On Convergence of a h-p Streamline Diffusion and Discontinuous Galerkin Methods for the Vlasov-Poisson-Fokker-Planck System

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Abstract. In this paper we investigate the basic ingredients for global superconvergence strategy of streamline diffusion (SD) and discontinuous Galerkin (DG) finite element approximations in H^1 and $W^{1,\infty}$-norms (see [1]) for the solution of the Vlasov–Poisson–Fokker–Planck system. This study is an extension of the results in [2]-[5], to finite element schemes including discretizations of the Poisson term, where we also introduce results of an extension of the h-versions of SD and DG to the corresponding hp-versions. Optimal convergence results presented in the paper relay on the estimates for the regularized Green's functions with memory terms where some interpolation postprocessing techniques play important roles, see [6].

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INTRODUCTION

Our purpose is to study the global superconvergence in L_2 and maximum norms for h and hp-versions of the streamline diffusion and discontinuous Galerkin finite element methods for the solution of the deterministic, multi-dimensional Vlasov–Poisson–Fokker–Planck (VPFP) system of Coulomb particles: given the initial distribution of particles $f_0(x,v) \geq 0$, in the phase–space variable $(x,v) \in \mathbb{R}^d \times \mathbb{R}^d$, $d = 1,2,3$, and the physical parameters $\beta \geq 0$ and $\sigma \geq 0$, find the distribution function $f(x,v,t)$ for $t > 0$, satisfying the nonlinear system of evolution equations

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \mathrm{div}_v[(E - \beta v) f] = \sigma \Delta_v f, & \text{in } \mathbb{R}^{2d} \times (0,\infty), \\ f(x,v,0) = f_0(x,v), & \text{for } (x,v) \in \mathbb{R}^{2d}, \\ E(x,t) = \dfrac{\theta}{|\mathscr{S}|^{d-1}} \dfrac{x}{|x|^d} *_x \rho(x,t), & \text{for } (x,t) \in \mathbb{R}^d \times (0,\infty), \\ \rho(x,t) = \displaystyle\int_{\mathbb{R}^d} f(x,v,t)\, dv, & E = \theta \tilde{E}, \text{ and } \theta = \pm 1, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^d$ is the position, $v \in \mathbb{R}^d$ is the velocity, and $t > 0$ is the time, $v \cdot \nabla_x = \sum_{i=1}^d v_i \partial / \partial x_i$. Finally $|\mathscr{S}|^{d-1} \sim 1/\omega_d$ is the surface area of the unit disc in $\mathbb{R}^d$, $\rho(x,t)$ is the spatial density and $*_x$ denotes the convolution in x. E and ρ can be interpreted as the electrical field, and charge, respectively. The macroscopic force field E can be of the form

$$E(x,t) = -\nabla_x \phi(x,t), \tag{2}$$

with $\phi(x,t)$ being the internal potential field. For a gradient field, E is divergence free and with no viscosity: $\beta = 0$, the first equation in (1) would become

$$\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = \sigma \Delta_v f, \tag{3}$$

which, with the rest of equations in (1), gives rise to the Vlasov–Fokker–Planck system. If in addition $\sigma = 0$, then we obtain the classical Vlasov–Poisson equation with $\phi(x,t)$ satisfying the Poisson equation

$$\Delta_x \phi(x,t) = \int_{\mathbb{R}^d} f(x,v,t) dv = \rho(x,t). \tag{4}$$

We shall concentrate on the following (modified) version of the VPFP equation

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f = \nabla_v \cdot (\beta v f + \sigma \nabla_v f). \tag{5}$$

The mathematical study of the VPFP system has been considered by several authors in various settings, see e.g. [7] The deterministic approach is based on controlling the behavior of the trajectories, i.e., the solutions of the ordinary differential equations underlying the Vlasov–Poisson equation. Compared to the analytical studies the numerical analysis of the VPFP system is much less developed. In the deterministic approaches the dominant part of numerical studies are using method of characteristics: basically particle methods developed for the Vlasov–Poisson equation, see [8].

Concerning hp finite element strategy: In the classical finite element method (h-version) convergence order improvement relies on mesh refinement while keeping the approximation order within the elements at a fixed low value (suitable for problems with highly singular solutions that require small mesh parameter). Some studies on the h-version of the SD finite element method can be found, e.g., in [9] for advection-diffusion, Navier-Stokes and first order hyperbolic equations, in [10] for Euler and Navier-Stokes equations, in [2] for the Vlasov-Poisson and in [3], for the Fokker-Planck and Fermi equations and in [11] for conservation laws. On the other hand in the spectral method the accuracy improvement is accomplished by raising the order of approximation polynomial rather than mesh refinement (advantageous in approximating smooth solutions). However, most realistic problems have local behavior (are locally smooth or locally singular), therefore a more realistic numerical approach would be a combination of mesh refinement in the vicinity of singularities (with lower order polynomial approximations), and higher order polynomial approximations in high regularity regions (with larger, non-refined, mesh parameter). This strategy, which can be viewed as a generalized adaptive approach, is the hp-version of the finite element method. For some basic$p$ and hp-finite element studies see, e.g., [12], [13] and [14].

In this paper we derive optimal error estimates for finite element approximation of (1) through the study of regularized Green's function for (4) combined with the SD and DG methods for (3) and (5). We also give optimal stability and convergence results for the hp-versions of the above approaches. We shall give the SD approach in some detail, however, to keep the presentation concise, for both DG and hp approaches we mention the main results and refer the reader to some current literature.

THE CONTINUOUS PROBLEM

With separate study of ϕ (and $\beta = 0$) we are left with the continuous problem called the Vlasov–Fokker–Planck system:

$$\begin{cases} \partial_t f + v\cdot\nabla_x f + E\cdot\nabla_v f - \sigma\Delta_v f = 0, & f(x,v,0) = f_0(x,v), \\ E(x,t) = C_d \int_{\mathbb{R}^d} \dfrac{x-y}{|x-y|^d}\rho(y,t)\,dy, & \rho(x,t) = \int_{\mathbb{R}^d} f(x,v,t)\,dv. \end{cases} \tag{6}$$

We split the study of problem (1) to solving the Poisson equation (4) for ϕ in order to determine the field E and then solve the following linear Fokker–Planck equation for f,

$$f_t + v\cdot\nabla_x f + E\cdot\nabla_v f - \sigma\Delta_v f = g, \quad f(x,v,0) = f_0(x,v), \tag{7}$$

where

$$E(x,v,t) = \Big(E_i(x,v,t)\Big)_{i=1}^{d},$$

is a given vector field and $f_0(x,v)$ and $g(x,v,t)$ are given functions. Existence, uniqueness, stability and regularity properties of the solution for the equation (7) are derived following 1D results in [6] for degenerate type equations.

In our studies $(x,v)\in\Omega := \Omega_x\times\Omega_v$, where $\Omega_x, \Omega_v \subset \mathbb{R}^d$ are bounded simply connected domains and we let $\Omega_T := \Omega_x\times\Omega_v\times(0,T]$. With these assumptions and $\beta\neq 0$ we consider the VPFP problem of finding (f,ϕ) satisfying

$$\begin{cases} \partial_t f + v\cdot\nabla_x f - \nabla_x\phi\cdot\nabla_v f = \nabla_v(\beta v f + \sigma\nabla_v f), & \text{in } \Omega_T, \\ f(x,v,0) = f_0(x,v), & \text{in } \Omega, \\ f(x,v,t) = 0, \quad \text{on } \Gamma^- := \{(x,v)\in\partial\Omega_x\times\partial\Omega_v : \mathbf{n}\cdot G < 0\}. \end{cases} \tag{8}$$

Here $G := (v, -\nabla_x\phi)$, and $\mathbf{n} = (\mathbf{n}(x), \mathbf{n}(v))$, with $\mathbf{n}(x)$ and $\mathbf{n}(v)$being outward unit normals to $\partial\Omega_x$ and $\partial\Omega_v$ at the point $(x,v)\in\partial\Omega_x\times\partial\Omega_v$. Further ϕ and f are associated through the Poisson equation

$$-\Delta_x\phi(x,t) = \int_{\Omega_v} f(x,v,t)\,dv, \qquad (x,t)\in\Omega_x\times(0,T] := \Omega_T, \tag{9}$$

where $\nabla_x \phi$ is uniformly bounded and $|\nabla_x \phi| \to 0$ as $x \to \partial\Omega_x$. We shall use the notation $\nabla f := (\nabla_x f, \nabla_v f)$ and

$$G(f) := (v, -\nabla_x \phi) = \Big(v_1, \dots, v_d, -\frac{\partial \phi}{\partial x_1}, \dots, -\frac{\partial \phi}{\partial x_d}\Big) = (G_1, \dots, G_{2d}),$$

leading to the following useful divergent free drift coefficient:

$$\operatorname{div} G(f) = \sum_{i=1}^{d} \frac{\partial G_i}{\partial x_i} + \sum_{i=d+1}^{2d} \frac{\partial G_i}{\partial v_{i-d}} = 0, \quad d = 1,2,3. \tag{10}$$

REGULARIZED GREEN'S FUNCTION

The Green's function plays a central role in the study of convergence of the finite element approximations for the elliptic equations and is usually considered as the solution of a dual problem. We apply this procedure to Poisson equation for ϕ by introducing its general framework below. To this end we recall the back-ward Gronwall's inequality:

Lemma 1. *Assume that ψ and φ are two non-negative functions defined on $[0,T]$. Then*

$$\psi(t) \le \varphi(t) + C\int_t^T \psi(s)\,ds \quad \Longrightarrow \quad \psi(t) \le C\Big\{\varphi(t) + \int_t^T \varphi(s)\,ds\Big\}, \quad t \in (0,T).$$

We start by introducing a finite element structure on $\Omega_x \times \Omega_v$. Let $T_h^x = \{\tau_x\}$ and $T_h^v = \{\tau_v\}$ denote finite element subdivisions of Ω_x and Ω_v, with elements τ_x and τ_v, respectively. Then $\mathscr{T}_h = T_h^x \times T_h^v = \{\tau_x \times \tau_v\} = \{\tau\}$ will be a finite element partition of $\Omega = \Omega_x \times \Omega_v$ into triangular or quadrilaterals with quasi-uniform elements $\tau = \tau_x \times \tau_v$. Let $V_h \subset H_0^1(\Omega)$ be the corresponding finite element space of order r. For a given point $z := (y,u) \in \Omega = \Omega_x \times \Omega_v$, let $\delta_h^z(p) \in V_h$, $p = (x,v)$ be the smoothed δ-function at z which satisfies

$$(\delta_h^z, g) = g(z), \quad g \in V_h. \tag{11}$$

Now we define the Green's function $\mathscr{G}^z(t) := \mathscr{G}^z(p,t;z) \in L^2((0,T); H_0^1(\Omega) \cap H^2(\Omega))$, to be the solution of the equation

$$A(t)\mathscr{G}^z(t) + \int_t^T \mathscr{B}(s,t)\mathscr{G}^z(s)\,ds = \delta_h^z \varphi(t), \quad \text{in } \Omega_T, \quad \varphi(t) \in C_0^\infty(0,T), \quad \|\varphi\|_{L_1(0,T)} \le 1, \tag{12}$$

where $\mathscr{B}$ is an integral kernel. Let l be any fixed hyperline direction and define the directional derivative

$$\partial_z \delta_h^z := \lim_{\Delta z \| l,\, \Delta z \to 0} \frac{\delta_h^{z+\Delta z} - \delta_h^z}{|\Delta z|}, \quad \text{satisfying } (\partial_z \delta_h^z, g) = \partial_z g(z), \quad g \in V_h. \tag{13}$$

We introduce the weight function $\mu(p) = \mu_z(p) := (|p-z|^2 + \nu^2)^{-1}$, with $\nu := \gamma h$ and $\gamma > 0$, and define

$$\|w\|_{\mu^\alpha}^2 := \int_\Omega \mu^\alpha |w|^2\,dxdv, \quad \|w\|_{m,\mu^\alpha}^2 := \sum_{|k| \le m} \|D^k w\|_{\mu^\alpha}^2, \quad m = 1,2,\dots, \alpha \in \mathbb{R}.$$

In this setting we have the estimate:

Lemma 2. *There is a constant C such that*

$$\|\mu^{-1}\partial_z \delta_h^z\|_0 = \|\partial_z \delta_h^z\|_{\mu^{-2}} \le C.$$

Similarly, we may define a Green's function of derivative type $\partial_z \mathscr{G}^z(t) \in L^2((0,T); H_0^1(\Omega) \cap H^2(\Omega))$ such that

$$A(t)\partial_z \mathscr{G}^z(t) + \int_t^T \mathscr{B}(s,t)\partial_z \mathscr{G}^z(s)\,ds = \partial_z \delta_h^z \varphi(t), \quad \text{in } \Omega_T. \tag{14}$$

Let $\mathscr{G}_h^z(t)$ and $\partial_z \mathscr{G}_h^z(t)$ be finite element approximations of the regularized Green's functions $\mathscr{G}^z$ and $\partial_z \mathscr{G}^z$, respectively. For these approximations we have the following error estimates (valid also for the solution of our Poisson equation):

Theorem 1. *Assume that $\mathscr{G}^z(t)$ and $\mathscr{G}_h^z(t)$ are the solutions of (12) and its finite element approximation, respectively. Then, there exists a constant C such that*

$$\|\mathscr{G}^z - \mathscr{G}_h^z\|_{1,1} \le Ch|\log h|(1+\varphi(t)), \qquad \|\mathscr{G}^z\|_{2,1} + \|\mathscr{G}_h^z\|_{2,1} \le C|\log h|(1+\varphi(t)),$$

where $\|g\|_{2,q} := \Big(\sum_\tau \|g\|_{2,q,\tau}\Big)^{1/2}$ with $1 \le q \le \infty$ for all elements $\tau \in \mathscr{T}_h$ and $g \in V_h$.

Theorem 2. *Assume that $\partial_z\mathscr{G}^z$ and $\partial_z\mathscr{G}_h^z$ are the solutions of* (14) *and its finite element approximation, respectively. Then, there exists a constant C such that*

$$||\partial_z\mathscr{G}^z - \partial_z\mathscr{G}_h^z||_{1,1} \le C(1+\varphi(t)), \qquad ||\partial_z\mathscr{G}z||_{1,1} \le C|\log h|(1+\varphi(t)),$$
$$||\partial_z\mathscr{G}^z||_0 + ||\partial_z\mathscr{G}_h^z||_0 \le C|\log h|^{1/2}(1+\varphi(t)).$$

These are superconvergence results, e.g. the first estimate in Theorem 1 is an almost $\mathscr{O}(h^2)$ L_1-norm convergence.

THE STREAMLINE DIFFUSION METHOD

The streamline diffusion (SD) method is a finite element method constructed for convection dominated convection–diffusion problems which (i) is higher order accurate and (ii) has good stability properties. The (SD) method was introduced by Hughes and Brooks [9] for the stationary problems. The mathematical analysis for this method in two settings (streamline diffusion and discontinuous Galerkin) are developed for ,e.g., two–dimensional incompressible Euler and Navier–Stokes equations in [10], for multi–dimensional Vlasov–Poisson equation in [2], for hyperbolic conservations laws in [11], and for the two–dimensional Fermi and Fokker–Planck in [3]. Here is the SD strategy: Let $0 = t_0 < t_1 < \cdots < t_M = T$ be a partition of the time interval $I = [0,T]$ into subintervals $I_m = (t_m, t_{m+1})$, $m = 0,1,\ldots,M-1$. Let $\mathscr{C}_h$ be the corresponding subdivision of $Q_T = \Omega \times [0,T]$ into elements $K := \tau \times I_m$, with the mesh parameter $h = \operatorname{diam} K$ and $P_k(K) = P_k(\tau_x) \times P_k(\tau_v) \times P_k(I_m)$ the set of polynomials in (x, v, t) of degree at most k on K. In the study of SD–method for the VPFP system given by (8), the trial functions are continuous in the x and v variables, but may assumed to be discontinuous in time. Below we introduce the basis finite element space

$$V_h = \left\{ g \in \mathscr{H}_0 : g\Big|_K \in P_k(\tau) \times P_k(I_m); \quad \forall K = \tau \times I_m \in \mathscr{C}_n,\ k = 0,1,\ldots \right\},$$

where $\mathscr{H}_0 = \prod_{m=0}^{M-1} H_0^1(S_m)$, $\quad S_m = \Omega \times I_m$, $\quad m = 0,1,\ldots,M-1$, $\qquad H_0^1 = \left\{ g \in H^1 : g \equiv 0 \text{ on } \partial\Omega_v^h \right\}$.
Further, $(f,g)_m = (f,g)_{S_m}$, $\quad ||g||_m = (g,g)_m^{1/2}$, $\quad <f,g>_m = (f(\cdot,\cdot,t_m),(g(\cdot,\cdot,t_m))_\Omega$ and $|g|_m = <g,g>_m^{1/2}$.
We also present the jump $[g] = g_+ - g_-$, where for $t \in I$,

$$g_\pm = \lim_{s\to 0\pm} g(x,v,t+s), \quad (x,v) \in \text{ Int } (\Omega_x) \times \Omega_v^h, \qquad g_\pm = \lim_{s\to 0\pm} g(x+sv,v,t+s), \quad (x,v) \in \partial\Omega_x \times \Omega_v^h,$$

and the boundary integrals defined by

$$<f_+,g_+>_{\Gamma^-} = \int_{\Gamma^-} f_+ g_+ (G^h\cdot\mathbf{n})\,dv, \qquad <f_+,g_+>_{\Gamma_m^-(\Gamma_I^-)} = \int_{I_m(I)} <f_+,g_+>_{\Gamma^-} dv,$$

with $G^h := G(f^h)$ defined as above. We use the discrete version of (10): div $G(f^h) = 0$, and for a given appropriate function $\tilde{f}$, define the trilinear form B by

$$\begin{aligned} B\big(G(\tilde{f});f,g\big) =& \big(f_t + G(\tilde{f})\nabla f, g + h(g_t + G(f^h)\nabla g)\big)_{Q_T} + \sigma\big(\nabla_v f,\nabla_v g\big)_{Q_T} - h\sigma\big(\Delta_v f, g_t + G(f^h)\nabla g\big)_{Q_T} \\ &+ \sum_{m=1}^{M-1} \langle [f], g_+\rangle_m + \langle f_+, g_+\rangle_0 - \langle f_+,g_+\rangle_{\Gamma_I^-}, \end{aligned}$$

and the bilinear form K by

$$K(f,g) = \big(\nabla_v(\beta v f), g + h(g_t + G(f^h)\nabla g)\big)_{Q_T}.$$

Note that both B and K depend implicitly on f^h (hence on h) through the term $G(f^h)$. We also define the linear form L

$$L(g) = \langle f_0, g_+\rangle_0.$$

Using this notation we can formulate the SD-problem in the following concise form: *find $f^h \in V_h$ such that*

$$B\big(G(f^h);f^h,g\big) - K\big(f^h,g\big) = L(g), \qquad \forall g \in V_h. \tag{15}$$

We shall give our stability and convergence estimates for (15) in a triple norm defined by

$$|||g|||^2 = \frac{1}{2}\left[2\sigma||\nabla_v g||^2_{Q_T} + |g|^2_M + |g|^2_0 + \sum_{m=1}^{M-1} |[g]|^2_m + 2h||g_t + G(f^h)\nabla g||^2_{Q_T} + \int_{\partial\Omega\times I} g^2|G^h\cdot\mathbf{n}|\,dv\,ds\right].$$

Lemma 3 (Stability I). *We have that*

$$\forall g \in \mathscr{H}_0, \qquad B\big(G(f^h); g, g\big) \geq \frac{1}{2}|||g|||^2.$$

Lemma 4 (Stability II). *For any constant $C_1 > 0$ we have for any $g \in \mathscr{H}_0$,*

$$||g||^2_{\Omega_T} \leq \left[\frac{1}{C_1}||g_t + G(f^h)\nabla g||^2_{Q_T} + \sum_{m=1}^{M} |g_-|^2_m + \int_{\partial\Omega\times I} g^2|G^h\cdot\mathbf{n}|\,dv\,ds\right] he^{C_1 h}.$$

For the proofs follow the argument in [2]-[3] (be constructive). Let $\tilde{f}^h \in V_h$ be an interpolant of f with the interpolation error denoted by $\eta = f - \tilde{f}^h$ and set $\xi = f^h - \tilde{f}^h$, so we have $e = f - f^h = \eta - \xi$. The objective in the error estimates is to dominate $|||\xi|||$ by the known interpolation estimates for $|||\eta|||$. Our main result in this section is as follows:

Theorem 3. *Assume that $f^h \in V_h$ and $f \in H^{k+1}(Q_T)$, $k \geq 1$, are the solutions of (15) and (8), respectively, such that*

$$||\nabla f||_\infty + ||G(f)||_\infty + ||\nabla\eta||_\infty \leq C. \qquad (16)$$

Then there exists a constant C such that

$$|||f - f^h||| \leq Ch^{k+\frac{1}{2}}||f||_{k+1,\Omega_T}.$$

In the proof of Theorem 3 we use two results estimating the forms B and K. Combining these results, with the estimates of the previous section for ϕ as a generalized Green's function, gives superconvergence for the SD estimate for VPFP. The discontinuous Galerkin counterpart assumes discontinuities, even, in x and v and follows similar pattern, however somewhat lengthy procedure where, in addition to the sum over jumps in the time direction, we also have a sum over the jumps over the enter-element boundaries, (see the formulation below).

DISCONTINUOUS GALERKIN AND HP RESULTS

Theorem 4. *Under the conditions of theorem 3, the discontinuous Galerkin approximation for solutions of (8), satisfies*

$$|||f - f^h|||_{DG} \leq Ch^{k+\frac{1}{2}}||f||_{k+1,\Omega_T},$$

where

$$|||f - f^h|||_{DG} = |||f - f^h||| + \sum_{K\in\mathscr{C}_h} \int_{\partial K_-(G'')} [u]^2|G^h\cdot n|,$$

with $\partial K_-(G'') = \{(x,v,t) \in \partial K_-(G') : n_t(x,v,t) = 0\}$ controls an additional term corresponding to the sum of the jump-discontinuities over enter-element boundaries.

As for the hp version (p is the degree of polynomial in spectral approximation, see the definition of $\mathscr{A}_p$ below. The accuracy of hp is measured in powers of (h/p) with small mesh parameter h and high spectral degree p, see [14]). Assume a partition $\mathscr{P}$ of Ω_T into open patches P which are image of the canonical cube $\hat{P} = (-1,1)^{2d+1}$, under smooth bijections F_P: $\forall P \in \mathscr{P}$; $P = F_P(\hat{P})$. For each P a mesh $\hat{\mathscr{T}}_P$ is obtained by subdividing $\hat{P}$ into quadrilaterals labeled $\hat{\tau}$ affine equivalent to $\hat{P}$,

$$\forall P \in \mathscr{P}; \quad \mathscr{T}_P := \{\tau | \tau = F_P(\hat{\tau}),\ \hat{\tau} \in \hat{\mathscr{T}}_P\}.$$

Each $\hat{\tau}$ is an image of $\hat{P}$ under affine mapping $A_{\hat{\tau}} : \hat{P} \to \hat{\tau}$. Let $\mathscr{T} := \cup_{P \in \mathscr{P}} \mathscr{T}_P$, $\quad F_{\mathscr{P}} = \{F_P : P \in \mathscr{P}\}$, $\quad F_\tau = F_P \circ A_{\hat{\tau}}$,

$$\mathscr{A}_p = \text{span}\{(\hat{x}, \hat{v})^\alpha : 0 \le \alpha_i \le p,\ 1 \le i \le 2d+1\}, \qquad (\hat{x}, \hat{v}) \in \hat{P}.$$

We skip the details and, with these notations, state a patch-wise optimal hp convergence result for the VPFP system.

Theorem 5. *The hp-estimate with piecewise polynomials of degree p for the SD method for solutions of* (8), *satisfies*

$$\|f - f\|^2_{SD,P} \le C \sum_{\tau \in \mathscr{T}_P} h_\tau^{2s_\tau+1} \frac{\Phi(p_\tau, s_\tau)}{p_\tau} \|\hat{f}\|^2_{s_\tau+1,\hat{\tau}} \le \left(\frac{h}{p}\right)^{2s_\tau+1} \|\hat{f}\|^2_{s_\tau+1,\hat{\tau}}, \quad \tau \in \mathscr{T},$$

where $\Phi(p_\tau, s_\tau) = \max(\Phi_1(p_\tau, s_\tau), \Phi_2(p_\tau, s_\tau))$, *and with parameters* $\alpha_p = \frac{1}{p(p+1)}$, $\beta_{|m|_k} = \frac{(p-s+|m|_k)!}{(p+s-|m|_k)!}$, *we have*

$$\Phi_1(p,s) = \mathscr{N} \sum_{i=1}^{\mathscr{N}} 2^{i-1} \sum_{|m|_{i-1} \le i-1} \alpha_p^{|m|_{i-1}+1} \beta_{|m|_{i-1}}, \quad \Phi_2(p,s) = \mathscr{N} \sum_{i=1}^{\mathscr{N}} \sum_{j=1}^{\mathscr{N}} 2^j \sum_{\substack{|m|_{j-1} \le j-1 \\ m_i = 1}} \alpha_p^{|m|_{j-1}} \beta_{|m|_{j-1}}.$$

A proof for can be obtained following the outlines in [5], using Stirling's formula (under certain assumption) to show:

$$\Phi(p_\tau, s_\tau) \le C p_\tau^{-2s_\tau}.$$

Similar estimates hold for the hp DG approximation including additional terms corresponding to the sum of the jump-discontinuities over enter-element boundaries. To summarize we have a convergence of order $\mathscr{O}(h/p)^{s+1/2}$ in $H^{s+1}(\Omega_T)$ which is an optimal result improving the classical convergence rate for hyperbolic problems by $\mathscr{O}(h/p)^{1/2}$.

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