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Convergence analysis for Backward-Euler and mixed discontinuous Galerkin methods for the Vlasov-Poisson system

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Abstract We construct and analyze a numerical scheme for the two-dimensional Vlasov-Poisson system based on a backward-Euler (BE) approximation in time combined with a mixed finite element method for a discretization of the Poisson equation in the spatial domain and a discontinuous Galerkin (DG) finite element approximation in the phase-space variables for the Vlasov equation. We prove the stability estimates and derive the optimal convergence rates depending upon the compatibility of the finite element meshes, used for the discretizations of the spatial variable in Poisson (mixed) and Vlasov (DG) equations, respectively. The error estimates for the Poisson equation are based on using Brezzi-Douglas-Marini (BDM) elements in L_2 and $H^{-s}, \ s > 0$, norms.

Keywords Vlasov-Poisson · Backward-Euler · Mixed finite element · Brezzi-Douglas-Marini elements · Discontinuous galerkin · Stability · Convergence

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1 Introduction

In this paper we study a numerical scheme approximating the solution of the deterministic two-dimensional Vlasov-Poisson (VP) system described below: Given the initial distribution of particles density $f_0(x, v)$, $(x, v) \in \Omega_x \times \mathbb{R}^2 \subset \mathbb{R}^2 \times \mathbb{R}^2$, find the evolution of a single species plasma formed by charged particles, at time t, in a bounded open set $\Omega_x \subset \mathbb{R}^2$ with a phase space density $f(x, v, t)$ satisfying

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x \varphi \cdot \nabla_v f = 0, & \text{in } \Omega \times [0, T], \\ f(x, v, 0) = f_0(x, v), & \text{in } \Omega = \Omega_x \times \mathbb{R}^2, \\ -\Delta_x \varphi = \displaystyle\int_{\mathbb{R}^2} f(x, v, t)\, dv, & \text{in } \Omega_x \times [0, T], \\ \varphi(x, t) = 0, & \text{on } \partial\Omega_x \times [0, T], \end{cases} \tag{1.1}$$

where $\cdot$ denotes the scalar product. To construct numerical methods for the system (1.1) we shall restrict the velocity variable v to a bounded domain $\Omega_v \subset \mathbb{R}^2$ and provide the equation with a Dirichlet type, inflow boundary condition, in the phase-space variable. We also split the equation system to separate the Poisson and Vlasov equations coupled with the potential φ. Thus we reformulate the problem (1.1) as follows: given the initial data $f_0(x, v)$ with $(x, v) \in \Omega_x \times \Omega_v \subset \mathbb{R}^2 \times \mathbb{R}^2$, find the density function $f(x, v, t)$ of the initial-boundary value problem for the Vlasov equation

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x \varphi \cdot \nabla_v f = 0, & \text{in } \Omega \times [0, T], \\ f(x, v, 0) = f_0(x, v), & \text{in } \Omega = \Omega_x \times \Omega_v, \\ f(x, v, t) = 0, & \text{on } \Gamma_v^- \quad \text{for } t \in [0, T]. \end{cases} \tag{1.2}$$

Here for every $v \in \Omega_v$, $\Gamma_v^- := \{x' \in \partial\Omega_x : \mathbf{n}(x') \cdot v < 0\}$ is the inflow boundary of Ω_x, (see [2]) and $\mathbf{n}(x')$ is the outward unit normal to $\partial\Omega_x$ at $x' \in \partial\Omega_x$, moreover, the potential φ satisfies the following Dirichlet problem for the Poisson equation

$$\begin{cases} -\Delta_x \varphi = \displaystyle\int_{\Omega_v} f(x, v, t)\, dv, & \text{in } \Omega_x \times [0, T], \\ \varphi(x, t) = 0, & \text{on } \partial\Omega_x \times [0, T]. \end{cases} \tag{1.3}$$

An analytic approach to solve the problem (1.3), is based on replacing f by a given suitable function g. Then inserting the solution, say φ_g, in (1.2) we obtain a new Vlasov equation as (1.2) with φ replaced by φ_g. Assume that we can solve this new Vlasov equation, then in this way we link its solution f_g to the given function g via, $f_g = \Lambda[g]$. Now a solution f for the original Vlasov equation is a fixed point of the operator Λ: $f = \Lambda[f]$, provided that Λ fulfills the conditions of a Schauder fixed point operator, see [32] for the details. For a discrete version, in a finite dimensional space, the argument relies on the Brouwer fixed point theorem, as in [2] and the reference therein. For simplicity, and due to the fact that the BDM method used in our Poisson scheme is given in 2D, we perform this study for the two dimensional case. In three dimensions some key estimates, based on Sobolev embedding theorems, and depending on the dimension, would become quasi-optimal, see, e.g. [3].

Positivity, existence, uniqueness and regularity properties for the continuous problem (1.1) in the full space $\mathbb{R}^{2d}$, $d = 2, 3$, are inherited from those derived for a bounded positive initial data $f_0 \in L_\infty(\mathbb{R}^{2d}) \geq 0$, with the bounded second phase-space moment: $\int_{\mathbb{R}^{2d}}(1 + |x|^2 + |v|^2) f_0\, dx dv < \infty$, see [11]. These are mainly indicating that f is non-negative and also we have bounded mass and energy.

Further analytic approaches are given, e.g. by Horst in [22]. For a general mathematical framework in this study we refer to the results by J. L. Lions in [26], and Baouendi and Grisvard in [10]. These are abstract results giving an idea about the behavior of the solution and, in the very special cases, are leading to the closed form analytical expressions. In this regard some specific studies for the Vlasov-type systems are, e.g. the global in time solutions of the two-dimensional Vlasov-Poisson system by Wollman [35], global symmetric solutions of the initial value problem of the stellar dynamics considered by Batt [8], and the classical solution in the large-in-time of two-dimensional Vlasov equation by Ukai and Okabe [32]. Finally, in three dimensions global existence of smooth solutions to the Vlasov-Poisson system is studied by Schaeffer [30].

Our goal is to construct and analyze a numerical scheme which yields approximate solutions (piecewise polynomials) sufficiently close to the exact solution in certain norms specified below. In our previous studies in, e.g. [2] and [4], we assumed a continuous Poisson solver. Then, using a streamline diffusion (SD) approach, see e.g. [24], we derived optimal convergence rates for both the SD and discontinuous Galerkin (DG) approximations for the Vlasov-Poisson and Vlasov-Fokker-Planck equations, respectively. This paper concerns a fully discrete scheme for a combined spatial discretization using a mixed DG method for the Poisson Eq. (1.3) with, a SD based, phase-space DG approximation for the Vlasov equation (1.2). Compared to [2] and [4], as well as [24], here we have separated the time discretization from the previous SD-based schemes and have employed a backward Euler (BE) method in time. The backward Euler method is well-correlated with the DG method, see [18]. In this setting, the local in time convergence rate is that of the BE method, whereas the discretization for the phase-space has a convergence rate governed by that of the less regular first order hyperbolic equation in (1.2) when the potential term is already replaced by its discretized counterpart.

More specifically, in this paper we shall consider the two-dimensional case and study the convergence of a fully discrete numerical scheme consisting of

(i) A mixed Brezzi-Douglas-Marini (BDM) finite element method for the spatial discretization for the Poisson Eq. (1.3).
(ii) A discontinuous Galerkin (DG) method for the space-velocity variables for the Vlasov equation (1.2).
(iii) A backward-Euler (BE) discretization in time for the Vlasov equation (1.2).

Problem (ii) being hyperbolic would require a finer mesh than the more regular elliptic problem (i). We shall correlate these meshes at the final combined step. However, the numerical approaches for the problems (i) and (ii) are chosen independently, therefore they are presented with the different (distinguishable) mesh sizes: $\hbar$ and h, respectively.

Our motivation for the choice of this combination lies in the fact that the BE is unconditionally stable. For the other choices: the Crank-Nicolson scheme is conditionally stable, and to derive the stability estimates for the Runge-Kutta (RK) method requires a rather involved and cumbersome procedure with no improvements compared to the BE approach (the merit of RK methods are in the fact that they are the only integrators compatible with an affine transformation in ODEs, but this is not in the scope of our study in here). Likewise the DG scheme for the phase-space approximation is more stable than the continuous standard Galerkin scheme. In general, using the DG method would cause a slight reduction in the accuracy whereas the stability is enhanced, see [18]. But the high accuracy with the weak stability has no use. Here, we have compromised through combining the BE scheme in time with the DG approach in the phase-space and achieved desirable stability properties.

We start with a continuous time variable and a (possibly coarse) spatial mesh of size $\hbar$ and solve (i) to obtain $\varphi_\hbar$. Then, in (ii), we replace φ by $\varphi_\hbar$, refine the mesh, if necessary, and obtain the discrete solution f_h. The approximation (f_h, φ_h) in (ii) may, roughly, be viewed as $(f_h, \varphi_h) \approx (f, \varphi_\hbar) \approx (f, \varphi)$. To perform (ii) we may formulate a linearized Vlasov equation:

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \varphi_\hbar \cdot \nabla_v f = 0, \quad \text{with} \quad -\Delta_{\hbar,x} \varphi_\hbar = \int_{\Omega_v} f_\hbar(x, v, t)\, dv, \tag{1.4}$$

where $\Delta_{\hbar,x}$ is the discrete Laplacian operator defined by $(-\Delta_{\hbar,x}\varphi_\hbar, u) = (\nabla\varphi_\hbar, \nabla u)$.

The backward Euler approximation in (iii) yields an iterative procedure. It starts from the initial data $f_0(x, t)$ and provides the phase-space solutions at each time level t_n, $n = 1, \ldots, N$. In performing the time discretization, φ_h^{n-1} (depending on f_h^{n-1}) is used to compute f_h^n on the next time level n, which yields a fully linearized, discrete in time, Vlasov equation. Observe that the mixed finite elements in BDM-spaces in (i), as described, are for the spatial approximation and do not involve time discretizations.

We derive sharp error bounds for (i) and (ii). The convergence rates for the discontinuous Galerkin (ii) and backward Euler (iii) methods, although each optimal, are of different order. Then, combining (i)-(iii), an optimal fully discrete method is constructed by assuming a compatibility condition on the mesh parameters. For the sharp approximations, regularity requirements of type, e.g. $\varphi \in W^{r,\infty}(\Omega_x)$, $r \geq 1$, and also a mesh compatibility relation like $h = \Delta t$, will be necessary.

Remark 1.1 Note that with implicit methods one can use large time steps, but still the error in time is of order $\mathcal{O}(\Delta t)$, regardless of the use of implicit or explicit Euler. The implicit method only means that we do not have stability conditions on the size of time step. We may use the arbitrary time step. If we use the higher order polynomials in the DG method, we can use the larger h to get the same error expansion in space and this means we can use the larger time step.

Early numerical studies for the Vlasov-Poisson and related equations have been dominated by the particle method approaches, e.g. by Cottet and Raviart in [16]; Ganguly, Lee and Victory in [20]; and Wollman, Ozizmir and Narasimhan in [34]. On the other hand, Raviart-Thomas (RT) and BDM approaches that are extensively used in the finite element approximation of the elliptic, parabolic, and parabolic integro-differential equations with memory, have substantially gained ground in more involved systems such as VP. Some related studies in this part are, e.g. the optimal L_∞ study of the finite element methods for irregular meshes by Scott in [31]; the two families of the mixed finite element methods for the second order elliptic problems by Brezzi, Douglas and Marini in [13], where the BDM spaces are introduced; the maximum norm estimates for the finite element approximation of the Stokes problem in 2D by Duran, Nochetto and Wang in [17]; the asymptotic expansions and L_∞ estimates for the mixed finite element methods for the second order elliptic problems by Wang in [33]; the maximum norm estimates for a Ritz-Volterra projection by Lin in [25]; the global superconvergence analysis in $W^{1,\infty}$-norm for the Galerkin FEMs of the integro-differential equations by Liu, Liu, Rao and Zhang in [27]; and the L_∞-error estimates and superconvergence in maximum norm of the mixed FEMs for nonfickian flows in porous media by Ewing, Lin, Wang and Zhang in [19].

For a DG approach for a semidiscrete Vlasov problem we refer to [21]. Some relevant "theoretical numerical analysis" for a higher order semi-lagrangian DG formulation, and the corresponding DG schemes for the Vlasov-Poisson equations are given by Jing-Mei Qui and Chi-Wang Shu in [28]. The corresponding DG schemes are studied by J. A. Rossmanith and D. C. Seal in [29]. The related numerical studies considering the Vlasov-Poisson can be found in the recent papers [6] and [15], and some references therein. These papers are considering a somewhat different approach, e.g. in [6] (more adequate to compare), the authors consider the Raviart-Thomas elements rather than the BDM ones which is the case in our study (see [13] for the crucial differences and motivations). The convergence rates in [6] are derived for the absolute values of certain error indicators $\mathcal{K}^i$, $i = 1, 2$ based on measuring the local quadrature-type estimates applied to interpolation operator. These estimates, controlling the absolute values of $\mathcal{K}^i$:s are sharp: $\mathcal{O}(h^{k+1})$ for $f \in H^{k+1}$. But such convergence rate is not possible for the DG method for the first order hyperbolic problems in L_p norms, see [23]. Under the same regularity assumption: $f \in H^{k+1}$, in [2], we derived *triple-norm estimates* (may be viewed as energy norm) with the *optimal convergence rate* of order $\mathcal{O}(h^{k+1/2})$, which is the best possible estimate one can hope for, cf [23]. Our particular approach would yield a convergence rate $\mathcal{O}(\Delta t^{1/2} + h^{k+1/2})$ for $f \in C^1([0, T], H^{k+1})$, confirming the compatibility of the BE with the DG(0), i.e. for $k = 0$.

In our step (i) in the present approach we use the results in [31], [13], and [33]. As for the discontinuous Galerkin approximation relevant in the Vlasov-Poisson estimates we refer to the articles by Johnson and Saranen for the Euler and Navier-Stokes equations in [24]; Asadzadeh for the Vlasov-Poisson equations in [2], Asadzadeh and Kowalczyk for the Vlasov-Fokker-Planck system in [4], and Asadzadeh and Sopasakis for the Vlasov-Poisson-Fokker-Planck system in [5].

An outline of this paper is as follows: in Section 2 we state the notations and preliminaries and derive L_2-norm error estimates for the Poisson (1.3) in mixed BDM spaces. In Section 3 we derive the L_2-stability for the DG method for the time discretized system at each time level. Section 4 is devoted to the error estimates for the DG method for the Vlasov-Poisson system.

In a forthcoming paper we shall study a posteriori error estimates of SD and DG methods for the fully discrete problem, where the approach in [2] as well as the scheme in the present work are considered.

In what follows the constants will be generic and not necessarily the same at each occurrence and independent of the other parameters, unless otherwise specified.

2 Mixed method for the poisson equation

We shall discretize the Poisson (1.3) using BDM spaces. To this end, we use the notation (vector functions will be denoted in bold face)

$$\begin{cases} -\Delta_x \varphi(x,t) = \displaystyle\int_{\Omega_v} f(x,v,t)\,dv =: \rho(x,t), \\ \Psi(x,t) := -\nabla_x \varphi(x,t), \end{cases}$$

and define a mixed form for (Ψ, φ) as

$$\begin{cases} \Psi + \nabla_x \varphi = 0, & \text{in} \quad \Omega_x, \\ \text{div } \Psi = \rho, & \text{in} \quad \Omega_x, \\ \varphi = \tilde{g}, & \text{on} \quad \partial\Omega_x, \end{cases} \tag{2.1}$$

where, to begin with, we ignore the time dependence in φ and Ψ. Note that we shall deal with homogeneous Dirichlet data, and the function $\tilde{g}$ in Eq. (2.1) is for showing the general form of the right hand side in Eq. (2.2) below. We shall use the following Hilbert space

$$S := H(\text{div}, \Omega_x) = \{\mathbf{u} \in [L_2(\Omega_x)]^2 : \text{div}\,\mathbf{u} \in L_2(\Omega_x)\},$$

associated with the norm

$$||\mathbf{u}||_S^2 = \|\mathbf{u}\|_2^2 + \|\text{div}\,\mathbf{u}\|_2^2.$$

The weak form for (2.1) reads as follows: find $(\Psi, \varphi) \in S \times L_2(\Omega_x)$ such that

$$\begin{cases} (\Psi, \mathbf{u}) - (\text{div } \mathbf{u}, \varphi) = -\langle \tilde{g}, \mathbf{u}\cdot\mathbf{n}\rangle, & \forall \mathbf{u} \in S, \\ (\text{div } \Psi, w) = (\rho, w), & \forall w \in L_2(\Omega_x), \end{cases} \tag{2.2}$$

where $(\cdot,\cdot)$ is the usual inner product in either $\left[L_2(\Omega_x)\right]^2$ or $L_2(\Omega_x)$ and $\langle\cdot,\cdot\rangle$ is the inner product in $L_2(\partial\Omega_x)$ and $\mathbf{n}$ is the outward unit normal to $\partial\Omega_x$. For $\tilde{g} \equiv 0$, the problems equations (1.3) and (2.2) are equivalent and the solubility of (2.2) is based on the inf-sup condition

$$\inf_{\Psi \in S} \sup_{w \in L_2} \frac{(\text{div } \Psi, w)}{\|\Psi\|_S \|w\|_2} \geq \lambda, \tag{2.3}$$

due to Babuška [7] and Brezzi [12] (known as Babuška-Brezzi condition) where λ is a positive constant.

For a general domain D with a triangulation $\{K\}$ and for a positive integer k, we define $P_k(K)$ as a set of scalar valued polynomials, of degree not greater than k, restricted to the element K and let $\mathbf{P}_k(K) = [P_k(K)]^2$ denote the restriction of the set of all vector valued polynomials of total degree not greater than k to K.

Now we consider a quasi-uniform triangulation of Ω_x as $\mathcal{T}_\hbar^x = \{\tau\}$ and define

$$\begin{aligned} S_\hbar^k : &= \{\mathbf{u} \in S : \mathbf{u}|_\tau \in \mathbf{P}_k(\tau), \quad \tau \in \mathcal{T}_\hbar^x\}, \\ W_\hbar^{k-1} : &= \{w \in L_2(\Omega_x) : w|_\tau \in P_{k-1}(\tau), \ \tau \in \mathcal{T}_\hbar^x\}. \end{aligned} \tag{2.4}$$

Then, $S_\hbar^k \times W_\hbar^{k-1} \subset S \times L_2(\Omega_x)$ is a mixed finite element space on the triangulation $\mathcal{T}_\hbar^x$ of Ω_x, for which the discrete version of the Babuška-Brezzi condition holds true:

$$\inf_{\Psi_\hbar \in S_\hbar^k} \sup_{w_\hbar \in W_\hbar^{k-1}} \frac{(\text{div } \Psi_\hbar, w_\hbar)}{\|\Psi_\hbar\|_S \|w_\hbar\|_2} \geq \tilde{\lambda}, \tag{2.5}$$

where $\tilde{\lambda}$ is independent of $\hbar$. Note that $W_\hbar^{k-1}$ is the space of piecewise polynomials of degree not greater than $k-1$. The mixed finite element method for (2.2) is now formulated as follows, see [13] and [33]: find $(\Psi_\hbar, \varphi_\hbar) \in S_\hbar^k \times W_\hbar^{k-1}$ such that

$$\begin{cases} (\Psi_\hbar, \mathbf{u}) - (\text{div } \mathbf{u}, \varphi_\hbar) = -\langle \tilde{g}, \mathbf{u} \cdot \mathbf{n} \rangle, & \forall \mathbf{u} \in S_\hbar^k, \\ (\text{div } \Psi_\hbar, w) = (\rho, w), & \forall w \in W_\hbar^{k-1}. \end{cases} \tag{2.6}$$

To simplify (2.6) it is customary, see [1], to employ a Lagrange multiplier to enforce the continuity of normal components of $\Psi_\hbar$ across inter-element boundaries.

Note that, a formal subtraction of the equations (2.6) and (2.2), in the subspaces $S_\hbar^k \times W_\hbar^{k-1} \subset S \times L_2$, yields the following Galerkin orthogonality for the mixed method:

$$\begin{cases} (\Psi - \Psi_\hbar, \mathbf{u}) - (\text{div } \mathbf{u}, \varphi - \varphi_\hbar) = 0, & \forall \mathbf{u} \in S_\hbar^k, \\ (\text{div } (\Psi - \Psi_\hbar), w) = 0, & \forall w \in W_\hbar^{k-1}. \end{cases} \tag{2.7}$$

2.1 Error estimates for the mixed method

Our main tools are existence of local projections $\Pi_\hbar = \Pi_\hbar^k : H(\text{div}, \Omega_x) \to S_\hbar^k$, and $\pi_\hbar = \pi_\hbar^{k-1} : L_2(\Omega_x) \to W_\hbar^{k-1}$: such that

$$\text{div} \circ \Pi_\hbar^k = \pi_\hbar^{k-1} \circ \text{div}, \tag{2.8}$$

and we have, the local, orthogonality

$$(w - \pi_\hbar^{k-1} w, \phi)_\tau = 0, \qquad \phi \in W_\hbar^{k-1}(\tau), \quad \tau \in \mathcal{T}_\hbar^x, \tag{2.9}$$

and, under certain conditions, the global orthogonality relations

$$(\text{div } (\mathbf{u} - \Pi_\hbar^k \mathbf{u}), w) = 0, \qquad w \in W_\hbar^{k-1}. \tag{2.10}$$

The relation (2.10) is identical to (2.7) in [13], which in turn relies on a rather involved globalization procedure for the projection operators.

Now, since div $S_\hbar^k = W_\hbar^{k-1}$,

$$(\text{div } \mathbf{u}, w - \pi_\hbar^{k-1} w) = 0, \qquad \mathbf{u} \in S_\hbar^k. \tag{2.11}$$

Then, it is well known that, for $0 \leq s \leq k$ and $0 \leq j \leq k$,

$$\|w - \pi_\hbar^{k-1} w\|_{H^{-s}(\Omega_x)} \leq C \left(\sum_\tau \hbar_\tau^{2(s+j)} \|w\|_{j,\tau}^2 \right)^{1/2}. \tag{2.12}$$

Moreover, for $1 \le r \le k+1$,

$$\|\mathbf{u} - \Pi_\hbar^k \mathbf{u}\|_{L_2(\Omega_x)} \le C \left(\sum_\tau \hbar_\tau^{2r} \|\mathbf{u}\|_r^2 \right)^{1/2}. \tag{2.13}$$

We shall use the following global form of the estimates (2.12) (with $s = 0$) and (2.13) in $L_2(\Omega_x)$-norm, justified by the construction of π_h^{k-1} and Π_h^k,

$$\|w - \pi_h^{k-1} w\|_{L_2(\Omega_x)} \le C h^k \|D^k w\|_{L_2(\Omega_x)}, \qquad \forall w \in H^k(\Omega_x), \tag{2.14}$$

$$\|\mathbf{u} - \Pi_h^k \mathbf{u}\|_{L_2(\Omega_x)} \le C h^{k+1} \|D^{k+1} \mathbf{u}\|_{L_2(\Omega_x)}, \qquad \forall \mathbf{u} \in \left[H^{k+1}(\Omega_x) \right]^d. \tag{2.15}$$

Below we gather the main error estimates of these approximations in the $L_2(\Omega_x)$-norm. In order to make the paper easy to follow, we shall give a brief outline to the proof of some key estimates. For more detailed proofs we refer the reader to [13] and [33].

Theorem 2.1 *Let $\{\Psi_\hbar, \varphi_\hbar\} \in S_\hbar^k \times W_\hbar^{k-1}$ be the solution of the mixed finite element scheme (2.6). Then, we have the following $L_2(\Omega_x)$ error estimates:*

$$\|\Psi - \Psi_\hbar\|_2 \le C \|\Psi - \Pi_\hbar^k \Psi\|_2 \le C \hbar^r \|\Psi\|_r, \quad 1 \le r \le k+1. \tag{2.16}$$

$$\|\rho - \rho_\hbar\|_2 = \|\rho - \pi_\hbar^{k-1} \rho\|_2 \le C \hbar^r \|\rho\|_r, \quad 0 \le r \le k. \tag{2.17}$$

$$\|\varphi_\hbar - \pi_\hbar^{k-1} \varphi_\hbar\|_2 \le C \hbar \|\Psi - \Pi_\hbar^k \Psi\|_2 + C \hbar^{\min(2,k)} \|\rho - \pi_\hbar^{k-1} \rho\|_2. \tag{2.18}$$

$$\|\varphi - \varphi_\hbar\|_2 \le C \hbar^r \left(\|\rho\|_{r-2} + |g|_{r-1/2} \right), \qquad 2 \le r \le k+2. \tag{2.19}$$

Proof Using (2.11) we may rewrite the first equation in (2.7) as

$$(\Psi - \Psi_\hbar, \mathbf{u}) - (\operatorname{div} \mathbf{u}, \pi_\hbar^{k-1} \varphi - \varphi_\hbar) = 0, \qquad \mathbf{u} \in S_\hbar^k. \tag{2.20}$$

Let now $\tilde{\mathbf{e}}_\hbar := \Pi_\hbar^k \Psi - \Psi_\hbar$ and in (2.20) take $\mathbf{u} = \tilde{\mathbf{e}}_\hbar$. Then,

$$\begin{aligned} \|\tilde{\mathbf{e}}_\hbar\|_{L_2(\Omega_x)}^2 &= (\Pi_\hbar \Psi - \Psi_\hbar, \tilde{\mathbf{e}}_\hbar) = (\Psi - \Psi_\hbar, \tilde{\mathbf{e}}_\hbar) - (\Psi - \Pi_\hbar \Psi, \tilde{\mathbf{e}}_\hbar) \\ &= (\operatorname{div} \tilde{\mathbf{e}}_\hbar, \pi_\hbar^{k-1} \varphi - \varphi_\hbar) - (\Psi - \Pi_\hbar \Psi, \tilde{\mathbf{e}}_\hbar) = -(\Psi - \Pi_\hbar \Psi, \tilde{\mathbf{e}}_\hbar), \end{aligned} \tag{2.21}$$

where, we used equations (2.20) and (2.11). Thus, using the Cauchy-Schwarz inequality

$$\|\tilde{\mathbf{e}}_\hbar\|_{L_2(\Omega_x)} \le \|\Psi - \Pi_\hbar \Psi\|_{L_2(\Omega_x)}, \tag{2.22}$$

and hence, by the well-known estimates for projection error, we get

$$\begin{aligned} \|\Psi - \Psi_\hbar\|_{L_2(\Omega_x)} &\le \|\tilde{\mathbf{e}}_\hbar\|_{L_2(\Omega_x)} + \|\Psi - \Pi_\hbar \Psi\|_{L_2(\Omega_x)} \\ &\le 2 \|\Psi - \Pi_\hbar \Psi\|_{L_2(\Omega_x)} \le C \hbar^r \|\Psi\|_r, \qquad 1 \le r \le k+1. \end{aligned} \tag{2.23}$$

This proves the first estimate (2.16) of the theorem.

Next, note that by the successive use of (2.10) and the second relation in Eq. (2.7),

$$(\operatorname{div} \tilde{\mathbf{e}}_\hbar, w) = (\operatorname{div} (\Psi - \Psi_\hbar), w) = 0, \qquad \forall\, w \in W_\hbar^{k-1}. \tag{2.24}$$

Taking $w = \operatorname{div} \tilde{\mathbf{e}}_\hbar$ we get $\operatorname{div} \tilde{\mathbf{e}}_\hbar = 0$. Thus, by the same calculations as in Eq. (2.23), and using projection error

$$\begin{aligned} \|\rho - \rho_\hbar\|_{L_2(\Omega_x)} &= \|\operatorname{div}(\Psi - \Psi_\hbar)\|_{L_2(\Omega_x)} = \|\operatorname{div}(\Psi - \Pi_\hbar \Psi)\|_{L_2(\Omega_x)} \\ &\le C\hbar^r \|\operatorname{div} \Psi\|_r = C\hbar^r \|\rho\|_r, \qquad 0 \le r \le k, \end{aligned} \tag{2.25}$$

which yields the second assertion (2.17) of the theorem.

Further, let $\mathcal{L}^\star \phi = \varrho$, where $\mathcal{L}^\star$ is the adjoint operator for $\mathcal{L} := -\Delta_x$, and we have that $\varrho \in L_2(\Omega_x)$ and $\phi \in H^2(\Omega_x) \cap H_0^1(\Omega_x)$. Then, we may write, see [13],

$$\left(\pi_\hbar^{k-1} \varphi - \varphi_\hbar\,,\, \varrho\right) = (\Psi - \Psi_\hbar, \nabla \phi - \Pi_\hbar(\nabla_x \phi)) + (\operatorname{div}(\Psi - \Psi_\hbar), \phi - \pi_\hbar \phi). \tag{2.26}$$

Then, by equations (2.14)–(2.15), together with the elliptic regularity of $\mathcal{L}^\star$, (2.26) yields (2.18).

Finally, using equations (2.16)–(2.18), and the projection error estimates equations (2.14) and (2.15),

$$\begin{aligned} \|\varphi - \varphi_\hbar\|_{L_2(\Omega_x)} &\le \|\pi_\hbar^{k-1} \varphi - \varphi_\hbar\|_{L_2(\Omega_x)} + \|\varphi - \pi_\hbar^{k-1} \varphi\|_{L_2(\Omega_x)} \\ &\le C\left(h^{r+2} \|\Psi\|_{r+1} + h^{\min(r+2,k)} \|\rho\|_r + h^{\min(r,k)} \|\varphi\|_r\right), \end{aligned} \tag{2.27}$$

which, using the elliptic regularity of $\mathcal{L}^\star$ is simplified to (2.19), (we omit the details), and the proof is complete. ☐

Below, we state some of the L_∞ results, due to Wang [33], for the error $\Psi - \Psi_\hbar$, based on the regularized Green's functions approach. These are intermediate steps in the L_∞ studies that are relevant in our L_2-error estimates.

Proposition 2.1 *Let (Ψ, φ) and $(\Psi_\hbar, \varphi_\hbar)$ be the exact solution for (2.2) and the mixed finite element approximations in the BDM space, respectively, and assume that $\varphi \in W^{1,\infty}(\Omega_x)$. Then*

$$\|\Psi - \Pi_\hbar \Psi\|_\infty \le C |\log \hbar|^{1/2} \Big(\|\Psi - \Pi_\hbar^k \Psi\|_\infty + \hbar\, |\log \hbar|^{\delta_{1k}/2} \|\rho - \pi_\hbar^{k-1} \rho\|_\infty \Big), \qquad k \ge 1$$

where δ_{1k} is the Kronecker function. An improved version of the above estimate for sufficiently smooth $\partial\Omega$ and $k > 1$ is given by

$$\|\Psi - \Pi_\hbar \Psi\|_\infty \le C \left(|\log \hbar|^{1/2} \|\Psi - \Pi_\hbar^k \Psi\|_\infty + \hbar \|\rho - \pi_\hbar^{k-1} \rho\|_\infty \right). \tag{2.28}$$

If in addition, $\varphi \in W^{k+2,\infty}(\Omega_x)$, then

$$\|\Psi - \Psi_\hbar\|_\infty \le C\hbar^{k+1} |\log \hbar|^{1/2} \left(\|\varphi\|_{k+2,\infty} + |\log \hbar|^{\delta_{k1}/2} \|\rho\|_{k,\infty} \right). \tag{2.29}$$

The estimates in Proposition 2.1 are used to derive the projection and finite element error estimates for $\|\varphi_\hbar - \pi_\hbar^{k-1} \varphi\|_\infty$ and $\|\varphi - \varphi_\hbar\|_\infty$. We use (2.29) in our estimates.

3 The discontinuous Galerkin method for Vlasov equation

In this section we consider the Vlasov equation (1.2), and insert the computed value Ψ_h, from the previous section, for the discrete gradient of the potential function φ. Thus, we study the following linearized version of the Vlasov equation: (1.4),

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \Psi_h \cdot \nabla_v f = 0, & \text{in } \Omega \times [0,T], \\ f(x,v,0) = f_0(x,v), & \text{in } \Omega = \Omega_x \times \Omega_v, \\ f(x,v,t) = 0, & \text{on } \Gamma_v^- \quad \text{for } t \in [0,T]. \end{cases} \tag{3.1}$$

We discretize (3.1) by the discontinuous Galerkin (DG) finite element method in (x, v), combined with the backward Euler (BE) method in t.

Remark 3.1 Note that the BE method is equivalent to a time discretization using the DG(0), i.e. the piecewise constant approximation in time. One can show that for the time discretization, the error of the continuous Galerkin of order one CG(1) approximation, is smaller than that of the DG(0) approximation (cf [18]). In particular, the CG(1) method converges more rapidly than the DG(0) method as the mesh is refined. On the other hand the stability properties of the DG(0) method is exactly as that of the BE, and hence yields better results, e.g. for parabolic problems, than the CG(1) method.

Let now $\mathcal{C}_h := \{K\} = \{\tau_x \times \tau_v\}$ be a family of quasi-uniform partition of the phase space domain $\Omega = \Omega_x \times \Omega_v$, with the mesh parameter $h (\sim h_x \sim h_v)$.

For the remaining part of the paper we let k be a positive integer and introduce the triangular finite element spaces of test and trial functions as

$$\begin{aligned}
&V_h = V_h^k := \{w \in L_2(\Omega) : w|_K \in P_k(K),\ \forall K \in \mathcal{C}_h\} \\
&V_h^{0,v} = V_h^{0,v,k} := \{w \in L_2(\Omega) : w|_K \in P_k(K),\ w|_{\partial K \cap \Gamma_v^-} = 0,\ \forall K \in \mathcal{C}_h\} \\
&\tilde{V}_h = \tilde{V}_h^{k} := \{w \in C([0,T], L_2(\Omega)) : w(t)|_K \in P_k(K),\ \forall K \in \mathcal{C}_h\} \\
&\tilde{W}_h = \tilde{W}_h^{k-1} := \left\{w \in C([0,T], L_2(\Omega_x)) : w(\cdot,t)|_{\tau_x} \in P_{k-1}(\tau_x),\ \forall \tau_x \in \mathcal{T}_{h_x}^x\right\} \\
&\tilde{S}_h = \tilde{S}_h^k := \left\{\mathbf{u} \in C([0,T], [L_2(\Omega_x)]^2) : \mathbf{u}(\cdot,t)|_{\tau_x} \in \mathbf{P}_k(\tau_x),\ \forall \tau_x \in \mathcal{T}_{h_x}^x\right\},
\end{aligned}$$

where in $\tilde{S}_h$; $\mathbf{u} \cdot \mathbf{n}_e$ are continuous across all interior edges e for $\tau_x \in \mathcal{T}_{h_x}^x$. Note that the space $\tilde{V}_h = C([0,T], V_h)$. Similar identifications for $\tilde{W}_h$ and $\tilde{S}_h$ would make the notation less concrete.

Next, we formulate the discontinuous Galerkin approximation of the Vlasov eqrefDG:VPsystem1 in x, v-variables as: given the initial data f_0, and an approximate potential $\Psi_\hbar \in \tilde{S}_\hbar$ (computed in Section 2), find $f_h \in \tilde{V}_h$ such that, for all $g \in V_h^{0,v}$,

$$(\partial_t f_h + G_h(\Psi_\hbar)\nabla f_h, g + hG_h(\Psi_\hbar)\nabla g)_\Omega + \sum_{K \in \mathcal{C}_h} \int_{\partial K_G^-} [f_h] g_+ |G_h(\Psi_\hbar) \cdot \mathbf{n}|\, d\nu = 0, \tag{3.2}$$

where for $(x, v) \in \partial K$, we use the jump notation $[w] = w_+ - w_-$ with

$$w_\pm(x,v) = \lim_{|s| \to 0} w((x,v) \pm s \cdot G_h(\Psi_\hbar)), \qquad s = (s_x, s_v), \quad s_x > 0, \quad s_v > 0,$$

and we suppress the inner product sign "$\cdot$", e.g. $G_h\nabla := G_h \cdot \nabla$, and $(\cdot,\cdot)_{\mathcal{D}}$ denotes the scalar product over the domain $\mathcal{D}$. Further we use the notation $G_h(\Psi_\hbar) := (v, \Psi_\hbar)$, $\nabla f = (\nabla_x f, \nabla_v f)$ and $\partial K_G^- = \{(x,v) \in \partial K : G_h(\Psi_\hbar)\cdot \mathbf{n}(x,v) < 0\}$. Note that the steps (i) and (ii) (see Section 1) are performed as follows: starting with $\Psi_\hbar$ (computed in Section 2) first we project it on the (possibly finer) mesh with parameter h and then compute f_h using (3.2). In practice one can choose $\hbar = h$ for all steps and for the clarity of presentation in what follows we use only h.

The boundary term in Eq. (3.2) is the sum of jump terms over the inter-element boundaries in (x,v)-variables. In case of no confusion we use ∂K_- and ∂K_+ for ∂K_G^- and ∂K_G^+, respectively.

Combining equations (3.2) and (2.6) we get the mixed discontinuous Galerkin method for the system (1.1) in x, v-variables: find $(f_h, \Psi_h, \varphi_h) \in \tilde{V}_h \times \tilde{S}_h \times \tilde{W}_h$ such that

$$\begin{cases} (\partial_t f_h + G_h(\Psi_h)\nabla f_h, g + hG_h(\Psi_h)\nabla g)_\Omega + \displaystyle\sum_{K\in\mathcal{C}_h}\int_{\partial K_G^-}[f_h]g_+|G_h(\Psi_h)\cdot\mathbf{n}|\,d\nu = 0,\\ (\Psi_h, \mathbf{u}) - (\operatorname{div}\mathbf{u}, \varphi_h) = 0,\\ \qquad (\operatorname{div}\Psi_h, w) = (\rho_h, w), \qquad \text{for all } g\in V_h^{0,v},\ \ \mathbf{u}\in S_h^k \ \text{ and } w\in W_h^{k-1}. \end{cases} \tag{3.3}$$

Finally, for a partition $0 = t_0 < t_1 \ldots < t_N = T$ of the time interval $[0,T]$, with $t_n = n\Delta t$ we define $f^n := f(t_n)$, $n = 0,1,\ldots,N$ and apply the backward Euler scheme in time. This gives a discrete in time formulation of (3.3). Thus, for each $n = 1,2,\ldots,N$, we have a variational formulation for a modified stationary Vlasov-Poisson system in (x,v)-domain, where data for the Poisson equation as well as the source term (initial data) of the Vlasov equation, both are equal to the computed solution of the Vlasov equation at the previous time level $n-1$. Then, the discrete system at the time level n reads as follows: given $f_h^{n-1} \in V_h$, find first $(\Psi_h^{n-1}, \varphi_h^{n-1}) \in S_h \times W_h$ (using the mixed method for the Poisson's equation) and then $f_h^n \in V_h$ such that

$$\begin{cases} \Big((f_h^n - f_h^{n-1})/\Delta t + G_h(\Psi_h^{n-1})\nabla f_h^n\,,\ g + hG_h(\Psi_h^{n-1})\nabla g\Big)_\Omega \\ \qquad\qquad + \displaystyle\sum_{K\in\mathcal{C}_h}\int_{\partial K_G^-}[f_h^n]g_+|G_h(\Psi_h^{n-1})\ \cdot\mathbf{n}\,|\,d\nu = 0,\\ (\Psi_h^{n-1}, \mathbf{u}) - (\operatorname{div}\mathbf{u}, \varphi_h^{n-1}) = 0,\\ \qquad (\operatorname{div}\Psi_h^{n-1}, w) = (\rho_h^{n-1}, w), \qquad \forall (g,\mathbf{u},w)\in V_h\times S_h\times W_h. \end{cases} \tag{3.4}$$

Note that since f_h^{n-1} is a piecewise polynomial, so its integral can be computed exactly using a sufficiently high order quadrature rule. Thus, the scheme (3.4) operates as follows: given $f_h^{n-1} \in V_h$; ρ_h^{n-1}, the quantities Ψ_h^{n-1} and φ_h^{n-1} are computed from the last two equations of (3.4). Then f_h^n is computed from the first equation of (3.4).

The first equation in the problem (3.4) can be formulated in a more concise form as

$$b(G_h(\Psi_h^{n-1}); f_h^n, g) = L(g), \qquad \forall g \in V_h^{0,v}, \tag{3.5}$$

where b and L are, respectively, the bilinear and linear forms defined by:

$$L(g) : = \Big(f_h^{n-1}, g + hG_h(\Psi_h^{n-1}) \cdot \nabla g\Big)_\Omega, \qquad \text{and} \tag{3.6}$$

$$b(\mu; f, g) : = \Big(f + \Delta t(\mu \cdot \nabla f), g + hG_h(\Psi_h^{n-1}) \cdot \nabla g)\Big) + \Delta t \sum_{K\in\mathcal{C}_h} \int_{\partial K_G^-} [f] g_+ |G_h(\Psi_h^{n-1}) \cdot \mathbf{n}|\, d\nu. \tag{3.7}$$

In contrary to $b(G(\varphi); f, g)$, with $G(\varphi) := (v, -\nabla_x \varphi)$, which appears in the variational formulation of the continuous problem equations (1.1) and (1.2), and is nonlinear (φ depends on f), $b(G_h(\Psi_h^{n-1}); f, g)$, with Ψ_h^{n-1} depending on f_h^{n-1}, is now linear. Recall that, for the composite phase-space schemes (3.4) the final meshes are chosen as $h = h_x$ and $h_v \sim h_x$. Therefore, in the sequel, we shall only use h as our phase-space parameter. Finally, we introduce the following triple norm

$$|||g|||_\omega^2 := \|g\|_\Omega^2 + h\Delta t \|G_h(\omega) \cdot \nabla g\|_\Omega^2 + \frac{h + \Delta t}{2} \times \times \left(\sum_{K\in\mathcal{C}_h} \int_{\partial K_G^-} [g]^2 |G_h(\omega) \cdot \mathbf{n}|\, d\nu + \int_{\partial\Omega_+} g^2 |G_h(\omega) \cdot \mathbf{n}|\, d\nu \right). \tag{3.8}$$

Below, we prove, L_2-based, stability estimates for (3.5), at an arbitrary time step n, in $|||g|||_{\Psi_h^{n-1}}$-norm. In Section 4, we shall derive the error estimates in the same norm.

3.1 L_2-stability estimates

Lemma 3.1 *Assume that the function g satisfies the homogeneous inflow boundary condition: $g|_{\Gamma_-} = 0$. Then the bilinear form $b(\cdot; \cdot, \cdot)$ is coercive (elliptic) with respect to $|||\cdot|||_{\Psi_h^{n-1}}$-norm, i.e.*

$$b(G_h(\Psi_h^{n-1}); g, g) \geq (1 - h/2)|||g|||_{\Psi_h^{n-1}}^2, \qquad \forall g \in V_h^0,$$

where, for simplicity, we restrict the domain of g to

$$V_h^0 := \{g \in L_2(\Omega) : g|_K \in H^1(K),\ \ g|_{\Gamma^-} = 0,\ \ g \text{ is piecewise discontinuous on } \mathcal{C}_h\}.$$

Proof Assume that Ψ_h^{n-1} is known from the previous steps (for simplicity, in this proof, we suppress all sub and superscripts: h and $n-1$ in G_h and Ψ_h^{n-1}), then

$$b(G(\Psi); f, g) = (f, g)_\Omega + (f, hG(\Psi) \cdot \nabla g)_\Omega + \Delta t(g, G(\Psi) \cdot \nabla f)_\Omega + \Delta t(G(\Psi)\cdot\nabla f, hG(\Psi)\cdot\nabla g)_\Omega + \Delta t \sum_{K\in\mathcal{C}_h} \int_{\partial K_-(G)} [f] g_+ |G(\Psi)\cdot\mathbf{n}|\, d\nu,$$

which, with $f = g$, yields

$$b(G(\Psi); g, g) = \sum_{K\in\mathcal{C}_h} \Big[\|g\|_K^2 + (h + \Delta t)(g, G(\Psi) \cdot \nabla g)_K + + \Delta t h\, \|G(\Psi) \cdot \nabla g\|_K^2 + \Delta t \int_{\partial K_G^-} [g] g_+ |G(\Psi) \cdot \mathbf{n}|\, d\nu \Big] := \sum_{i=1}^4 T_i. \tag{3.9}$$

Hence we only need to estimate the terms T_2 and T_4. Now, using the Green's formula

$$
\begin{aligned}
(g, G(\Psi)\cdot\nabla g)_K &= \frac{1}{2}\int_{\partial K}(G(\Psi)\cdot\mathbf{n})g^2\,d\nu \\
&= \frac{1}{2}\int_{\partial K_+} g_-^2|G(\Psi)\cdot\mathbf{n}|\,d\nu - \frac{1}{2}\int_{\partial K_-} g_+^2|G(\Psi)\cdot\mathbf{n}|\,d\nu. \quad (3.10)
\end{aligned}
$$

Next, we write $[g]g_+ = g_+^2 - g_-g_+$ to get

$$
\int_{\partial K_G^-}[g]g_+|G(\Psi)\cdot\mathbf{n}|\,d\nu = \int_{\partial K_G^-} g_+^2|G(\Psi)\cdot\mathbf{n}|\,d\nu - \int_{\partial K_G^-} g_-g_+|G(\Psi)\cdot\mathbf{n}|\,d\nu. \quad (3.11)
$$

Combining equations (3.10), (3.11) and the identity

$$
\sum_{K\in\mathcal{C}_h}\int_{\partial K_+} g_-^2\bullet = \sum_{K\in\mathcal{C}_h}\int_{\partial K_-} g_-^2\,\bullet - \int_{\Gamma_-} g_-^2\,\bullet + \int_{\Gamma_+} g_-^2\bullet, \quad (3.12)
$$

we can write (note that below the added $(h+\Delta t)$-term is identically zero),

$$
\begin{aligned}
T_2 + T_4 = \sum_{K\in\mathcal{C}_h}\Bigg[&\tfrac{h+\Delta t}{2}\left(\int_{\partial K_-}[g_-^2|G(\Psi)\cdot\mathbf{n}| - g_+^2|G(\Psi)\cdot\mathbf{n}|]\,d\nu\right) \\
&+\Delta t\int_{\partial K_-} g_+^2|G(\Psi)\cdot\mathbf{n}|\,d\nu - \Delta t\int_{\partial K_-} g_+g_-|G(\Psi)\cdot\mathbf{n}|\,d\nu \\
&+(h+\Delta t)\left(\int_{\partial K_-} g_+g_-|G(\Psi)\cdot\mathbf{n}|\,d\nu - \int_{\partial K_-} g_+g_-|G(\Psi)\cdot\mathbf{n}|\,d\nu\right)\Bigg] \\
&+\tfrac{h+\Delta t}{2}\left(\int_{\Gamma_+} g_-^2|G(\Psi)\cdot\mathbf{n}|\,d\nu - \int_{\Gamma_-} g_-^2|G(\Psi)\cdot\mathbf{n}|\,d\nu\right).
\end{aligned}
$$

By the assumption $g|_{\Gamma_-} = 0$, the identity above can be written as

$$
\begin{aligned}
T_2 + T_4 =& \sum_{K\in\mathcal{C}_h}\Bigg[\frac{h+\Delta t}{2}\int_{\partial K_-}[g]^2|G(\Psi)\cdot\mathbf{n}| - (h+\Delta t)\int_{\partial K_-} g_+^2|G(\Psi)\cdot\mathbf{n}|\,d\nu \\
&+\Delta t\int_{\partial K_-} g_+^2|G(\Psi)\cdot\mathbf{n}|\,d\nu - \Delta t\int_{\partial K_-} g_+g_-|G(\Psi)\cdot\mathbf{n}|\,d\nu \\
&+(h+\Delta t)\int_{\partial K_-} g_+g_-|G(\Psi)\cdot\mathbf{n}|\,d\nu\Bigg] + \tfrac{h+\Delta t}{2}\int_{\Gamma_+} g_-^2|G(\Psi)\cdot\mathbf{n}|\,d\nu \\
=& \sum_{K\in\mathcal{C}_h}\left[\frac{h+\Delta t}{2}\int_{\partial K_-}[g]^2|G(\Psi)\cdot\mathbf{n}| - h\int_{\partial K_-}[g]g_+|G(\Psi)\cdot\mathbf{n}|\,d\nu\right] \\
&+\tfrac{h+\Delta t}{2}\int_{\Gamma_+} g_-^2|G(\Psi)\cdot\mathbf{n}|\,d\nu.
\end{aligned}
$$

Using $-[g]g_+ \geq -[g]^2/2 - g_+^2/2$, the negative term above is bounded below,

$$
-h\int_{\partial K_-}[g]g_+|G(\Psi)\cdot\mathbf{n}| \geq -\frac{h}{2}\int_{\partial K_-}[g]^2|G(\Psi)\cdot\mathbf{n}| - \frac{h}{2}\int_{\partial K_-} g_+^2|G(\Psi)\cdot\mathbf{n}|.
$$

Now we use the trace estimate, see e.g. [26], $\int_{\partial K_-} g_+^2|G(\Psi)\cdot\mathbf{n}| \le C_K\|g\|_K^2$, (where for a convex domain K; $C_K < 1$), to obtain the bound

$$-h\int_{\partial K_-}[g]g_+|G(\Psi)\cdot\mathbf{n}| \ge -\frac{h}{2}\int_{\partial K_-}[g]^2|G(\Psi)\cdot\mathbf{n}| - C_K\frac{h}{2}\|g\|_K^2. \tag{3.13}$$

Inserting (3.13) in the last equality for T_2+T_4 we get the estimate

$$T_2+T_4 \ge \sum_{K\in\mathcal{C}_h}\left[\frac{\Delta t}{2}\int_{\partial K_-}[g]^2|G(\Psi)\cdot\mathbf{n}| - C_K\frac{h}{2}\|g\|_K^2\right]+\frac{h+\Delta t}{2}\int_{\Gamma_+} g_-^2|G(\Psi)\cdot\mathbf{n}|\,d\nu.$$

Thus with a kick back argument and due to the presence of the small coefficients h_K and also the fact that $C_K < 1$, the contribution from the negative term can be hidden in the first term: $\|g\|_K$ in the triple-norm and we get, recalling (3.9), the desired result:

$$b(G(\Psi); g, g) \ge (1-h/2)|||g|||_\Psi^2,$$

and the proof is complete. □

Remark 3.2 Note that the above lemma may be proved similarly for the continuous case, where the appearance of the $G(\varphi^{n-1})$-terms (instead of $G_h(\Psi_h^{n-1})$) would require some more caution.

4 Error estimates

Following the standard procedure, we let $\tilde{f}_h^n$ to be the interpolant of f with the interpolation error denoted by $\eta^n = f^n - \tilde{f}_h^n$ and set $\xi^n = f_h^n - \tilde{f}_h^n$, so that $e^n = f^n - f_h^n = \eta^n - \xi^n$. We shall use the following well-known results:

Proposition 4.1 *Assume that Ω is a sufficiently smooth domain and let the function $f \in C^1([0,T], W^{k,\infty}(\Omega)\cap W^{k+1,2}(\Omega))$. Then we have the interpolation error estimates*

$$\|\eta\|_{L_2(\Omega,|G_h(\Psi_h^{n-1})\cdot\boldsymbol{n}|)} \le C_\Psi^v h^{k+1}\|f\|_{k+1}, \qquad \max_{1\le n\le N}|||\eta^n|||_{\Psi_h^{n-1}} \le C_i h^{k+1}\|f\|_{k+1}, \tag{4.1}$$

where $C_\Psi^v = C_i|\Omega_v|\|\Psi\|_\infty$ with $\Psi \in S$ and C_i is the interpolation constant.

Proposition 4.2 (Trace theorem) *Suppose that K is a Lipschitz domain. Then there is a constant $C_K = C|K|$ such that*

$$\|w\|_{L_2(\partial K)} \le C_K\|w\|_{L_2(K)}^{1/2}\|w\|_{H^1(K)}^{1/2}.$$

Proposition 4.1 can be proved as Theorem 4.4.3 in [14], see also [24]. For a proof of Proposition 4.2, see Brenner and Scott [9].

In the sequel we shall assume $h = \Delta t$.

Lemma 4.1 *For each $n = 1, 2, \ldots, N$, and with η^n and ξ^n defined as above, there are the positive constants c and C' such that*

$$|b(G_h(\Psi_h^{n-1}); \eta^n, \xi^n)| \le ch|||\xi^n|||^2_{\Psi_h^{n-1}} + C'h^{-1}\|\eta^n\|_2^2 + C'|||\eta^n|||^2_{\Psi_h^{n-1}}. \tag{4.2}$$

Proof We use the definition of the triple norm and estimate the bilinear form as

$$\begin{aligned}
|b(G_h(\Psi_h^{n-1}); \eta^n, \xi^n)| = \Bigg| &\Big(\eta^n + \Delta t G_h(\Psi_h^{n-1})\nabla\eta^n, \xi^n + hG_h(\Psi_h^{n-1})\nabla\xi^n\Big) \\
&+\Delta t \sum_{K\in\mathcal{C}_h}\int_{\partial K_G^-}[\eta^n]\xi_+^n|G_h(\Psi_h^{n-1})\cdot\mathbf{n}|\,d\nu\Bigg| \\
\le Ch^{-1}\|\eta^n\|_2^2 &+ ch\|\xi^n\|_2^2 + C\Delta t\|G_h(\Psi_h^{n-1})\nabla\eta^n\|_2^2 + c(\Delta t)h^2\|G_h(\Psi_h^{n-1})\nabla\xi^n\|_2^2 \\
&+ Ch^{-1}\|\eta^n\|_2^2 + ch^3\|G_h(\Psi_h^{n-1})\nabla\xi^n\|_2^2 + c\Delta t\|\xi^n\|_2^2 \\
&+C\Delta t\|G_h(\Psi_h^{n-1})\nabla\eta^n\|_2^2+\Delta t\left|\sum_{K\in\mathcal{C}_h}\int_{\partial K_G^-}[\eta^n]\xi_+^n|G_h(\Psi_h^{n-1})\cdot\mathbf{n}|\,d\nu\right|.
\end{aligned}$$

We use the Proposition 2.1, assumptions, and inverse inequality to bound the term

$$\begin{aligned}
\|G_h(\Psi_h^{n-1})\nabla\eta^n\|_2 &\le C_v\|\Psi^{n-1} - \Psi_h^{n-1}\|_\infty\|\nabla\eta^n\|_2 + C_v\|\Psi^{n-1}\|_\infty\|\nabla\eta^n\|_2 \\
&\le C_v h^{-1}\|\eta^n\|_2.
\end{aligned}$$

Moreover, for the contribution from the boundary terms we use the trace estimate as

$$\begin{aligned}
\Delta t&\left|\sum_{K\in\mathcal{C}_h}\int_{\partial K_-(G)}[\eta^n]\xi_+^n|G_h(\Psi_h^{n-1})\cdot\mathbf{n}|\,d\nu\right| \le C\Delta t\sum_{K\in\mathcal{C}_h}\int_{\partial K_G^-}[\eta^n]^2|G_h(\Psi_h^{n-1})\cdot\mathbf{n}|\,d\nu \\
&+c\Delta t\sum_{K\in\mathcal{C}_h}\int_{\partial K_G^-}|\xi^n|^2|G_h(\Psi_h^{n-1})\cdot\mathbf{n}|\,d\nu \le C|||\eta^n|||^2_{\Psi_h^{n-1}} \\
+c\Delta t \max C_K &\left(\sum_{K\in\mathcal{C}_h}\|\xi^n\|^2_{L_2(K,|G_h(\Psi_h^{n-1})\cdot\mathbf{n}|)}\right)^{1/2}\left(\sum_{K\in\mathcal{C}_h}\|\nabla\xi^n\|^2_{L_2(K,|G_h(\Psi_h^{n-1})\cdot\mathbf{n}|)}\right)^{1/2} \\
&\le C|||\eta^n|||^2_{\Psi_h^{n-1}} + c_1\Delta t|||\xi^n|||^2_{\Psi_h^{n-1}} + c_1(\Delta t)h^2\|\nabla\xi^n\|^2_{L_2(\Omega,|G_h(\Psi_h^{n-1})\cdot\mathbf{n}|)} \\
&\le C|||\eta^n|||^2_{\Psi_h^{n-1}} + c_2\Delta t|||\xi^n|||^2_{\Psi_h^{n-1}}.
\end{aligned}$$

Now we may add up all η and ξ-terms in the corresponding $\|\eta^n\|_2$ or $|||\xi^n|||_{\Psi_h^{n-1}}$-norms, use the assumption that $h \approx \Delta t$ and represent the final coefficient by a new constant C' to conclude that

$$|b(G_h(\Psi_h^{n-1}); \eta^n, \xi^n)| \le ch|||\xi^n|||^2_{\Psi_h^{n-1}} + C'h^{-1}\|\eta^n\|_2^2 + C'|||\eta^n|||^2_{\Psi_h^{n-1}},$$

which is the desired result. □

Our main result is the following, local in time, error estimate.

Theorem 4.1 *Let $(f_h^n, \Psi_h^{n-1}, \varphi_h^{n-1}) \in V_h \times S_h \times W_h$ be the mixed discontinuous Galerkin finite element approximation of (3.4), and (f, Ψ, φ) be the exact solution* *of the system equations (1.2)–(1.3) and (2.1), such that* $\|\nabla f^n\|_2 + \|\nabla f^n\|_\infty \le C$, $\varphi^{n-1} \in W^{k+2,\infty}$ *for* $n = 1, \dots, N$, *and* $f \in C^1([0,T], W^{k,\infty} \cap W^{k+1,2})$. *Then, there are the positive constants* C_1, C_2, c, *independent of* h, φ *and* f, *but may depend on the size of the velocity domain* Ω_v, *such that for sufficiently small* h *and for each* $n = 1, 2, \dots, N$

$$|||\xi^n|||^2_{\Psi_h^{n-1}} \le \frac{1}{1-ch}\left(C_1 h^{2k+1} + C_2 \Delta t \|\Theta^n\|_2^2\right) + \frac{1}{1-ch}\|\xi^{n-1}\|_2^2,$$

where $\Theta^n = \frac{f^n - f^{n-1}}{\Delta t} - f_t^n = o(\Delta t)$, *with* $o(\Delta t)$ *interpreted due to the amount of regularity of* f *in time.*

Proof The exact solution f^n at time $t = t_n$ satisfies

$$\begin{aligned} b^e(G(\varphi^n); f^n, g) &:= b(G(\varphi^n); f^n, g) - \left(\Delta t \Theta^n, g + hG_h(\Psi_h^{n-1})\nabla g\right) \\ &= \left(f^{n-1}, g + hG_h(\Psi_h^{n-1})\nabla g\right), \end{aligned}$$

where, for the sake of simplicity, the right hand side is denoted by b^e. Hence, using the Lemma 3.1 and (3.5), we may write

$$\begin{aligned} (1 - h/2)|||\xi^n|||^2_{\Psi_h^{n-1}} &\le b(G_h(\Psi_h^{n-1}); f_h^n - \tilde{f}_h^n, \xi^n) \\ &= b(G_h(\Psi_h^{n-1}); f_h^n, \xi^n) - b(G_h(\Psi_h^{n-1}); \tilde{f}_h^n, \xi^n) \\ &= \left(f_h^{n-1}, \xi^n + hG_h(\Psi_h^{n-1})\nabla\xi^n\right) - b(G_h(\Psi_h^{n-1}); \tilde{f}_h^n, \xi^n) \\ &= b^e(G(\varphi^n); f^n, \xi^n) - b(G_h(\Psi_h^{n-1}); \tilde{f}_h^n, \xi^n) \\ &\quad + \left(\xi^{n-1} - \eta^{n-1}, \xi^n + hG_h(\Psi_h^{n-1})\nabla\xi^n\right) \\ &= \left[b^e(G(\varphi^n); f^n, \xi^n) - b(G_h(\Psi_h^{n-1}); f^n, \xi^n)\right] \\ &\quad + b(G_h(\Psi_h^{n-1}); \eta^n, \xi^n) \\ &\quad + \left(\xi^{n-1} - \eta^{n-1}, \xi^n + hG_h(\Psi_h^{n-1})\nabla\xi^n\right) := J_1 + J_2 + J_3. \end{aligned}$$

Here, Lemma 4.1 gives a bound for the J_2-term. J_1 and J_3 are combined error indicators for the mixed finite element (Ψ_h is computed, using the DG approximated f_h) DG and BE approximations. Below, we estimate J_1 and J_3 separately. As for the J_1-term, using the definition of b^e and (3.5),

$$\begin{aligned} |J_1| &= \Big|\left(f^n + \Delta t G(\varphi^n)\nabla f^n - \Delta t \Theta^n, \xi^n + hG_h(\Psi_h^{n-1})\nabla\xi^n\right) \\ &\quad - \left(f^n + \Delta t G(\Psi_h^{n-1})\nabla f^n, \xi^n + hG_h(\Psi_h^{n-1})\nabla\xi^n\right)\Big| \\ &\le \left|\left(\Delta t[G(\varphi^n) - G_h(\Psi_h^{n-1})]\nabla f^n, \xi^n + hG_h(\Psi_h^{n-1})\nabla\xi^n\right)\right| \\ &\quad + \left|\left(\Delta t \Theta^n, \xi^n + hG_h(\Psi_h^{n-1})\nabla\xi^n\right)\right| := J_{11} + J_{12}. \end{aligned}$$

Evidently, we may write

$$
\begin{aligned}
|J_{11}| = & \left| \left(\Delta t [G(\varphi^n) - G(\varphi^{n-1}) + G(\varphi^{n-1}) - G_h(\Psi_h^{n-1})] \nabla f^n, \xi^n + h G_h(\Psi_h^{n-1}) \nabla \xi^n \right) \right| \\
& \le \Delta t \left| \left([G(\varphi^n) - G(\varphi^{n-1})] \nabla f^n, \xi^n + h G_h(\Psi_h^{n-1}) \nabla \xi^n \right) \right| \\
& \quad + \Delta t \left| \left([G(\varphi^{n-1}) - G_h(\Psi_h^{n-1})] \nabla f^n, \xi^n + h G_h(\Psi_h^{n-1}) \nabla \xi^n \right) \right| .
\end{aligned}
$$

Further, using the Hölder and Young's inequalities, combined with the assumptions in the theorem, the last estimate (2.29) of Proposition 2.1, and the constant contribution from the linearized term $G_h(\varphi_h^{n-1}) - G_h(\Psi_h^{n-1})$, yield

$$
\begin{aligned}
|J_{11}| & \le \Delta t \|\nabla_x(\varphi^n - \varphi^{n-1})\|_2 \|\nabla f^n\|_\infty \|\xi^n + h G_h(\Psi_h^{n-1}) \nabla \xi^n\|_2 \\
& \quad + \Delta t \|\nabla_x(\varphi^{n-1} - \varphi_h^{n-1})\|_\infty \|\nabla f^n\|_2 \|\xi^n + h G_h(\Psi_h^{n-1}) \nabla \xi^n\|_2 \\
& \le C \Delta t \|f^n - f^{n-1}\|_2 \|\nabla f^n\|_\infty \|\xi^n + h G_h(\Psi_h^{n-1}) \nabla \xi^n\|_2 \\
& \quad + \Delta t \|\Psi^{n-1} - \Psi_h^{n-1}\|_\infty \|\nabla f^n\|_2 \|\xi^n + h G_h(\Psi_h^{n-1}) \nabla \xi^n\|_2 \\
& \le C'(\Delta t)^2 \|\xi^n + h G_h(\Psi_h^{n-1}) \nabla \xi^n\|_2 + C(\Delta t) h^{k+1} |\log h|^{1/2} \\
& \qquad \cdot \left(\|\varphi^{n-1}\|_{k+2,\infty} + |\log h|^{\delta_{k1}/2} \|\rho^{n-1}\|_{k,\infty} \right) \|\xi^n + h G_h(\Psi_h^{n-1}) \nabla \xi^n\|_2 \\
& \le C'(\Delta t)^3 + \left(C_1 + \tfrac{1}{4C_{11}} \right) \Delta t \left(\|\xi^n\|_2^2 + h^2 \|G_h(\Psi_h^{n-1}) \nabla \xi^n\|_2^2 \right) \\
& \quad + C_{11}(\Delta t) h^{2k+2} |\log h| \left(\|\varphi^{n-1}\|_{k+2,\infty}^2 + |\log h|^{\delta_{k1}} \|\rho^{n-1}\|_{k,\infty}^2 \right) .
\end{aligned}
$$

As for the J_{12}-term

$$
\begin{aligned}
|J_{12}| & \le \Delta t \|\Theta^n\|_2 \|\xi^n + h G_h(\Psi_h^{n-1}) \nabla \xi^n\|_2 \\
& \le \tfrac{1}{2} \Delta t \|\Theta^n\|_2^2 + \tfrac{1}{2} \Delta t \left(\|\xi^n\|_2^2 + h^2 \|G_h(\Psi_h^{n-1}) \nabla \xi^n\|_2^2 \right) .
\end{aligned}
\tag{4.3}
$$

Next, for the term J_3 we have

$$
\begin{aligned}
|J_3| & \le \|\eta^{n-1} - \xi^{n-1}\|_2 \|\xi^n + h G_h(\Psi_h^{n-1}) \nabla \xi^n\|_2 \\
& \le \tfrac{1}{2} \left(\|\eta^{n-1}\|_2^2 + \|\xi^{n-1}\|_2^2 \right) + \tfrac{1}{2} \left(\|\xi^n\|_2^2 + h^2 \|G_h(\Psi_h^{n-1}) \nabla \xi^n\|_2^2 \right) .
\end{aligned}
\tag{4.4}
$$

Now adding the estimates for the terms J_{11}, J_{12}, J_2, J_3, using the mesh compatibility relation $\Delta t = h$ and hiding the terms involving $c|||\xi^n|||_{\Psi_h^{n-1}}^2$, $c\|\xi^n\|_2^2$, and the second term on the right hand side of the estimates for $|J_{11}|$ and (4.3), and $\frac{1}{2} \left(\|\xi^n\|_2^2 + h^2 \|G_h(\Psi_h^{n-1}) \nabla \xi^n\|_2^2 \right)$ from the right hand side of (4.4), in the triple norm on the left, we end up with the bound

$$
\begin{aligned}
\left(\tfrac{1}{2} - c'h \right) |||\xi^n|||_{\Psi_h^{n-1}}^2 & \le C_{11}(\Delta t) h^{2k+2} \left(\|\varphi^{n-1}\|_{k+2,\infty}^2 + |\log h|^{\delta_{k1}} \|\rho^{n-1}\|_{k,\infty}^2 \right) \\
& \quad + C' h^{-1} \|\eta^n\|_2^2 + C' |||\eta^n|||_{\Psi_h^{n-1}}^2 \\
& \quad + \tfrac{1}{2} \|\eta^{n-1}\|_2^2 + \tfrac{1}{2} \|\xi^{n-1}\|_2^2 + C_2 \Delta t \|\Theta^n\|_2^2 \\
& \le \tilde{C} h^{2k+1} + \tfrac{1}{2} \|\xi^{n-1}\|_2^2 + \tilde{C}_2 \Delta t \|\Theta^n\|_2^2 ,
\end{aligned}
\tag{4.5}
$$

where in the last step we used the first and second interpolation error estimates in Eq. 4.1, the fact that the second term on the right hand side is dominating the first one, and $c' = 1 + \left(C_1 + \frac{1}{4C_{11}}\right) + c$. Hence, multiplying both sides of (4.5) by 2 and dividing by $1 - 2c'h$, the proof is complete. □

Corollary 4.1 *Under the assumptions of Theorem 4.1, there exist the constants C_1 and C_2, depending on T, such that*

$$|||f^N - f_h^N|||_{\Psi_h^{N-1}} \leq C_1 h^k + C_2 \max_{1\leq n\leq N} \|\Theta^n\|_2.$$

Proof From Theorem 4.1 we get for $n = 1, 2, \ldots, N$,

$$|||\xi^n|||^2_{\Psi_h^{n-1}} \leq \frac{1}{1-ch}\left(C_1 h^{2k+1} + C_2 \Delta t\|\Theta^n\|_2^2\right) + \frac{1}{1-ch}|||\xi^{n-1}|||^2_{\Psi_h^{n-2}}. \tag{4.6}$$

For the clarity of further estimations, we simplify the notation using $\mu_n := C_1 h^{2k+1} + C_2\Delta t\|\Theta^n\|_2^2$ and $e_n := |||\xi^n|||^2_{\Psi_h^{n-1}}$. Hence the inequality (4.6) takes the form

$$e_n \leq \frac{1}{1-ch}\mu_n + \frac{1}{1-ch}e_{n-1}.$$

For sufficiently small h we have $ch \leq \frac{1}{2}$, which gives $(1-ch)^{-1} \leq e^{2ch}$. Thus

$$e_N \leq \frac{1}{1-ch}e_{N-1} + \frac{1}{1-ch}\mu_N \leq e_{N-1}e^{2ch} + \mu_N e^{2ch}. \tag{4.7}$$

Iterating (4.7) and using the fact that, by Theorem 4.1, $e_1 \leq \mu_1 e^{2ch} + \tilde{e}_0$, with $\tilde{e}_0 = \|\xi^0\|_2^2 = 0$, we get

$$\begin{aligned} e_N &\leq e^{4ch}\left(e_{N-2} + \mu_{N-1}\right) + \mu_N e^{2ch} \\ &\leq e^{6ch}e_{N-3} + e^{6ch}\mu_{N-2} + e^{4ch}\mu_{N-1} + e^{2ch}\mu_N \leq \cdots \leq \\ &\leq e^{2cNh}\tilde{e}_0 + \sum_{n=1}^N e^{2c(N-n+1)h}\mu_n = \sum_{n=1}^N e^{2c(N-n+1)h}\mu_n. \end{aligned} \tag{4.8}$$

Denoting $\tau_n := t_N - t_{n-1} = (N-n+1)h$, we obviously have $\tau_n = \tau_{n+1} + h$. Hence, since $ch \leq 1/2$, we have $2c\tau_n \leq 2c\tau_{n+1} + 1$. Further for $\tau_{n+1} \leq \tau \leq \tau_n$, we can write

$$e^{2c\tau_n}h = \int_{\tau_{n+1}}^{\tau_n} e^{2c\tau_n}d\tau \leq \int_{\tau_{n+1}}^{\tau_n} e^{(2c\tau_{n+1}+1)}d\tau \leq e\int_{\tau_{n+1}}^{\tau_n} e^{2c\tau}d\tau.$$

Summing the above inequality over n we get

$$\begin{aligned} \sum_{n=1}^N e^{2c\tau_n} &\leq \frac{e}{h}\left(\sum_{n=1}^N \int_{\tau_{n+1}}^{\tau_n} e^{2c\tau}d\tau\right) = \frac{e}{h}\int_{\tau_{N+1}}^{\tau_1} e^{2c\tau}d\tau = \frac{e}{h}\int_0^{t_N} e^{2c\tau}d\tau \\ &= \frac{e}{2ch}\left(e^{2ct_N} - 1\right). \end{aligned}$$

Now, using the above estimate and (4.8), we have that

$$e_N \leq \left(\sum_{n=1}^N e^{2c\tau_n}\right)\max_{1\leq n\leq N}\mu_n \leq \frac{e}{2ch}\left(e^{2ct_N} - 1\right)\max_{1\leq n\leq N}\mu_n.$$

Hence, for each n and for *sufficiently small h* we have the error bound

$$|||\xi^N|||^2_{\Psi_h^{N-1}} \leq C_0 h^{2k} + \tilde{C}_2 \max_{1\leq n\leq N} \|\Theta^n\|_2^2.$$

Now recalling that the second interpolation error estimate, cf (4.1), is of order h^{k+1}, the proof is complete. □

Remark 4.1 The term $\max_{1\leq n\leq N} \|\Theta^n\|_2$ tends to zero with $\Delta t \to 0$. Assuming more regularity in time, $f \in C^2([0,T], W^{k,\infty} \cap W^{k+1,2})$, we have for each n that $\|\Theta^n\|_2 \leq C\Delta t$, i.e. the optimal convergence rate for backward Euler. Using higher order approximations in time we get the optimal result as in [2].

5 Conclusions

In this paper we presented a numerical scheme for the solution of the two-dimensional Vlasov-Poisson equation based on a hybrid of three different numerical approaches consisting of i) the mixed finite element approximation for the Poisson equation combined with ii) the discontinuous Galerkin finite element method for the Vlasov equation and iii) the backward Euler scheme for time discretization. The choice of the BE is mainly for its *unconditional stability* and its compatibility with the DG method. We proved that our composite scheme possesses good stability and high accuracy properties and yields the optimal convergence rate.

In a forthcoming paper we shall study a posteriori error estimates of SD and DG methods for the fully discrete problem. We also plan to study some relativistic models of the three-dimensional problem.

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