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STREAMLINE DIFFUSION METHODS FOR THE VLASOV-POISSON EQUATION (*)

Mohammad ASADZADEH (1)

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Abstract. — *We prove error estimates for the streamline diffusion and the discontinuous Galerkin finite element methods for discretization of the Vlasov-Poisson equation.*

Résumé. — *Nous démontrons des estimations d'erreur pour la méthode de Galerkin discontinue pour la discrétisation de l'équation de Vlasov-Poisson.*

0. INTRODUCTION

In this paper we prove error estimates for the streamline diffusion and the discontinuous Galerkin finite element methods for discretization of the 1, 2 and 3 dimensional Vlasov-Poisson equation. This extends results of Johnson and Saranen for the two-dimensional incompressible Euler and Navier Stokes equations [17].

The initial value problem for the Vlasov equation reads as follows : given the initial data $f_0^\pm$, find the potential of the electric field ϕ and the densities of ions (+) and electrons (–) $f^\pm$ of a plasma such that

$$(0.1)\begin{cases} \dfrac{\partial f^\pm}{\partial t} + v \,.\, \nabla_x f^\pm + \alpha^\pm \nabla_x \phi \,.\, \nabla_v f^\pm = 0\,, & (x, v, t) \in R^n \times R^n \times R^+\,, \\ \Delta_x \phi = \beta \displaystyle\int_{R^n} (f^+(x, v, t) - f^-(x, v, t))\, dv\,, & (x, t) \in R^n \times R^+\,, \\ f^\pm(x, v, 0) = f_0^\pm(x, v)\,, & (x, v) \in R^n \times R^n\,, \\ \quad \nabla_x \phi \text{ is uniformly bounded and } \nabla_x \phi \to 0 \text{ as } |x| \to \infty\,, \end{cases}$$

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(1) Chalmers University of Technology and The University of Göteborg, Department of Mathematics, S-412 96 Göteborg, Sweden.

where $\nabla_x = (\partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_n)$, $\nabla_v = (\partial/\partial v_1, \partial/\partial v_2, \ldots, \partial/\partial v_n)$, and $.$ is the inner product in R^n, $\alpha^{\pm} = \mp e/m^{\pm}$, $\beta = -4\pi e$, where e is the unit of electric charge and $m^{\pm}$ are the masses of ions (+) and electrons (−).

If we assume that $f^+ = 0$, then (0.1) reduces to the following initial value problem for the Liouville-Newton equation

$$(0.2a)\quad \frac{\partial f}{\partial t} + v \,.\, \nabla_x f + \alpha \nabla_x \phi \,.\, \nabla_v f = 0\,,\qquad (x, v, t) \in R^n \times R^n \times [0, T] :\equiv Q_T,\ T > 0\,,$$

$$(0.2b)\quad -\Delta_x \phi = \beta \int_{R^n} f(x, v, t)\, dv\,,\qquad (x, t) \in R^n \times [0, T] :\equiv \Omega_T,\ T > 0\,,$$

$$(0.2c)\quad f(x, v, 0) = f_0(x, v)\,,\qquad (x, v) \in R^n \times R^n\,,$$

(0.2d) $\nabla_x \phi$ is uniformly bounded and $\nabla_x \phi \to 0$ as $|x| \to \infty$,

where $\beta = 4\pi\gamma m$, with γ being the gravitational constant and m the particle mass.

The Vlasov equation, in its complete form (0.1) emerging from application of many-particle theory to plasma physics, was introduced and studied by Vlasov in [23].

The existence of a unique classical solution for (0.1) for all time has been proved by Iordanskii [13] in the one-dimensional case $n = 1$, and for $n = 2$ by Ukai and Okabe [22], who also discuss existence of local in time classical solutions for $n \geqslant 3$. For the three-dimensional case existence of weak solutions for all time and existence of local in time classical solutions have been studied by Arsen'ev in [1] and [2], respectively.

The global in time results of [13] and [22] depend on Sobolev type estimates and can not be extended to higher dimensions. These methods do not need any restriction on the size of the initial data, which have only to be smooth enough. Global in time solutions for the three dimensional problem are given by Bardos and Degond [3], who consider small initial data and use the dispersive effect of the linearized equation to derive the existence of a global unique solution. A recent survey of a diffusion process approach to existence of a unique solution is given by Wollman [24].

Particle type methods have so far been the dominating numerical methods in plasma physics. These methods are known as vortex methods in fluid mechanics. For a mathematical analysis of vortex methods, we refer to [4], and surveys on particle methods can be found in [11]. Particle methods for the initial value problem (0.2) have been studied by Cottet and Raviart [6] and [7] for the one-dimensional case. Convergence of a particle in-cell method in one, two and three dimensional cases is studied by Neunzert and

Wick [20]. Long-time-scale particle simulations are considered by Denavit [8].

In this paper, we study the streamline diffusion and the discontinuous Galerkin finite element methods using piecewise polynomials of degree k, for the one, two and three-dimensional Vlasov-Poisson equation (0.2) in a domain $\Omega = \Omega_x \times R^n$, $n = 1, 2, 3$, where $\Omega_x \subset R^n$ is bounded and simply connected and f_0 is compactly supported in $\Omega_v :\equiv R^n$. Following the techniques of Johnson and Saranen in [17], we derive error estimates of order $O(h^{k+1/2})$ assuming sufficient regularity of the exact solution.

An outline of this note is as follows. In Section 1 we briefly review the existence of a unique solution for the continuous problem (0.2). In Section 2 we introduce notation and assumptions which will be used through the paper. Section 3 is devoted to the streamline diffusion method and in the concluding Section 4 we study the discontinuous Galerkin finite element method.

1. THE CONTINUOUS PROBLEM

In this section we review an analytic approach for existence of a unique classical solution for (0.2), in the large in time for $n = 1, 2$ and local in time for $n \geqslant 3$, assuming sufficiently smooth initial data f_0 with suitable decay at infinity. For a global existence theorem for $n = 3$ with small initial data we refer to [3].

We start by splitting (0.2) in two parts.

(I) The Poisson equation (0.2b) with f replaced by a given function g and the electrostatic potential ϕ satisfying (0.2d);

(II) The Vlasov equation (0.2a) with initial condition (0.2c).

By solving ϕ from (I) and replacing this ϕ in (II) we assign a function f to a given function g which we will denote by $f = \Lambda[g]$. A fixed point of the mapping Λ on a certain set S will give us a classical solution of (0.2).

Let us describe the steps (I) and (II) in more detail.

I. Given $g = g(x, v, t)$ find the solution $\phi = \phi(x, t)$ of Poisson equation

$$(1.1)\quad \begin{cases} -\Delta_x \phi = \beta \int_{R^n} g(x, v, t)\, dv\,, & (x, t) \in \Omega_T = R^n \times [0, T],\ T > 0\,, \\ \nabla_x \phi \text{ is uniformly bounded and } \nabla_x \phi \to 0, \text{ as } |x| \to \infty\,. \end{cases}$$

The solution ϕ of (1.1) is given by

$$(1.2)\qquad \phi(x, t) = \beta \int_{R^n} K(x - x') \left[\int_{R^n} g(x', v, t)\, dv \right] dx'\,,$$

where K is the fundamental solution of $-\Delta_x$ in R^n, $n \geq 2$ (the case $n = 1$ will be considered separately below),

$$K(x) = \begin{cases} \dfrac{1}{(2-n)\,\omega_n} \dfrac{1}{|x|^{n-2}}, & n \geq 3 \\ \dfrac{1}{2\pi} \log |x|, & n = 2. \end{cases} \tag{1.3}$$

Here ω_n is the surface area of the unit sphere in R^n, $(n \geq 3)$. With this ϕ we then solve the following initial value problem.

II. Given ϕ and f_0 find the solution $f = f(x, v, t)$ of Vlasov equation

$$\begin{cases} \dfrac{\partial f}{\partial t} + v \cdot \nabla_x f + \alpha \nabla_x \phi \cdot \nabla_v f = 0, & (x, v, t) \in Q_T = \\ & = R^n \times R^n \times [0, T],\ T > 0, \\ f(x, v, 0) = f_0(x, v), & (x, v) \in R^n \times R^n. \end{cases} \tag{1.4}$$

The problem II is equivalent to solving the following Hamiltonian system (characteristic equations of (1.4))

$$\begin{cases} \dfrac{dX(s)}{ds} = V(s), \\ \dfrac{dV(s)}{ds} = \alpha(\nabla_x \phi)(s, X), \end{cases} \tag{1.5}$$

for $(X, V) \in R^n \times R^n$. For ϕ sufficiently smooth (see assumptions on $E = \nabla_x \phi$ in [3]) and with the Cauchy data

$$\begin{cases} X(t\,; x, v, t) = x, \\ V(t\,; x, v, t) = v, \end{cases} \tag{1.6}$$

(1.5) has a unique solution which we shall denote by

$$s \to (X(s\,; x, v, t), V(s\,; x, v, t)).$$

The solution of (1.4) is then given by

$$f(x, v, t) = f_0(X(0\,; x, v, t), V(0\,; x, v, t)). \tag{1.7}$$

It remains to construct a set S of functions g in such a way that the map Λ defined on this set can be shown to have a fixed point $f = \Lambda f$, $f \in S$. For this purpose we define for $0 \leq \sigma \leq 1$ the following class of functions

$$B^{\ell+\sigma}(A) = \{g \in C_b(A) : D^{\alpha}_{|\alpha| \leq \ell}\, g \in C_b(A) \text{ and } D^{\alpha}_{|\alpha| = \ell}\, g \in C^{\sigma}_{uH}(A)\},$$

where $\ell \in Z^+$, $C_b(A)$ is the class of continuous and bounded functions in A and $C^{\sigma}_{uH}(A)$ is the set of uniformly Hölder continuous functions in A of

order σ. $B^{\ell+\sigma}(A)$ is a Banach space with the obvious $\|\cdot\|_{B^{\ell+\sigma}}$ norm. Now let $S \subset B^0(Q_T)$ be a set consisting of all functions $g = g(x, v, t)$ which satisfy the following conditions

$$\begin{array}{lll} \text{(i)} & g \in B^{\delta}(Q_T), & \delta \in (0, 1), \\ \text{(ii)} & |g(x, v, t)| \leq M_1(1 + |x|)^{-\gamma}(1 + |v|)^{-\gamma}, & \\ & & (x, v, t) \in Q_T,\ \gamma > n, \\ \text{(iii)} & \displaystyle\int_{R^n \times R^n} |g(x, v, t)|\, dx\, dv \leq M_2, & t \in [0, T], \\ \text{(iv)} & \displaystyle\int_{R^n} |g(x, v, t)|\, dv \leq M_0(t), & (x, t) \in \Omega_T, \end{array} \tag{1.8}$$

where γ, M_1 and M_2 are positive constants, and $M_0(t)$ is a positive nondecreasing function of t on $[0, T]$. Then by Propositions 3.1 and 7.1 of [22]:

1) S is a compact convex subset of $B^0(Q_T)$,

2) Λ maps S into itself continuously in the topology of $B^0(Q_T)$.

Thus by Schauder's fixed point theorem Λ has a fixed point f in S, see Dugundji [9, p. 415].

On the other hand Propositions 4.1 and 6.1 of [22] guarantee that any fixed point of Λ in S gives a classical solution of (0.2), provided that f_0 satisfies the condition (1.9) below

$$\begin{array}{ll} \text{(i)} & f_0 \in B^1(R^n \times R^n), \\ \text{(ii)} & |f_0(x, v)| \leq k_0(1 + |x|)^{-2\gamma}(1 + |v|)^{-2\gamma},\ \gamma > n,\ k_0 \geq 0. \end{array} \tag{1.9}$$

For uniqueness results we refer to [6], [22] and [3] in one, two and three dimensions respectively.

Remark 1.1 : For the case $n = 1$ assuming a periodicity on x, the Poisson equation (1.1) becomes

$$\begin{cases} -\dfrac{\partial^2 \phi}{\partial x^2} = 1 - \displaystyle\int_{-\infty}^{\infty} f(x, v, t)\, dv, \\ \phi(0, t) = \phi(p, t),\ t \geq 0, \end{cases} \tag{1.1$'$}$$

and the kernel K being the Green function

$$K(x - x') = \begin{cases} x\left(1 - \dfrac{x'}{p}\right), & 0 \leq x \leq x', \\ \left(1 - \dfrac{x}{p}\right) x', & x' \leq x \leq p, \end{cases} \tag{1.3$'$}$$

where p is the period, i.e. $f(0, v, t) = f(p, v, t)$. For more details in the one-dimensional case we refer to [6], [7] and Iordanskii [13].

2. NOTATION AND ASSUMPTIONS

We assume now that $(x, v) \in \Omega = \Omega_x \times \Omega_v \subset R^n \times R^n$, $n = 1, 2, 3$, where $\Omega_v :\equiv R^n$ and Ω_x is a bounded simply connected domain. We further assume that

$$f_0 \text{ is compactly supported in } \Omega_v = R^n . \tag{2.1}$$

We shall consider the following variant of the initial value problem (0.2) : given $T > 0$ find (f, ϕ) such that

$$\begin{cases} \dfrac{\partial f}{\partial t} + v \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f = 0 , & (x, v, t) \in \Omega \times [0, T] :\equiv Q_T , \\ f(x, v, 0) = f_0(x, v) , & (x, v) \in \Omega = \Omega_x \times R^n , \\ f(x, v, t) = 0 , & (x, v, t) \in \Gamma_v^- \times R^n \times [0, T] , \end{cases} \tag{2.2}$$

with ϕ satisfying

$$\begin{cases} - \Delta_x \phi = \displaystyle\int_{R^n} f(x, v, t)\, dv , & (x, t) \in R^n \times [0, T] :\equiv \Omega_T , \\ \nabla_x \phi \text{ is uniformly bounded and } \nabla_x \phi \to 0, \text{ as } |x| \to \infty , \end{cases} \tag{2.3}$$

and for $v \in R^n$,

$$\Gamma_v^- = \{ x \in \partial\Omega_x : n_x(x) \cdot v < 0 \} ,$$

where $n_x(x)$ is the outward unit normal to $\partial\Omega_x$ at the point $x \in \partial\Omega_x$. We assume that a solution f of (2.2) exists on the time interval $[0, T]$.

Observe that Poisson equation (2.3) is considered in the whole space R^n ($x \in R^n$). Thus we may first solve ϕ as in Section 1 and then take the restriction of this ϕ to $\Omega_x \times [0, T]$ in (2.2).

Remark 2.1 : The condition (2.1) implies that for $T > 0$ there is a constant C such that $f(x, v, t) = 0$ for $|v| \geq C$, $x \in \Omega_x$, $t \in [0, T]$. For the analysis of the original problem (1.4) if we assume that f is compactly supported in $R^n \times R^n$, then taking Ω large enough, both Ω_x and Ω_v can be assumed to be bounded and zero boundary condition may be imposed, see [3]. The analysis of this case is included in our case below if all boundary integrals are dropped. □

Introducing the notation

$$\nabla f :\equiv (\nabla_x f, \nabla_v f) = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}, \frac{\partial f}{\partial v_1}, \dots, \frac{\partial f}{\partial v_n} \right), \; n = 1, 2, 3$$

$$G(f) :\equiv (v, -\nabla_x \phi) = \left(v_1, \dots, v_n, -\frac{\partial \phi}{\partial x_1}, \dots, -\frac{\partial \phi}{\partial x_n} \right) = (G_1, \dots, G_{2n}) ,$$

(2.2) can be rewritten as

$$\begin{cases} \dfrac{\partial f}{\partial t} + G(f) \cdot \nabla f = 0 , & \text{in } \Omega \times I = Q_T , \\ f(x, v, 0) = f_0(x, v) , & \text{in } \Omega , \\ f(x, v, t) = 0 , & \text{on } \Gamma_v^- \times \Omega_v \times I , \end{cases} \tag{2.4}$$

where $I :\equiv [0, T]$, $G(f) = (v, -\nabla_x \phi)$ and ϕ satisfies (2.3). Note that

$$\operatorname{div} G(f) = \sum_{i=1}^{n} \frac{\partial G_i}{\partial x_i} + \sum_{i=n+1}^{2n} \frac{\partial G_i}{\partial v_{i-n}} = 0, \; n = 1, 2, 3 . \tag{2.5}$$

We now introduce a finite element structure on $\Omega_x \times \Omega_v$. Let $T_h^x = \{\tau_x\}$ and $T_h^v = \{\tau_v\}$ be finite element subdivisions of Ω_x with elements τ_x and $\Omega_v (= R^n)$ with elements τ_v, respectively. Then $T_h = T_h^x \times T_h^v = \{\tau_x \times \tau_v\} = \{\tau\}$ will be a subdivision of $\Omega = \Omega_x \times \Omega_v$ with $\tau = \tau_x \times \tau_v$ as elements. Moreover we let $0 = t_0 < t_1 < \cdots < t_M = T$ be a subdivision of the time interval $I = [0, T]$ into sub-intervals $I_m = (t_m, t_{m+1})$, $m = 0, 1, \dots, M-1$. Further let $\mathscr{C}_h$ be the corresponding subdivision of $Q_T = \Omega \times [0, T]$ into elements $K = \tau \times I_m$, with $h = \operatorname{diam} K$ as the mesh parameter and $P_k(K) = P_k(\tau_x) \times P_k(\tau_v) \times P_k(I_m)$ the set of polynomials in x, v and t of degree at most k on K.

Given a domain Q we denote by $(\cdot\,,\cdot)_Q$ the usual $L_2(Q)$ scalar product and $\|\cdot\|_Q$ the corresponding norm. $H^s(Q)$, for s a positive integer, will denote the usual Sobolev space with norm $\|\cdot\|_{s,Q}$. Further for piecewise polynomials w_i defined on the triangulation $\mathscr{C}_h' = \{K\}$ where $\mathscr{C}_h' \subset \mathscr{C}_h$ and for D_i some differential operators, we use the notation

$$(D_1 w_1, D_2 w_2)_{Q'} = \sum_{K \in \mathscr{C}_h'} (D_1 w_1, D_2 w_2)_K, \; Q' = \bigcup_{K \in \mathscr{C}_h'} K .$$

Finally, C denotes a positive constant subject to change without notice.

3. THE STREAMLINE DIFFUSION METHOD

3.1. Stability

The streamline diffusion method is a finite element method for convection dominated convection-diffusion problems which (i) is higher order accurate and (ii) has good stability properties. The method was introduced by Hughes and Brooks [12] in the case of stationary problems. The mathematical analysis of this method was begun in Johnson [14] and Johnson and Nävert [15], and was continued in Johnson, Nävert and Pitkäranta [16] and Nävert [21], where also the method was extended to time dependent problems. SD (streamline diffusion)-method for two-dimensional time-dependent incompressible Euler and Navier-Stokes equations are studied in Johnson and Saranen [17]. Computational results for the cases considered in [17] are given in Hansbo [10]. Applications of the SD-method to Burgers' equation together with computational results are given in Johnson and Szepessy [18].

In this section we consider the SD-method for the Vlasov-Poisson equation (2.4), with the trial functions being continuous in the x and v variables. Since f_0 has compact support in $\Omega_v = R^n$ we have $f(x, v, t) = 0$ for v large and thus the analysis can be restricted to a bounded domain Ω_v^h with all SD-test functions vanishing on $\partial\Omega_v^h$. We shall also use the following notation : for $k = 0, 1, 2, \ldots$, let

$$V_h = \{g \in \mathscr{H}_0 : g|_K \in P_k(\tau) \times P_k(I_m) ; \forall K = \tau \times I_m \in \mathscr{C}_h\} ,$$

where

$$\mathscr{H}_0 = \prod_{m=0}^{M-1} H_0^1(S_m) , \quad S_m = \Omega \times I_m , \quad m = 0, 1, \ldots, M-1 .$$

and

$$H_0^1 = \{g \in H^1 : g \equiv 0 \text{ on } \partial\Omega_v^h\} .$$

Further we write

$$(f, g)_m = (f, g)_{S_m} , \quad \|g\|_m = (g, g)_m^{1/2} ,$$

and

$$\langle f, g \rangle_m = (f(\cdot, \cdot, t_m), g(\cdot, \cdot, t_m))_\Omega , \quad |g|_m = \langle g, g \rangle_m^{1/2} .$$

Also

$$[g] = g_+ - g_- ,$$

where

$$g_\pm = \lim_{s \to 0\pm} g(x, v, t+s) , \quad \text{for} \quad (x, v) \in \text{Int}\, \Omega_x \times \Omega_v^h , \ t \in I ,$$

$$g_\pm = \lim_{s \to 0\pm} g(x + sv, v, t+s) , \quad \text{for} \quad (x, v) \in \partial\Omega_x \times \Omega_v^h , \ t \in I ,$$

and

$$\langle f_+, g_+ \rangle_{\Gamma^-} = \int_{\Gamma^-} f_+ \, g_+ \, G^h . n \, d\sigma ,$$

$$\langle f_+, g_+ \rangle_{\Gamma_m^-} = \int_{I_m} \langle f_+, g_+ \rangle_{\Gamma^-} \, ds ,$$

$$\langle f_+, g_+ \rangle_{\Gamma_I^-} = \int_I \langle f_+, g_+ \rangle_{\Gamma^-} \, ds ,$$

with $G^h \equiv G(f^h)$ defined in (3.1) below and

$$\Gamma_- = \{(x, v) \in \Gamma = \partial(\Omega_x \times \Omega_v^h) : G^h . n < 0\} ,$$

where $n = (n_x, n_v)$ with n_x and n_v being outward unit normals to $\partial\Omega_x$ and $\partial\Omega_v^h$ respectively. Finally in this section $\Omega = \Omega_x \times \Omega_v^h$.

The streamline diffusion method for (2.4) can now be formulated as follows : find $f^h \in V_h$ such that for $m = 0, 1, \ldots, M-1$.

$$(3.1) \quad (f_t^h + G(f^h) . \nabla f^h, g + h(g_t + G(f^h) . \nabla g))_m + \\ + \langle f_+^h, g_+ \rangle_m - \langle f_+^h, g_+ \rangle_{\Gamma_m^-} = \langle f_-^h, g_+ \rangle_m , \quad \forall g \in V_h ,$$

where $G(f^h) = (v, -\nabla_x \phi^h)$ and ϕ^h satisfies the Poisson equation (2.3) with f replaced by f^h, i.e.,

$$(3.2) \quad \begin{cases} -\Delta_x \phi^h = \int_{R^n} f^h(x, v, t) \, dv , & (x, t) \in R^n \times I \equiv \Omega_T , \\ \nabla_x \phi^h \text{ is uniformly bounded and } \nabla_x \phi^h \to 0 \text{ as } |x| \to \infty , \end{cases}$$

and $f_-^h(x, v, 0) = f_0(x, v)$.

We introduce the notation

$$(3.3) \quad B(G(\tilde{f}) ; f, g) = \sum_{m=0}^{M-1} [(f_t + G(\tilde{f}) . \nabla f, \ g + h(g_t + G(f^h) . \nabla g))_m \\ - \langle f_+, g_+ \rangle_{\Gamma_m^-}] + \sum_{m=1}^{M-1} \langle [f], g_+ \rangle_m + \langle f_+, g_+ \rangle_0 ,$$

$$L(g) = \langle f_0, g_+ \rangle_0 .$$

Observe the f^h dependence of B in the first term on the right hand side of (3.3). The problem (3.1) can now be more concisely formulated as follows : find $f^h \in V_h$ such that

$$B(G(f^h)\,;f^h, g) = L(g)\,, \quad \forall g \in V_h\,, \tag{3.4}$$

where $G(f^h) = (v, -\nabla_x \phi^h)$ and ϕ^h satisfies (3.2).

We shall use a stability estimate for (3.4) in a norm $|||\,.\,|||$ defined by

$$|||g|||^2 = \frac{1}{2}\left[|g|_M^2 + |g|_0^2 + \sum_{m=1}^{M-1} |[g]|_m^2 + 2h\|g_t + G(f^h)\,.\,\nabla g\|_{Q_T}^2 + \int_{\partial\Omega\times I} g^2 |G^h\,.\,n|\, d\sigma\, ds\right].$$

LEMMA 3.1 : *We have*

$$B(G(f^h)\,;g, g) = |||g|||^2\,, \quad \forall g \in \mathscr{H}_0\,.$$

Proof: Using the definition of B we have

$$\begin{aligned} B(G(f^h)\,;g,g) = {} & (g_t, g)_{Q_T} + (G(f^h)\,.\,\nabla g, g)_{Q_T} + \\ & + h\|g_t + G(f^h)\,.\,\nabla g\|_{Q_T}^2 \\ & + \sum_{m=1}^{M-1} \langle [g], g_+\rangle_m + \langle g_+, g_+\rangle_0 - \langle g_+, g_+\rangle_{\Gamma_I^-}\,. \end{aligned} \tag{3.5}$$

Integrating by parts we get

$$\begin{aligned} (g_t, g)_{Q_T} + \sum_{m=1}^{M-1} \langle [g], g_+\rangle_m + \langle g_+, g_+\rangle_0 = \\ = \frac{1}{2}\left[|g|_M^2 + |g|_0^2 + \sum_{m=1}^{M-1} |[g]|_m^2\right]. \end{aligned} \tag{3.6}$$

Using Green's formula

$$\begin{aligned} (G(f^h)\,.\,\nabla g, g)_\Omega = {} & \int_{\partial\Omega} (G(f^h)\,.\,n)\, g^2\, d\sigma - \\ & - (G(f^h)\,.\,\nabla g, g)_\Omega + (\operatorname{div} G(f^h)\, g, g)_\Omega\,, \end{aligned}$$

and recalling (2.5) we have

$$\begin{aligned} (G(f^h)\,.\,\nabla g, g)_\Omega - \langle g_+, g_+\rangle_{\Gamma^-} & = \\ = \frac{1}{2}\int_{\partial\Omega} g^2 (G(f^h)\,.\,n)\, d\sigma & - \int_{\Gamma^-} g^2(G(f^h)\,.\,n)\, d\sigma \\ = \frac{1}{2}\int_{\partial\Omega} g^2 |G(f^h)\,.\,n|\, d\sigma\,. & \end{aligned} \tag{3.7}$$

Now the proof follows by (3.5)-(3.7). □

LEMMA 3.2 : *For any constant* $C_1 > 0$, *we have for* $g \in \mathscr{H}_0$,

$$\|g\|_{Q_T}^2 \leqslant \left[\frac{1}{C_1}\|g_t + G(f^h)\cdot\nabla g\|_{Q_T}^2 + \sum_{m=1}^{M} |g_-|_m^2 + \int_{\partial\Omega\times I} g^2|G^h\,.\,n|\, d\sigma\, ds\right] he^{C_1 h}\,.$$

Proof: For $t_m < t < t_{m+1}$, we have using (3.7)

$$\begin{aligned} \|g(t)\|_\Omega^2 & = |g_-|_{m+1}^2 - \int_t^{t_{m+1}} \frac{d}{dt}\|g(s)\|_\Omega^2\, ds \\ & = |g_-|_{m+1}^2 - 2\int_t^{t_{m+1}} \left[(g_t + G(f^h)\,.\,\nabla g, g)_\Omega - \frac{1}{2}\int_{\partial\Omega} g^2 |G^h\,.\,n|\, d\sigma - \langle g_+, g_+\rangle_{\Gamma^-}\right] ds \\ & \leqslant |g_-|_{m+1}^2 + \frac{1}{C_1}\|g_t + G(f^h)\,.\,\nabla g\|_m^2 + \int_{\partial\Omega\times I_m} g^2|G^h\,.\,n|\, d\sigma\, ds + C_1\int_t^{t_{m+1}} \|g(s)\|_\Omega^2\, ds\,. \end{aligned}$$

Thus by Grönwall's inequality for $t_m < t < t_{m+1}$,

$$\|g(t)\|_\Omega^2 \leqslant \left[\frac{1}{C_1}\|g_t + G(f^h)\,.\,\nabla g\|_m^2 + |g_-|_{m+1}^2 + \int_{\partial\Omega\times I_m} g^2|G^h\,.\,n|\, d\sigma\, ds\right] e^{C_1 h}\,. \tag{3.8}$$

Integrating over $t_m < t < t_{m+1}$ and summing over $m = 0, \ldots, M-1$, we obtain the desired result. □

LEMMA 3.3 (EXISTENCE THEOREM) : *For any* $h > 0$ *the problem* (3.4) *admits at least one solution.*

The proof is similar to that of Lemma 2.4 in [17], where a Brouwer's type fixed point theorem as in [19] is used. □

3.2. Error estimates

Let $\tilde{f}^h \in V_h$ be an interpolant of the exact solution f and set $\eta = f - \tilde{f}^h$ and $\xi = f^h - \tilde{f}^h$. Then we have

$$e :\equiv f - f^h = (f - \tilde{f}^h) - (f^h - \tilde{f}^h) = \eta - \xi\,.$$

Our main result is

THEOREM 3.1 : *If $f^h \in V_h$ satisfies* (3.4) *and the exact solution f of* (2.4) *satisfies*

(3.9) $$\|\nabla f\|_\infty + \|G(f)\|_\infty + \|\nabla\eta\|_\infty + \|f\|_{k+1,Q_T} \leq C ,$$

then there is a constant C such that

$$|||f - f^h||| \leq Ch^{k+\frac{1}{2}} .$$

Proof: Since f satisfies (2.4) we have for $g \in V_h$,

$$B(G(f); f, g) = L(g) ,$$

so that by (3.4) and Lemma 3.1

(3.10) $$|||\xi|||^2 = B(G(f^h); f^h - \tilde{f}^h, \xi) = L(\xi) - B(G(f^h); \tilde{f}^h, \xi) = \\ = B(G(f); f, \xi) - B(G(f^h); f - \eta, \xi) = T_1 + T_2 ,$$

where

$$T_1 := B(G(f^h); \eta, \xi)$$
$$T_2 := B(G(f); f, \xi) - B(G(f^h); f, \xi) .$$

We now estimate T_1 and T_2 separately. Integrating by parts, using (2.5) and the same argument as in the proof of Lemma 3.1 we find that

(3.11) $$|T_1| = \Big| - (\eta, \xi_t + G(f^h) . \nabla\xi)_{Q_T} + \langle \eta_-, \xi_- \rangle_M - \sum_{m=1}^{M-1} \langle \eta_-, [\xi] \rangle_m + \\ + \int_{\partial\Omega \times I} \eta\xi |G^h . n| \, d\sigma \, ds + h(\eta_t + G(f^h) . \nabla\eta, \xi_t + G(f^h) . \nabla\xi)_{Q_T} \Big| \\ \leq \frac{1}{8} |||\xi|||^2 + C \Big[\int_{\partial\Omega \times I} \eta^2 |G^h . n| \, d\sigma \, ds + h^{-1} \|\eta\|^2_{Q_T} + \sum_{m=1}^{M} |\eta_-|^2_m \\ + h \|\eta_t + G(f^h) . \nabla\eta\|^2_{Q_T} \Big],$$

where we have also used the fact that f and f^h and consequently η and ξ have compact support in Ω_v^h. Moreover since Ω_x is bounded there exists $r > 0$, such that $\Omega_x \subset \{x \in R^n : \ |x| < r\}$. Using (3.2) with $\int_{\Omega_v^h} f(x, v, t) \, dv = \rho(x, t)$ and (1.2) for $x \in \Omega_x$, we have

$$\nabla_x \phi(x, t) = C \int_{\Omega_x} \frac{(x - y)}{|x - y|^n} \rho(y, t) \, dy := CK' * \rho ,$$

where

$$K'(z) = \frac{z}{|z|^n} , \quad z \in D \subset \{x \in R^n, |x| < 2r\} .$$

Using Young's inequality

$$\|\nabla_x \phi\|_{\Omega_T} = \|\nabla_x \phi\|_{L_2(\Omega_T)} \leq C \|K'\|_{L_1(D)} \|\rho\|_{L_2(\Omega_T)} ,$$

we obtain

$$\|\nabla_x \phi\|_{\Omega_T} \leq C \|\rho\|_{\Omega_T} \leq C \|f\|_{Q_T} .$$

Now, since f and f^h are compactly supported in $R^n \times R^n$, (Ω_v^h and $\bar{\Omega}_x$ are bounded) and

$$G(f^h) - G(f) = -(0, 0, \dots, 0, \nabla_x(\phi^h - \phi)) ,$$

we have

(3.12) $$\|G(f^h) - G(f)\|_{\Omega_T} \leq C \|f - f^h\|_{Q_T} \leq C(\|\xi\|_{Q_T} + \|\eta\|_{Q_T}) ,$$

and consequently

(3.13) $$\|\eta_t + G(f^h) . \nabla\eta\|_{Q_T} \leq \\ \leq \|\eta_t + G(f) . \nabla\eta\|_{Q_T} + \|(G(f^h) - G(f)) . \nabla\eta\|_{Q_T} \\ \leq \|\eta_t\|_{Q_T} + \|G(f)\|_\infty \|\nabla\eta\|_{Q_T} + C \|\nabla\eta\|_\infty (\|\xi\|_{Q_T} + \|\eta\|_{Q_T}) .$$

Next

$$T_2 = ((G(f) - G(f^h)) . \nabla f, \xi)_{Q_T} + \\ + h((G(f) - G(f^h)) . \nabla f, \xi_t + G(f^h) . \nabla\xi)_{Q_T}$$

so that by (3.12),

(3.14) $$|T_2| \leq C(\|\xi\|_{Q_T} + \|\eta\|_{Q_T}) \|\nabla f\|_\infty \|\xi\|_{Q_T} + \\ + Ch(\|\xi\|_{Q_T} + \|\eta\|_{Q_T})^2 \|\nabla f\|^2_\infty + Ch \|\xi_t + G(f^h) . \nabla\xi\|^2_{Q_T} .$$

Estimating $\|\xi\|^2_{Q_T}$ from Lemma 3.2 and hiding the terms as $Ch\|\xi_t + G(f^h) . \nabla\xi\|^2_{Q_T}$, in (3.14) in $|||\xi|||^2$, a combination of (3.9)-(3.11), (3.13), (3.14) and

Lemma 3.2 with C_1 large enough gives

$$\|\!|\xi\|\!|^2 \leq C\left[\int\int_{\partial\Omega \times I} \eta^2 |G^h . n| \, d\sigma \, ds + h^{-1}\|\eta\|_{Q_T}^2 + \right.$$
$$\left. + \sum_{m=1}^{M} |\eta_-|_m^2 + h\|\eta\|_{1,Q_T}^2 + \sum_{m=1}^{M} |\xi_-|_m^2 h\right].$$

Finally, by standard interpolation theory we have (see e.g. Ciarlet [6], p. 123).

$$\left[h\int_{\partial\Omega \times I} \eta^2 |G^h . n| \, d\sigma \, ds + \|\eta\|_{Q_T}^2 + h\sum_{m=1}^{M} |\eta_-|_m^2 + h^2\|\eta\|_{1,Q_T}^2\right]^{1/2} \leq$$
$$\leq C h^{k+1} \|f\|_{k+1,Q_T}.$$

Thus by (3.9)

$$\|\!|\xi\|\!|^2 \leq C h^{2k+1} + C_1 \sum_{m=1}^{M} |\xi_-|_m^2 h. \tag{3.15}$$

We shall now use the following discrete Grönwall's estimate. If

$$y(\,.\,, t_m) \leq C + C_1 \sum_{j \leq m} |y(\,.\,, t_j)| \, h, \tag{3.16}$$

then

$$y(t_m) \leq C e^{C_1 t} \leq C e^{C_1 T}.$$

This is an analogue of the following continuous Grönwall's estimate. If

$$y(t) \leq C + C_1 \int_0^t y(s) \, ds,$$

then

$$y(t) \leq C e^{C_1 t}.$$

Obviously (3.15) implies that

$$|\xi_-|_m^2 \leq C h^{2k+1} + C_1 \sum_{m=1}^{M} |\xi_-|_m^2 h,$$

so that using (3.16)

$$|\xi_-|_m^2 \leq C h^{2k+1} e^{C_1 T}. \tag{3.17}$$

By (3.15) and (3.17)

$$\|\!|\xi\|\!|^2 \leq C h^{2k+1} + C_1 \sum_{m=1}^{M} (C h^{2k+1} e^{C_1 T}) h \leq C(T) h^{2k+1}$$

where

$$C(T) = C e^{C_1 T}.$$

Recalling that the interpolation error is of the order $h^{k+1/2}$ the proof of Theorem 3.1 is complete. □

A uniqueness result for the Vlasov-Poisson equation is obtained in a similar way as for the Euler equation in [17]. □

4. DISCONTINUOUS GALERKIN

In this section we use trial functions which may be discontinuous across interelement boundaries also in the space and velocity variables.

To define a finite element method using discontinuous trial functions we introduce the following notation : if $\beta = (\beta_1, \beta_2, \dots, \beta_{2n})$, $n = 1, 2, 3$ is a given smooth vector field on Q_T we define for $K \in \mathscr{C}_h$

$$\partial K_{\pm}(\beta) = \{(x, v, t) \in \partial K : n_t(x, v, t) + n(x, v, t) . \beta(x, v, t) \lesseqgtr 0\} \tag{4.1}$$

where $(n, n_t) = (n_x, n_v, n_t)$ denotes the outward unit normal to $\partial K \subset Q_T$. We also introduce for $k \geq 0$,

$$W_h = \{g \in L_2(Q_T) : g|_K \in P_k(K), \ \forall K \in \mathscr{C}_h\}.$$

The discontinuous Galerkin finite element method for (2.4) can now be formulated as follows : find $f^h \in W_h$ such that

$$(f_t^h + \beta . \nabla f^h, g + h(g_t + \beta . \nabla g))_{Q_T} + \tag{4.2}$$
$$+ \sum_{K \in \mathscr{C}_h} \int_{\partial K_-(\beta)} [f^h] \, g_+ |n_t + n . \beta| \, d\sigma = 0,$$
$$\forall g \in W_h$$

where $\beta = G(f^h) = (v, -\nabla_x \phi^h)$, with ϕ^h satisfying (3.2), $f_-^h(x, v, 0) = f_0(x, v)$ and $[g] = g_+ - g_-$, with $g_{\pm} = \lim_{s \to 0\pm} g((x, v) + G(f^h) s, t + s)$.

Recall that since β is divergence free, $\beta . n = G(f^h) . n$ is continuous across the interelement boundaries of $\mathscr{C}_h$ and thus $\partial K_{\pm}(\beta)$ is well defined.

To write (4.2) on more compact form we introduce the notation

$$B(G; f, g) = (f_t + G \cdot \nabla f, g + h(g_t + \beta \cdot \nabla g))_{Q_T} + \sum_K \int_{\partial K_-(\beta)'} [f] g_+ |n_t + n \cdot \beta| \, d\sigma + \langle f_+, g_+ \rangle_0 \tag{4.3}$$

and

$$L(g) = \langle f_0, g_+ \rangle_0 ,$$

where $\partial K_-(\beta)' = \partial K_-(\beta) \backslash \Omega \times \{0\}$. Then (4.2) can be written in the following form: find $f^h \in W_h$ such that

$$B(G(f^h); f^h, g) = L(g), \quad \forall g \in W_h . \tag{4.4}$$

This method is analyzed in a similar way to the SD-method and in particular we have the following analogues of Lemmas 3.1 and 3.2.

LEMMA 4.1 : *We have with* $\beta = G(f^h)$ *and* B *defined by* (4.3)

$$B(G(f^h); g, g) = |||g|||^2 \quad \forall g \in W_h$$

where

$$|||g|||^2 = \frac{1}{2}\left[|g|_M^2 + |g|_0^2 + \sum_K \int_{\partial K_-(\beta)'} [g]^2 |n_t + n \cdot \beta| \, d\sigma + 2h \|g_t + \beta \cdot \nabla g\|_{Q_T}^2 + \int_{\partial\Omega_+ \times I} g^2 |n \cdot \beta| \, d\sigma \right]. \tag{4.5}$$

The proof is similar to that of Lemma 3.1 using here the equality

$$\sum_{K \in \mathscr{C}_h} \left[(g_t, g)_K + (\beta \cdot \nabla g, g)_K + \int_{\partial K_-(\beta)'} [g] g_+ |n_t + n \cdot \beta| \, d\sigma \right] + |g|_0^2 = \frac{1}{2}\left[|g|_M^2 + |g|_0^2 + \sum_K \int_{\partial K_-(\beta)'} [g]^2 |n_t + n \cdot \beta| \, d\sigma + \int_{\partial\Omega_+ \times I} g^2 |n \cdot \beta| \, d\sigma \right]. \quad \square$$

LEMMA 4.2 : *For any constant* $C_1 > 0$ *we have for* $\beta = G(f^h)$ *and* $g \in W_h$

$$\|g\|_{Q_T} \leqslant \left[\frac{1}{C_1} \|g_t + \beta \cdot \nabla g\|_{Q_T}^2 + \sum_{m=1}^{M} |g_-|_m^2 + \sum_K \int_{\partial K_-(\beta)''} [g]^2 |n \cdot \beta| \, d\sigma + \int_{\partial\Omega_+ \times I} g^2 |n \cdot \beta| \, d\sigma \right] h e^{C_1 h},$$

where

$$\partial K_-(\beta)'' = \{(x, v, t) \in \partial K_-(\beta)' : n_t(x, v, t) = 0\} .$$

Proof: We have for $t_m < t < t_{m+1}$, $K = \tau \times I_m$,

$$\|g(t)\|_\tau^2 = |g_-|_{m+1,\tau}^2 - \int_t^{t_{m+1}} \frac{d}{dt} \|g(s)\|_\tau^2 \, ds = |g_-|_{m+1,\tau}^2 - 2\int_t^{t_{m+1}} \left[(g_t + \beta \cdot \nabla g, g)_\tau - \int_{\partial\tau_-} g^2 n \cdot \beta \, d\sigma - \frac{1}{2}\int_{\partial\tau} g^2 |n \cdot \beta| \, d\sigma \right],$$

where $|g_-|_{m+1,\tau}$ is the obvious restriction of $|g_-|_{m+1}$ to τ. Summing over $\tau \in T$, we obtain

$$\|g(t)\|_\Omega^2 = |g_-|_{m+1}^2 - 2\int_t^{t_{m+1}} (g_t + \beta \cdot \nabla g, g)_\Omega + \sum_K \int_{\partial K_-(\beta)'' \cap \{s : t < s < t_{m+1}\}} [g]^2 |n \cdot \beta| \, d\sigma + \int_{\partial\Omega_+ \times \{s : t < s < t_{m+1}\}} g^2 |n \cdot \beta| \, d\sigma$$

$$\leqslant |g_-|_{m+1}^2 + \frac{1}{C_1} \|g_t + \beta \cdot \nabla g\|_m^2 + C_1 \int_t^{t_{m+1}} \|g(s)\|_\Omega^2 + \sum_K \int_{\partial K_-(\beta)'' \cap I_m} [g]^2 |n \cdot \beta| \, d\sigma + \int_{\partial\Omega_+ \times I_m} g^2 |n \cdot \beta| \, d\sigma .$$

Now using Grönwall's inequality we find that

$$\|g(t)\|_\Omega^2 \leqslant \left[[g_-]_{m+1}^2 + \frac{1}{C_1} \|g_t + \beta \cdot \nabla g\|_m^2 + \sum_K \int_{\partial K_-(\beta)'' \cap I_m} [g]^2 |n \cdot \beta| \, d\sigma + \int_{\partial\Omega_+ \times I_m} g^2 |n \cdot \beta| \, d\sigma \right] e^{C_1 h}.$$

Integration over I_m and summation for $m = 0, \ldots, M-1$, complete the proof. $\square$

THEOREM 4.1 : *Let* f *and* f^h *be as in Theorem* 3.1 *and* $\|f\|_{k+1,\infty} \leqslant C$, *then we have the following error estimate for the problem* (4.2),

$$|||f - f^h||| \leqslant C h^{k+\frac{1}{2}}$$

where $\|\cdot\|_{k+1,\infty}$ *denotes the* $W_\infty^{k+1}(Q_T)$*-norm.*

Proof: We have as in the proof of Theorem 3.1,

$$|||\xi|||^2 = B(G(f^h);\eta,\xi) + [B(G(f);f,\xi) - B(G(f^h);f,\xi)] := T_1 + T_2$$

where ξ and η are the same as in Section 3. Integration by parts in the term T_1 leads to appearence of a term of the form

$$T_3 = \sum_K \int_{\partial K_-(\beta)''} [\xi]\,\eta_+\,|n\,.\,\beta|\,d\sigma\,,$$

where $\beta = G(f^h)$. Using Cauchy's inequality we have for $\delta > 0$

$$|T_3| \leqslant \frac{C}{\delta}\sum_K \int_{\partial K_-(\beta)''} |\eta_+|^2\,|n\,.\,\beta|\,d\sigma + C\delta \sum_K \int_{\partial K_-(\beta)''} [\xi]^2\,|n\,.\,\beta|\,d\sigma\,.$$

Here the last sum can be hidden in $|||\xi|||^2$, and we estimate the first one as follows

$$(4.6)\quad \sum_K \int_{\partial K_-(\beta)''} |\eta_+|^2\,|n\,.\,\beta|\,d\sigma \leqslant$$
$$\leqslant \|\eta\|_\infty^2 \sum_K \left[\int_{\partial K_-(\beta)''} |n\,.\,\beta|^2\,ds + \int_{\partial K_-(\beta)''} d\sigma\right]$$
$$\leqslant C\,\|\eta\|_\infty^2 \sum_K [Ch^{-1}\|\beta\|_K^2 + Ch^{2n}]\,,\quad n = 1, 2, 3\,.$$

Here we have used the fact that

$$\int_{\partial K} |g\,.\,n|^2\,d\sigma \leqslant Ch^{-1}\int_K g^2\,dy\,,\quad \forall g \in P_k(K)\,.$$

Now by (3.12) we have

$$(4.7)\qquad \|\beta\|_{Q_T} = \|G(f^h)\|_{Q_T} \leqslant C(\|\xi\|_{Q_T} + \|\eta\|_{Q_T}) + \|G(f)\|_{Q_T}\,.$$

Moreover the interpolation error η satisfies

$$(4.8)\qquad \|\eta\|_\infty = \|f - \tilde{f}^h\|_\infty \leqslant Ch^{k+1}\|f\|_{k+1,\infty}\,.$$

Thus (4.6)-(4.8) imply that

$$|T_3| \leqslant \frac{1}{8}\,|||\xi|||^2 + Ch^{2k+2}\|f\|_{k+1,\infty}^2 \times$$
$$\times\,[h^{-1}(\|\xi\|_{Q_T}^2 + \|\eta\|_{Q_T}^2 + \|G(f)\|_{Q_T}^2) + h^{2n}]$$

and by the assumption of the theorem

$$|T_3| \leqslant Ch^{2k+1} + \frac{1}{8}\,|||\xi|||^2\,.$$

The remaining terms are estimated by similar arguments as in the proof of Theorem 3.1 and the proof is complete. □

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THÉORIE DE LA PÉNALISATION EXACTE (*)

Joseph Frédéric BONNANS (1)

Communiqué par J. CEA

Résumé. — *Depuis les travaux de Pschenichnyi et Han montrant comment leur utilisation permet de globaliser certains algorithmes, les fonctionnelles de pénalisation exacte jouent en optimisation un rôle croissant. L'objet de cet article est de rappeler les principaux résultats en améliorant certains d'entre eux. Nous étudions en particulier la théorie de la normalité et l'utilisation de conditions du deuxième ordre faibles.*

Abstract. — *Since the work of Pschenichnyi and Han concerning their use in the globalization of algorithms, exact penalty functions have in the field of optimization an increasing importance. The purpose of this paper is to review the main results and to improve some of them. We focus on the normality theory and on weak second-order sufficient conditions.*

1. INTRODUCTION

Nous considérons des problèmes d'optimisation du type

$$\text{(P)} \quad \min f(x) \, ; \quad g_i(x) \leqq 0 \, , \; i \in I \, , \; g_j(x) = 0 \, , \; j \in J \, ,$$

où I et J sont des ensembles finis, les applications f et g_i $(i \in I \cup J)$ étant de $\mathbb{R}^n$ dans $\mathbb{R}$. Étant donné $K \in I \cup J$, on posera

$$g_K(x) = \{g_i(x) \, ; i \in K\} \, .$$

La partie positive z^+ d'un vecteur z étant définie en prenant la partie positive de chaque composante, nous appellerons fonctionnelle pénalisée exacte une fonction du type

$$\theta_r(x) = f(x) + r \| (g_I(x)^+ , g_J(x)) \| \, ,$$

(*) Reçu en 1987.
(1) INRIA, Rocquencourt, 78153 Le Chesnay, France.