4b) we find that

$$\|(T-T_N)T_N^3\|\to 0,\quad \text{as } N\to\infty$$

and using Proposition 1.1 the proof is complete. □

We now return to the

Proof of Lemma 3.1. By an orthogonal coordinate transformation we may assume that $\mu=(1,0,0)$. If $(\mu,\nu,\gamma)\in I'_\epsilon$ then

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial\mu},$$
$$\frac{\partial u}{\partial y}=\left[\frac{\gamma_1\nu_3-\gamma_3\nu_1}{\gamma_3\nu_2-\nu_3\gamma_2}\frac{\partial u}{\partial\mu}+\frac{\gamma_3}{\gamma_3\nu_2-\nu_3\gamma_2}\frac{\partial u}{\partial\nu}\,\frac{\nu_3}{\gamma_3\nu_2-\nu_3\gamma_2}\frac{\partial u}{\partial\gamma}\right],$$

where $\frac{\partial u}{\partial\mu}=\mu\cdot\nabla u$ and similarly for ν and γ. There is a similar relation for $\frac{\partial u}{\partial z}$. Now

$$|\gamma_3\nu_2-\nu_3\gamma_2|=|\mu\cdot(\gamma\times\nu)|\ge\epsilon$$

so that using (1.10)

$$\|\nabla(T_\mu T_\nu T_\gamma g)\|\le\frac{C}{\epsilon}\left[\|\frac{\partial}{\partial\mu}(T_\mu T_\nu T_\gamma g)\|+\|\frac{\partial}{\partial\nu}(T_\mu T_\nu T_\gamma g)\|\right.\tag{3.1}$$
$$\left.+\|\frac{\partial}{\partial\gamma}(T_\mu T_\nu T_\gamma g)\|\right]\le\frac{C}{\epsilon}\left[\|g\|+\|\frac{\partial}{\partial\nu}(T_\mu T_\nu T_\gamma g)\|+\|\frac{\partial}{\partial\gamma}(T_\mu T_\nu T_\gamma g)\|\right].$$

Recalling (1.3) we have

$$T_\mu T_\nu T_\gamma g(x)=\int_0^{d(x,\mu)}e^{-s}T_\nu T_\gamma g(x-s\mu)\,ds,$$

and thus

$$\frac{\partial}{\partial\nu}(T_\mu T_\nu T_\gamma g(x))=e^{-d}T_\nu T_\gamma g(\bar x)\,\frac{\partial d}{\partial\nu}+\int_0^{d(x,\mu)}e^{-s}\frac{\partial}{\partial\nu}(T_\nu T_\gamma g(x-s\mu))\,ds,\tag{3.2}$$

where $d=d(x,\mu)$ and $\bar x=x-s\mu$.

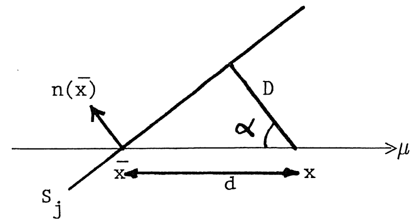

Figure 3.1

On the other hand as one can see from the Figure 3.1.

$$d=\frac{D}{\cos\alpha}=\frac{D}{-\mu\cdot n},$$

where D is the distance from the point x to the side S_j of Ω, i.e. for $y\in S_j$

$$D=\frac{n\cdot(y-x)}{|n|}=n\cdot y-n\cdot x=C-n\cdot x.$$

Thus

$$d=\frac{C-n\cdot x}{-\mu\cdot n},$$

so that

$$\frac{\partial d}{\partial\nu}=\frac{\nu\cdot n}{\mu\cdot n},$$

where $n=(n_1,n_2,n_3)$ is the outward unit normal to Γ at $\bar x\in\Gamma$. Since μ is parallel to x_1, we have in (3.2) that $\bar x$ depends only on x_2 and x_3. By a rotation of coordinate system we may choose the x_2-axis on a hyperplane parallel to S_j. Let $\Omega_j=\{x\in\Omega:\ \bar x\in S_j\}$. Squaring (3.2) and integrating over Ω_j, using the fact that in the first term $dx=|\mu\cdot n|\,d\sigma\,dx_1$ and summing over j, we have

$$\|\frac{\partial}{\partial\nu}(T_\mu T_\nu T_\gamma g)\|^2\le C\int_\Gamma|T_\nu T_\gamma g(\bar x)|^2\left|\frac{\nu\cdot n}{\mu\cdot n}\right|^2|\mu\cdot n|\,d\sigma$$
$$+C\int_\Omega\int_0^{d(x,\mu)}e^{-2s}\left(\frac{\partial}{\partial\nu}T_\nu T_\gamma g(x-s\mu)\right)^2ds\,dx$$
$$\le C\,\epsilon^{-1}\|g\|^2,$$

where we have repeatedly used (1.10) and the fact that $|\mu\cdot n| > C\epsilon$.

Thus

$$\left\|\frac{\partial}{\partial\nu}(T_\mu T_\nu T_\gamma g(x))\right\| \le C\epsilon^{-1/2}\|g\|. \tag{3.3}$$

Similarly

$$\frac{\partial}{\partial\gamma}(T_\mu T_\nu T_\gamma g(x)) = e^{-d} T_\nu T_\gamma g(\bar{x})\frac{\partial d}{\partial\gamma} + \int_0^{d(x,\mu)} e^{-s}\frac{\partial}{\partial\gamma}(T_\nu T_\gamma g(x-s\mu))ds,$$

and by the same argument as for $\frac{\partial}{\partial\nu}$, we obtain

$$\begin{aligned}\left\|\frac{\partial}{\partial\gamma}(T_\mu T_\nu T_\gamma g(x))\right\|^2 &\le C\int_\Gamma |T_\nu T_\gamma g(\bar{x})|^2\left|\frac{\gamma\cdot n}{\mu.n}\right|^2|\mu\cdot n|ds \\ &\quad + C\int_\Omega\int_0^{d(x,\mu)} e^{-2s}\frac{\partial}{\partial\gamma}(T_\nu T_\gamma g(x-s\mu))^2 ds\,dx \\ &\le C\left[\int_\Gamma |T_\nu T_\gamma g(\bar{x})|^2\left|\frac{\gamma\cdot n}{\mu\cdot n}\right|^2\left|\frac{\nu\cdot n}{\nu\cdot n}\right||\mu\cdot n|ds + \left\|\frac{\partial}{\partial\gamma}(T_\nu T_\gamma g(\cdot))\right\|^2\right] \\ &\le C\left\|\frac{\partial}{\partial\gamma}(T_\nu T_\gamma g(\cdot))\right\|^2 + C\epsilon^{-2}\|g\|^2.\end{aligned} \tag{3.4}$$

Here we have also used $\frac{\partial d}{\partial\gamma} = \frac{\gamma\cdot n}{\mu\cdot n}$. To estimate $\left\|\frac{\partial}{\partial\gamma}(T_\nu T_\gamma g(\cdot))\right\|$ we note that

$$T_\nu T_\gamma g(x-s\mu) = \int_0^{d(x-s\mu,\nu)} e^{-t} T_\gamma g(x-s\mu-t\nu)dt,$$

so that

$$\frac{\partial}{\partial\gamma}(T_\nu T_\gamma g(x-s\mu)) = e^{-d_1} T_\gamma g(x-s\mu-d_1\nu)\frac{\partial d_1}{\partial\gamma} + \int_0^{d(x-s\mu,\nu)} e^{-t}\frac{\partial}{\partial\gamma}(T_\gamma g(x-s\mu-t\nu))dt, \tag{3.5}$$

where $d_1 = d(x-s\mu,\nu)$. Squaring (3.5) and integrating over Ω using the fact that $\frac{\partial d_1}{\partial\gamma} = \frac{\gamma\cdot n}{\nu\cdot n}$ and $|\nu.n| \ge C\epsilon$ we find that

$$\begin{aligned}\left\|\frac{\partial}{\partial\gamma}(T_\nu T_\gamma g(x-s\mu))\right\|^2 &\le \frac{C}{\epsilon^2}\int_\Gamma |T_\gamma g(\bar{\bar{x}})|^2|\gamma.n|d\sigma \\ &\quad + C\int_\Omega\int_0^{d_1} e^{-2t}\left|\frac{\partial}{\partial\gamma}T_\gamma g(x-s\mu-t\nu)\right|^2 dtdx \\ &\le \frac{C}{\epsilon^2}\|g\|^2 + C\|g\|^2,\end{aligned} \tag{3.6}$$

where $\bar{\bar{x}} = x-s\mu-d_1\nu$, and where we have applied Fubini's theorem and (1.10). By (3.4) and (3.6) we have

$$\left\|\frac{\partial}{\partial\gamma}(T_\mu T_\nu T_\gamma g(x))\right\| \le C\epsilon^{-1}\|g\|. \tag{3.7}$$

Since by (1.10) $\|T_\mu T_\nu T_\gamma\| \le C\|g\|$, we obtain by (3.1), (3.3) and (3.7) the desired result. □

For the proof of Lemma 3.2 we shall use the following result.

<u>Lemma 3.3.</u> There exists a constant C such that if $u(x,\alpha,\beta) = T_\mu g(x)$ with $g \in H^1(\Omega)$, then

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}\left\|\frac{\partial}{\partial\alpha}u(\cdot,\alpha,\beta)\right\|_{L_2(\Omega)} d\alpha d\beta \le C\|g\|_1, \tag{3.8}$$

and

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sum_\ell \omega_{\alpha_\ell}(\beta)\left\|\frac{\partial}{\partial\beta}u(\cdot,\alpha,\beta)\right\|_{L_2(\Omega)} d\beta \le C\|g\|_1, \tag{3.9}$$

where $\mu = (\cos\beta\cos\alpha, \cos\beta\sin\alpha, \sin\beta)$.

<u>Proof.</u> We have

$$u(x,\mu) = u(x,\alpha,\beta) = \int_0^{d(x,\alpha,\beta)} e^{-s} g(x-s\mu)ds,$$

with $d(x,\alpha,\beta) = d(x,\mu)$, so that

$$\begin{aligned}\frac{\partial u}{\partial\alpha} &= e^{-d} g(x-d\mu)\frac{\partial d}{\partial\alpha} + \int_0^d e^{-s}\frac{\partial}{\partial\alpha}g(x-s\mu)ds \\ &= e^{-d} g(\bar{x})\frac{\partial d}{\partial\alpha} + \int_0^d e^{-s} s\frac{\partial}{\partial\mu'}g(x-s\mu)ds,\end{aligned} \tag{3.10}$$

where $\bar{x} = x - d\mu$ and since $x-s\mu=(x_1 - s\cos\beta\cos\alpha, x_2 - s\cos\beta\sin\alpha, x_3 - s\sin\beta)$ we have $\frac{\partial}{\partial\alpha}g = \mu'\cdot\nabla g$ with $\mu'=(\sin\beta\sin\alpha, -\sin\beta\cos\alpha, 0) \in S^2$ being orthogonal to μ. Let us now estimate $\frac{\partial d}{\partial\alpha}$ in each of subdomains Ω_j defined by

$$\Omega_j = \{x \in \Omega: \bar{x} \in S_j\},$$

where the S_j are the sides of Ω. Let $\psi_j := \psi_j(\alpha,\beta)$ be the angle between S_j and μ and let for $x \in \Omega_j$, $a_j(x)$ be the distance from x to the plane S_j. Now since

$$d(x,\alpha,\beta) = \frac{a_j(x)}{\sin \psi_j(\alpha,\beta)}$$

we have

$$\frac{\partial d}{\partial \alpha} = a_j(x)\frac{\partial}{\partial\alpha}\left[\frac{1}{\sin \psi_j(\alpha,\beta)}\right] = a_j(x)\left[\frac{-(\cos\psi_j)\frac{\partial\psi_j}{\partial\alpha}}{\sin^2\psi_j}\right].$$

Moreover, since Ω is bounded we have $a_j(x) \le C \sin\psi_j(\alpha,\beta)$ and thus

$$\left|\frac{\partial}{\partial\alpha} d(x,\alpha,\beta)\right| \le C\left|\frac{-\cos\psi_j(\alpha,\beta)\frac{\partial}{\partial\alpha}\psi_j(\alpha,\beta)}{\sin\psi_j(\alpha,\beta)}\right|.$$

Hence, squaring (3.10), integrating over Ω_j and using an orthogonal coordinate system (ξ_1, ξ_2, ξ_3) with $(\xi_1, \xi_2) \in S_j$ we get

$$\text{(3.11)}\qquad \int_{\Omega_j} \left|\frac{\partial}{\partial\alpha} u(x,\alpha,\beta)\right|^2 dx \le$$

$$\le C\Big[\int_{\Omega_j} \Big(|g(\bar{x})|^2\left|\frac{\cos\psi_j \frac{\partial}{\partial\alpha}\psi_j}{\sin\psi_j}\right|^2 + \int_0^d e^{-2s}s^2|\mu'\cdot\nabla g|^2 ds\Big)d\xi_1 d\xi_2 d\xi_3\Big]$$

$$\le C\Big[\int_{S_j} |g|^2 d\sigma \int_0^{C|\sin\psi_j|} \left|\frac{\cos\psi_j \frac{\partial}{\partial\alpha}\psi_j}{\sin\psi_j}\right|^2 d\xi_3 + \|\nabla g\|^2\Big]$$

$$\le C\Big[\|g\|_\Gamma^2 |\sin\psi_j|^{-1}|\cos\psi_j|^2\left|\frac{\partial}{\partial\alpha}\psi_j\right|^2 + \|\nabla g\|^2\Big],$$

where $\|\cdot\|_\Gamma$ denotes the $L_2(\Gamma)$-norm. Using the trace estimate

$$\|g\|_{L_2(\Gamma)} \le C\|g\|_1,$$

integrating both sides of (3.11) with respect to α and β in the first octant we find that

$$\text{(3.12)}\qquad \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \left\|\frac{\partial}{\partial\alpha}u(\cdot,\alpha,\beta)\right\|_{L_2(\Omega_j)} d\alpha\, d\beta \le$$

$$\le C\Big(\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{|\cos\psi_j(\alpha,\beta)\frac{\partial}{\partial\alpha}\psi_j(\alpha,\beta)|}{\sqrt{\sin\psi_j(\alpha,\beta)}} d\alpha\, d\beta\Big)\|g\|_1.$$

We see that the integral on the right hand side of (3.12) is bounded, since

$$\frac{\partial}{\partial\alpha}\psi_j(\alpha,\beta) = \frac{\partial}{\partial\alpha}(\arcsin(\sin\psi_j(\alpha,\beta)))$$

$$= \frac{\frac{\partial}{\partial\alpha}(\sin\psi_j(\alpha,\beta))}{\sqrt{1-\sin^2\psi_j(\alpha,\beta)}}.$$

Thus

$$\frac{|\cos\psi_j(\alpha,\beta)\frac{\partial}{\partial\alpha}\psi_j(\alpha,\beta)|}{\sqrt{\sin\psi_j(\alpha,\beta)}} = \frac{\left|\frac{\partial}{\partial\alpha}(\sin\psi_j(\alpha,\beta))\right|}{\sqrt{\sin\psi_j(\alpha,\beta)}}.$$

Now let $I^+ = \{\alpha\colon \frac{\partial}{\partial\alpha}(\sin\psi_j(\alpha,\beta)) \ge 0\}$ and $I^- = [0,\frac{\pi}{2}] \setminus I^+$. Then there are α_0 and $\alpha_1 \in [0,\frac{\pi}{2}]$ such that

$$\text{(3.13)}\qquad \int_0^{\frac{\pi}{2}} \frac{|\frac{\partial}{\partial\alpha}(\sin\psi_j(\alpha,\beta))|}{\sqrt{\sin\psi_j(\alpha,\beta)}} d\alpha = \Big(\int_{I^+} - \int_{I^-}\Big)\frac{\frac{\partial}{\partial\alpha}(\sin\psi_j(\alpha,\beta))}{\sqrt{\sin\psi_j(\alpha,\beta)}} d\alpha.$$

Further

$$\int_{I^+} \frac{\frac{\partial}{\partial\alpha}(\sin\psi_j(\alpha,\beta))}{\sqrt{\sin\psi_j(\alpha,\beta)}} d\alpha = 2\left(\sqrt{\sin\psi_j(\alpha_1,\beta)} - \sqrt{\sin\psi_j(\alpha_0,\beta)}\right),$$

which is integrable with respect to β. Similarly, the integral over I^- in the left hand side of (3.13) is integrable with respect to β. Thus summing over j in (3.12), we obtain (3.8) for the first octant and hence by symmetry for the whole sphere. To prove (3.9) we have similarly

$$\frac{\partial u}{\partial\beta} = e^{-d}g(\bar{x})\frac{\partial d}{\partial\beta} + \int_0^d e^{-s}s\frac{\partial}{\partial\mu''}g(x-s\mu)ds,$$

where $\mu'' = (\sin\beta\cos\alpha,\ \sin\beta\sin\alpha,\ -\cos\beta) \in S^2$ is orthogonal to μ and

$$\left|\frac{\partial d}{\partial\beta}\right| = \left|\frac{\partial}{\partial\beta}d(x,\alpha,\beta)\right| \le C\left|\frac{-\cos\psi_j(\alpha,\beta)\frac{\partial}{\partial\beta}\psi_j(\alpha,\beta)}{\sin\psi_j(\alpha,\beta)}\right|$$

Now by the same calculation as in (3.11) we obtain

$$\int_{\Omega_j}\left|\frac{\partial}{\partial\beta}u(x,\alpha,\beta)\right|^2dx \le C[\|g\|_\Gamma^2|\sin\psi_j|^{-1}|\cos\psi_j|\,|\frac{\partial}{\partial\beta}\psi_j|^2 + \|\nabla g\|^2], \tag{3.14}$$

where

$$\frac{\partial}{\partial\beta}\psi_j = \frac{\partial}{\partial\beta}(\arcsin(\sin\psi_j(\alpha,\beta))) = \frac{\frac{\partial}{\partial\beta}(\sin\psi_j(\alpha,\beta))}{\sqrt{1-\sin^2\psi_j(\alpha,\beta)}}.$$

Multiplying (3.14) by $\omega_{\alpha_\ell}(\beta)$, summing over ℓ and integrating with respect to $\beta\in I_k$ we find that

$$\begin{aligned}\int_{I_k}\sum_\ell\omega_{\alpha_\ell}(\beta)\|\frac{\partial}{\partial\beta}u(\cdot,\alpha_\ell,\beta)\|_{L_2(\Omega_j)}d\beta &\le C\Big[\int_{I_k}(\sum_\ell\omega_{\alpha_\ell}(\beta_k)\frac{|\frac{\partial}{\partial\beta}(\sin\psi_j(\alpha_{\ell,k},\beta))|}{\sqrt{\sin\psi_j(\alpha_{\ell,k},\beta)}})d\beta\Big]\|g\|_1\\
&\le C\Big[\int_{I_k}(\int_0^{\frac{\pi}{2}}\frac{|\frac{\partial}{\partial\beta}(\sin\psi_j(\alpha,\beta))|}{\sqrt{\sin\psi_j(\alpha,\beta)}}d\alpha\\
&\quad+\int_0^{\frac{\pi}{2}}\frac{\partial}{\partial\alpha}(\omega_{\alpha_\ell}(\beta_k)\frac{|\frac{\partial}{\partial\beta}(\sin\psi_j(\alpha,\beta))|}{\sqrt{\sin\psi_j(\alpha,\beta)}})d\alpha)d\beta\Big]\|g\|_1\\
&\le C\Big[\int_0^{\frac{\pi}{2}}\int_{I_k}\frac{|\frac{\partial}{\partial\beta}(\sin\psi_j(\alpha,\beta))|}{\sqrt{\sin\psi_j(\alpha,\beta)}}d\beta d\alpha\\
&\quad+\int_0^{\frac{\pi}{2}}\frac{\partial}{\partial\alpha}(\int_{I_k}\omega_{\alpha_\ell}(\beta_k)\frac{|\frac{\partial}{\partial\beta}(\sin\psi_j(\alpha,\beta))|}{\sqrt{\sin\psi_j(\alpha,\beta)}}d\beta)d\alpha\Big]\|g\|_1,\end{aligned}\tag{3.15}$$

where we have used the fact that $\omega_{\alpha_\ell}(\beta)$ is piecewise constant, so that

the derivative with respect to β can be transformed to u, and Fubini's theorem. Treating the inner integrals on the right hand side of (3.15) in a similar way as in (3.13) and then summing over j and k, we obtain (3.9) for $\beta\in[0,\frac{\pi}{2}]$ and hence by symmetry the proof is complete. □

<u>Proof of Lemma 3.2.</u> Writing $u(x,\alpha,\beta) = T_\mu g(x)$, with $\mu = (\cos\beta\cos\alpha,\ \cos\beta\sin\alpha,\ \sin\beta)$ we have using (2.4a),

$$\begin{aligned}\|(T-T_N)g\| &= \|\int_{S^2}T_\mu g(\cdot)d\mu - \Sigma_{\mu\in Q}T_\mu g(\cdot)\omega_\mu\|\\
&= \|\int_{S^2}u(\cdot,\mu)d\mu - \Sigma_{\mu\in Q}u(\cdot,\mu)\omega_\mu\|\\
&\le CN^{-1/2}\Big[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}\|\frac{\partial}{\partial\alpha}u(\cdot,\alpha,\beta)\|d\alpha d\beta\\
&\quad+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sum_\ell\omega_{\alpha_\ell}(\beta)\|\frac{\partial}{\partial\beta}u(\cdot,\alpha,\beta)\|\,|\cos\beta|d\beta\\
&\quad+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sum_\ell\omega_{\alpha_\ell}(\beta)\|u(\cdot,\alpha,\beta)\|\,|\sin\beta|d\beta\Big],\end{aligned}$$

and thus the desired result follows from Lemma 3.3. and (1.10). □

<u>Error estimate</u>. We have the following quadrature error for the scalar flux U,

$$U - U_N = (I-\lambda T_N)^{-1}e_N(x).$$

Now if $\lambda^{-1}\notin\sigma(T)$ and N is sufficiently large then by Proposition 3.1, $(I-\lambda T_N)^{-1}$ is uniformly bounded and since

$$\|e_N(x)\| = \|(T-T_N)(\lambda U + f)(x)\| \le CN^{-1/2}(\lambda\|U\|_1 + \|f\|_1)$$

we have

$$\|U-U_N\| \le C_1 N^{-1/2}(\lambda\|U\|_1 + \|f\|_1). \quad \square$$

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THE LINEARIZED BOLTZMANN EQUATION WITH REFLECTING BOUNDARY CONDITIONS
I. THE SPACE OF CONTINUOUS FUNCTIONS

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ABSTRACT

We consider the linearized Boltzmann equation with special reflecting boundary conditions. Both the Boltzmann operator and the reflecting conditions are time dependent. It seems to be adequate to use locally convex spaces. The basic idea is a transformation of the boundary value problem into an initial data problem. Our aim is the formulation of an existence and uniqueness theorem. In this paper we describe the problem in a space of continuous functions.

1. SOLUTION OF THE BOUNDARY VALUE PROBLEM

We start with the integro-differential equation

$$(1.1)\qquad \frac{\partial n(x,v,t)}{\partial t} = -v\frac{\partial n(x,v,t)}{\partial x} - \sigma(x,v,t)n(x,v,t) + \int_{-1}^{1} dv' k(x,v,v',t)n(x,v',t)$$

$$(x,v)\in \left]0,1\right[\times \left[-1,1\right] \quad ,t\geq 0$$