Discontinuous Galerkin for transport equation and critical eigenvalues

M. Asadzadeh([1]) (*) and L. Thevenot([2]) (**)

([1]) *Department of Mathematics, Chalmers University of Technology and Göteborg University, SE 412 96 Göteborg, Sweden.*
([2]) *Department of Mathematics, University of Besancon, France.*

Summary. — In this paper we study the discontinuous Galerkin (DG) finite element method for approximate solution of the *mono-energetic, critical* transport equation in an infinite cylindrical domain $\widetilde{\Omega}$ in $\mathbf{R}^3$ with a polygonal convex cross-section Ω. Assuming that all involved functions are constant in the direction of the symmetry axis of the cylinder, the problem is reduced to $\mathbf{R}^2$ by projection along the symmetry axis. By embedding between Sobolev and Besov interpolation spaces, see [3], we derive superconvergence estimates for fully discrete scalar flux and the critical eigenvalue. The velocity discretization rely on a special quadrature rule developed in [1]-[2]. For the critical eigenvalues studies see, *e.g.* [5]-[7]

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1. – Description

The critical eigenvalue is a positive parameter λ for which there exists a nonnegative function $\varphi = \varphi(x, v)$ satisfying the, so-called, critical transport equation:

$$(1) \qquad \begin{cases} -v \cdot \nabla_x \varphi - \Sigma\varphi + \displaystyle\int_V \sigma_s \varphi(x, v')d\mu(v') + \frac{1}{\lambda}\int_V \sigma_f \varphi(x, v')d\mu(v') = 0, \\ \varphi = 0 \text{ on } \Gamma^- := \big\{(x, v) \in \partial\Omega \times V : \ v \cdot n(x) < 0\big\}, \end{cases}$$

where the space variable x is in an open subset $\Omega \subset \mathbf{R}^d$, the domain of the core of the reactor, and the velocity variable v is in a closed subset $V \subset \mathbf{R}^d$, the admissible velocity

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domain. Further, Γ^- denotes the inflow boundary and $n(x)$ is the outward unit normal at the point $x \in \partial\Omega$. The kernels $\sigma_s := \sigma_s(x, v, v')$ and $\sigma_f := \sigma_f(x, v, v')$ describe the pure scattering and fission, respectively, while $\Sigma := \Sigma(x, v)$ represents the total cross-section.

In this note we study the numerical solution of the *mono-energetic* critical equation in a cylindrical domain $\widetilde{\Omega}$ in $\mathbf{R}^3$ with a polygonal convex cross-section Ω. Thus the velocity domain is the unit sphere $S^2 \subset \mathbf{R}^3$. All involved functions are assumed to be constant in the direction of the symmetry axis of the cylinder. This allows us to reduce the problem to $\mathbf{R}^2$ by projection along the symmetry axis of the cylinder. Therefore we study the *mono-energetic* version of the (1) in a bounded convex polygonal domain $\Omega \subset \mathbf{R}^2$, where due to the projection the integration over velocity domain $\mathbf{D} \subset \mathbf{R}^2$ is now associated by the measure $(1 - |\eta|^2)^{-1/2} d\eta$. Furthermore, we assume that the kernels satisfy

$$\Sigma(x, v) = \Sigma(|v|), \quad \sigma_s(x, v, v') = \sigma_s(v, v') \quad \text{and} \quad \sigma_f(x, v, v') = \sigma_f(|v|, |v'|) = 1.$$

Since Σ and σ_f depend only on $|v|$, thus for the mono-energetic model they are constant. We may normalize σ_f to 1, and use the same notation for λ by a corresponding "stretch" mapping $\lambda \to \lambda|\sigma_f|$.

2. – The continuous problem

By geometry the projection of the mono-energetic version of the critical equation (1) onto the cross-section Ω of $\widetilde{\Omega}$ satisfies the equation

$$\text{(2)} \quad \begin{cases} -\mu \cdot \nabla_x \varphi - \Sigma\varphi + \displaystyle\int_{\mathbf{D}} \sigma_s(\mu, \eta)\varphi(x, \eta) \frac{d\eta}{\sqrt{1 - |\eta|^2}} + \frac{1}{\lambda} \int_{\mathbf{D}} \varphi(x, \eta) \frac{d\eta}{\sqrt{1 - |\eta|^2}} = 0 \\ \varphi = 0 \text{ on } \Gamma^- := \big\{(x, \mu) \in \partial\Omega \times \mathbf{D} : \ \mu \cdot n(x) < 0\big\}. \end{cases}$$

In contrary to the mono-energetic version of (1), where $\mu \in S^2 \Rightarrow |\mu| = 1$, the projected equation (2) allows small velocities as well and we have $|\mu| \leq 1$. cross-section and all kernels are space homogeneous. Our functional space setting will be

$$\text{(3a)} \quad L_w^p(\Omega \times \mathbf{D}) = L^p\Big(\Omega \times \mathbf{D}, w \ dx d\mu\Big), \qquad 1 \leq p < \infty, \qquad w(\mu) = \frac{1}{\sqrt{1 - |\mu|^2}},$$

$$\text{(3b)} \quad W_w^p(\Omega \times \mathbf{D}) = \Big\{\varphi \in L_w^p(\Omega \times \mathbf{D}), \ \mu \cdot \nabla_x \varphi \in L_w^p(\Omega \times \mathbf{D})\Big\},$$

The total cross-section Σ is split into the scattering (Σ_s) and fission (Σ_f) cross-sections: $\Sigma = \Sigma_s + \Sigma_f$, with $\Sigma_s > 0$ and $\Sigma_f > 0$, and Σ_s is given by

$$\text{(4)} \qquad \Sigma_s = \int_{\mathbf{D}} \sigma_s(\eta, \mu) \frac{d\eta}{\sqrt{1 - |\eta|^2}}.$$

Further, using the notation $C_s(\varphi) := \Sigma_s \varphi(x, \mu) - \displaystyle\int_{\mathbf{D}} \sigma_s(\mu, \eta)\varphi(x, \eta) w(\eta) \ d\eta$, we also get $\Big(C_s\varphi(\cdot, \mu), |\varphi(\cdot, \mu)|^{p-2}\varphi(\cdot, \mu)\Big)_{w(\mu)} \geq 0, \ 1 \leq p < \infty$. That is, $\forall \ \varphi \in L_w^p(\Omega \times \mathbf{D})$

$$\text{(5)} \quad \int_{\Omega \times \mathbf{D}} \Sigma_s \frac{|\varphi(x, \mu)|^p \ d\mu}{\sqrt{1 - |\mu|^2}} dx \geq \int_{\Omega \times \mathbf{D}^2} \sigma_s(\mu, \eta)|\varphi(x, \mu)|^{p-2} \frac{\varphi(x, \mu) \, d\mu}{\sqrt{1 - |\mu|^2}} \cdot \frac{\varphi(x, \eta) d\eta}{\sqrt{1 - |\eta|^2}} dx.$$

To continue let $L^p_w := L^p_w(\Omega \times \mathbf{D})$, and define the operators S, K_s, K_f and A by

$$S\varphi = -\mu \cdot \nabla_x \varphi - \Sigma\varphi, \qquad \mathcal{D}(S) = \{\varphi : \varphi \in L^p_w,\ \mu \cdot \nabla_x \varphi \in L^p_w,\ \varphi = 0 \text{ on } \Gamma_-\},$$
$$K_s\varphi(x,\mu) = \int_{\mathbf{D}} \sigma_s(\mu,\eta)\varphi(x,\eta)\frac{d\eta}{\sqrt{1-|\eta|^2}}, \qquad K_f\varphi(x,\mu) = \int_{\mathbf{D}} \varphi(x,\eta)\frac{d\eta}{\sqrt{1-|\eta|^2}},$$
$$A\varphi = S\varphi + K_s\varphi, \qquad \mathcal{D}(A) = \mathcal{D}(S).$$

Note that the operators K_s and K_f are bounded on $L^p_w(\Omega \times \mathbf{D})$. We also recall that the operators S and A generate strongly continuous semigroups on $L^p_w(\Omega \times \mathbf{D})$ denoted by $\{e^{tS},\ t \geq 0\}$ and $\{e^{tA},\ t \geq 0\}$, respectively. In the sequel, we may replace the conservative assumption (4) by a somewhat stronger one, viz $\exists\delta > 0$ such that

$$\Sigma_s \geq \int_{\mathbf{D}} \sigma_s(\eta,\mu)\frac{d\eta}{\sqrt{1-|\eta|^2}} + \delta. \tag{6}$$

In the numerical approximations we combine a quadrature rule for the angular variable with a discontinuous Galerkin finite element scheme for the spatial discretization.

3. – The semi-discrete problem - Quadrature rule

Let $\Delta_n = \{\mu_i\}_{i=1}^n \subset \mathbf{D}$ be a discrete set of quadrature points associated with the quadrature weights w_{μ_i} and introduce the discrete operators K_s^n and K_f^n, approximating the operators K_s and K_f, respectively

$$K_s^n\varphi(x,\mu) := \sum_{\eta\in\Delta_n} \sigma_s(\mu,\eta)\varphi(x,\eta)w_\eta \approx \int_{\mathbf{D}} \sigma_s(\mu,\eta)\varphi(x,\eta)\frac{d\eta}{\sqrt{1-|\eta|^2}}, \tag{7a}$$

$$K_f^n\varphi(x,\mu) := \sum_{\eta\in\Delta_n} \varphi(x,\eta)w_\eta \approx \int_{\mathbf{D}} \varphi(x,\eta)\frac{d\eta}{\sqrt{1-|\eta|^2}}. \tag{7b}$$

We also introduce the semi-discrete $l^2_w(\Delta_n; L^2(\Omega))$ space associated with the norm

$$\Big(\sum_{\mu\in\Delta_n} w_\mu \int_\Omega |\varphi(x,\mu)|^2 dx\Big)^{1/2}.$$

Note that the operators K_s^n and K_f^n are bounded on $l^2_w(\Delta_n; L^2(\Omega))$ and we have

$$\|K_s^n\| \leq \sup_{(\mu,\eta)\in\mathbf{D}^2}\Big(\sigma_s(\mu,\eta)\Big)\Big(\sum_{\eta\in\Delta_n} w_\eta\Big), \qquad \|K_f^n\| \leq \Big(\sum_{\eta\in\Delta_n} w_\eta\Big).$$

More specifically writing $\eta \in \Delta$ in polar coordinates as $\eta = r(\cos\theta, \sin\theta)$, $r = |\eta|$ we may choose a uniform quadrature rule on θ with a uniform weight of $2\pi/M$, where M is the number of quadrature points in θ (unit circle). Here, for the radial quadrature, we choose a particular Gauss rule on $(0,1)$ with the quadrature points and weights given by (r_k, A_k), $k = 1, \ldots, N$, where N is the number of quadrature points in $r \in (0,1)$. We let $n = MN$ be the total number of quadrature points on $\mathbf{D}$, then we can prove that

Lemma 3.1. Let $f \in \mathcal{C}^{4,r}_{2,\theta}(\mathbf{D}, L_1(\Omega))$, *then* $\exists$ *constants* $C > 0$ *and small* $\varepsilon_1 > 0$,

$$\left| \int_{\mathbf{D}} f(x,\mu,\eta) \frac{d\eta}{\sqrt{1-|\eta|^2}} - \sum_{i=1}^{n} f(x,\mu,\eta_i) w_{\eta_i} \right| \leq C\Big(\frac{1}{N^4} + \frac{1}{M^{2-\varepsilon_1}}\Big) ||f||_{L_1(\Omega)},$$

where $\mathcal{C}^{4,r}_{2,\theta}(\mathbf{D}, L_1(\Omega))$ *denotes the space functions, defined in* $\mathbf{D} \times \Omega$ *that are in* $L_1(\Omega)$ *and are continuously differentiable 4 times in* r *and twice in* θ.

Lemma 3.2. Assume (6) then for sufficiently large n *and all* $\mu \in \mathbf{D}$ *we have that*

$$\Sigma_s \geq \max\Big(\sum_{\eta\in\Delta_n} (\sigma_s(\eta,\mu) w_\eta \ , \ \sum_{\eta\in\Delta_n} \sigma_s(\mu,\eta) w_\eta \Big). \tag{8}$$

The proof is based on (6) and Lemma 1 which for sufficiently large n, yields

$$\begin{aligned} \sum_{\eta\in\Delta_n} \sigma_s(\eta,\mu) w_\eta \ &\leq \ \left| \sum_{\eta\in\Delta_n} \sigma_s(\eta,\mu) w_\eta - \int_{\mathbf{D}} \frac{\sigma_s(\eta,\mu)\, d\eta}{\sqrt{1-|\eta|^2}} \right| + \int_{\mathbf{D}} \frac{\sigma_s(\eta,\mu)\, d\eta}{\sqrt{1-|\eta|^2}} \\ &\leq \ C\Big(\frac{1}{N^4} + \frac{1}{M^{2-\varepsilon_1}}\Big) - \delta + \Sigma_s \ \leq \ \Sigma_s. \end{aligned}$$

4. – The Fully-Discrete Problem - Discontinuous Galerkin method

Let $\{\mathcal{C}_h\}$ be a family of quasiuniform triangulations $\mathcal{C}_h = \{K\}$ of Ω indexed by the parameter h, the maximum diameter of triangles $K \in \mathcal{C}_h$ and introduce the finite element space V_h of functions which are allowed to be discontinuous over enterelement boundaries:

$$V_h = \Big\{ v \in L^2(\Omega) : \ v\Big|_K \text{ is linear, } \forall K \in \mathcal{C}_h \Big\}.$$

Given $\mu \in$ D and $g \in L_2(\Omega)$, we define $T^h_\mu g \in V_h$ as the solution $u(.,\mu) \in V_h$ for

$$\begin{cases} \sum_{K\in\mathcal{C}_h} \Big[\big(\mu.\nabla u + \Sigma u, v\big)_K + \int_{\partial K_-} [u] v_+ |\mu\cdot n| d\sigma \Big] = \int_\Omega g v dx, & \forall v \in V_h \\ u = 0, \qquad \text{on} \quad \Gamma^-_\mu := \big\{ x \in \partial\Omega : \ \mu\cdot n(x) < 0 \big\}, \end{cases} \tag{9}$$

where

$$\begin{aligned} &\big(u,v\big)_K = \int_K u v dx, \qquad \partial K_- = \big\{ x \in \partial K : \ \mu\cdot n(x) < 0 \big\}, \\ &[v] = v_+ - v_-, \ v_\pm(x) = \lim_{s\to 0_\pm} v(x + s\mu) \text{ for } x \in \partial K, \end{aligned}$$

$n = n(x)$ is the outward unit normal to ∂K at $x \in \partial K$, $d\sigma$ is the surface measure on ∂K.

To continue we need to introduce the adjoint operator $\big(T^h_\mu\big)^\star$ of T^h_μ. For a given $\mu \in$ D and $f \in L^2(\Omega)$, we define $\big(T^h_\mu\big)^\star f \in V_h$ as the solution $u(\cdot,\mu) \in V_h$ of the dual problem

$$\begin{cases} \sum_{K\in\mathcal{C}_h} \Big[\big(-\mu\cdot\nabla u + \Sigma u, v\big)_K - \int_{\partial K_-} [u] v_- |\mu\cdot n| d\sigma \Big] = 0, & \forall v \in V_h \\ u = \tilde{g}, \qquad \text{on} \quad \Gamma^+_\mu := \big\{ x \in \partial\Omega : \ \mu\cdot n(x) > 0 \big\}, \qquad (\tilde{g} \text{ is given}), \end{cases}$$

$(T^h_\mu)^\star$ is well defined adjoint of the operator T^h_μ in $L^2(\Omega)$. We simplify the notation by introducing $T = (-S)^{-1}$ on L^p_w. Then the critical eigenvalue problem is formulated as

$$\varphi - TK_s\varphi = \frac{1}{\lambda}TK_f\varphi.$$

The fully discrete scheme: Find the parameter $\lambda^h_n > 0$ and a nonnegative function $\varphi^h_n \in l^2_w(\Delta_n; L^2(\Omega))$ such that

$$\text{(10)}\qquad \begin{cases} \varphi^h_n - T^h_n K^n_s \varphi^h_n = \frac{1}{\lambda^h_n} T^h_n K^n_f \varphi^h_n, \\ T^h_n\varphi(x,\mu) = T^h_\mu\varphi(\cdot,\mu) \in V_h, \qquad \forall\mu \in \Delta_n, \qquad \forall\varphi \in l^2_w(\Delta_n; L^2(\Omega)). \end{cases}$$

According to [1]-[3], the discrete operator T^h_n is bounded on $l^2_w(\Delta_n; L^2(\Omega))$, *i.e.* λ^h_n and $\varphi^h_n(.,\mu) \in V_h$, $\forall\mu \in \Delta_n$, are solution of the fully discrete critical eigenvalue equation

$$\begin{cases} \displaystyle\sum_{K\in\mathcal{C}_h} \Big[\big(\mu.\nabla\varphi^h_n + \Sigma\varphi^h_n, v\big)_K + \int_{\partial K_-} [\varphi^h_n]v_+|\mu\cdot n|d\sigma\Big] - \int_\Omega v(x) \sum_{\eta\in\Delta_n} \sigma_s(\mu,\eta)\varphi^h_n(x,\eta)w_\eta \\ \displaystyle -\frac{1}{\lambda^h_n}\int_\Omega v(x)\sum_{\eta\in\Delta_n}\varphi^h_n(x,\eta)w_\eta\, dx = 0, \qquad \forall\mu \in \Delta_n, \quad \forall v \in V_h \\ u = 0 \quad \text{on } \Gamma^-_\mu := \{x \in \partial\Omega:\ \mu\cdot n(x) < 0\}, \quad \forall\mu \in \Delta_n. \end{cases}$$

Lemma 4.1. For sufficiently large n, the operators $T^h_nK^n_f$ and $T^h_nK^n_s$ are uniformly bounded on $l^2_w(\Delta_n; L^2(\Omega))$. Moreover, there exists a constant $0 < \alpha < 1$ such that $\|T^h_nK^n_s\| < \alpha$. Consequently the operator $(Id - T^h_nK^n_s)$ is invertible on $l^2_w(\Delta_n; L^2(\Omega))$ and the inverse operator $(Id - T^h_nK^n_s)^{-1}$ is uniformly bounded.

Proof. Let $\tau \in l^2_w(\Delta_n; L^2(\Omega))$ and $u = T^h_nK^n_s\tau$. For a given $\mu \in \Delta_n$ it follows from the definition of T^h_n, with the choice of u as a test function in (9), that

$$\text{(11)}\qquad \int_\Omega uK^n_s\tau dx = \sum_{K\in\mathcal{C}_h}\Big[\big(\mu\cdot\nabla u + \Sigma u, u\big)_K + \int_{\partial K_-}[u]u_+|\mu\cdot n|d\sigma\Big].$$

Let $\mathcal{E} = \cup\partial K$, $\partial K \subset \Omega\setminus\partial\Omega$, *i.e.* $\mathcal{E}$ is the set of all the sides of the triangles $K \in \mathcal{C}_h$ which are not included in $\partial\Omega$. By using Green's formula we get that

$$\begin{aligned} &\sum_{K\in\mathcal{C}_h}\Big[\big(\mu\cdot\nabla u, u\big)_K + \int_{\partial K_-}[u]u_+|\mu\cdot n|d\Gamma\Big] \\ &= \frac{1}{2}\sum_{K\in\mathcal{C}_h}\Big[\int_{\partial K_+}|\mu\cdot n||u_-|^2d\Gamma - \int_{\partial K_-}|\mu\cdot n||u_+|^2d\Gamma + \int_{\partial K_-}[u]u_+|\mu\cdot n|d\Gamma\Big] \\ &= \sum_{\partial K_-\in\mathcal{E}}\Big[\frac{1}{2}\int_{\partial K_-}|\mu\cdot n||u_-|^2d\Gamma + \frac{1}{2}\int_{\partial K_-}|\mu\cdot n||u_+|^2d\Gamma - \int_{\partial K_-}|\mu\cdot n|u_-u_+d\Gamma\Big] \\ &+\frac{1}{2}\int_{\Gamma^+_\mu}|\mu\cdot n||u_-|^2d\Gamma = \sum_{\partial K_-\in\mathcal{E}}\Big[\frac{1}{2}\int_{\partial K_-}|\mu\cdot n|\ [u]^2d\Gamma\Big] + \frac{1}{2}\int_{\Gamma^+_\mu}|\mu\cdot n||u_-|^2d\Gamma \ge 0. \end{aligned}$$

Consequently, summing (11) over Δ_n, it follows that

$$\sum_{\mu\in\Delta_n}\int_\Omega u(x,\mu)K_s^n\tau(x,\mu)dx\ w_\mu \geq \Sigma\sum_{\mu\in\Delta_n}\int_\Omega |u(x,\mu)|^2dx\ w_\mu. \tag{12}$$

On the other hand by the repeated use of Cauchy-Schwartz inequality, and Lemma 3,

$$\begin{aligned}
&\sum_{\mu\in\Delta_n}\int_\Omega u(x,\mu)K_s^n\tau(x,\mu)dx\ w_\mu = \sum_{\mu\in\Delta_n}\int_\Omega u(x,\mu)\sum_{\eta\in\Delta_n}\sigma_s(\mu,\eta)\tau(x,\eta)w_\eta w_\mu\,dx\\
&\leq \int_\Omega\sum_{\mu\in\Delta_n}|u(x,\mu)|\Big(\sum_{\eta\in\Delta_n}\sigma(\mu,\eta)w_\eta\Big)^{1/2}\times\Big(\sum_{\eta\in\Delta_n}\sigma_s(\mu,\eta)|\tau(x,\eta)|^2w_\eta\Big)^{1/2}w_\mu\,dx\\
&\leq\Big(\int_\Omega\sum_{\mu\in\Delta_n}\sum_{\eta\in\Delta_n}|u(x,\mu)|^2\sigma_s w_\mu w_\eta\,dx\Big)^{1/2}\times\Big(\int_\Omega\sum_{\mu\in\Delta_n}\sum_{\eta\in\Delta_n}\sigma_s|\tau(x,\eta)|^2w_\mu w_\eta\,dx\Big)^{1/2}\\
&\leq\Sigma_s\Big(\int_\Omega\sum_{\mu\in\Delta_n}|u(x,\mu)|^2w_\mu\,dx\Big)^{1/2}\times\Big(\int_\Omega\sum_{\eta\in\Delta_n}|\tau(x,\eta)|^2w_\eta\,dx\Big)^{1/2}.
\end{aligned}$$

Hence from the inequality (12) we deduce that

$$\Big(\int_\Omega\sum_{\mu\in\Delta_n}|u(x,\mu)|^2w_\mu\Big)^{1/2}\leq\frac{\Sigma_s}{\Sigma}\Big(\int_\Omega\sum_{\eta\in\Delta_n}|\tau(x,\eta)|^2w_\eta\Big)^{1/2}.$$

Therefore the operator norm of $T_n^hK_s^n$ is strictly smaller than $\Sigma_s\Sigma^{-1}<1$. A similar, but simpler, calculus yields $\|T_n^hK_f^n\|<\Sigma^{-1}$.

Lemma 4.2. Given μ in D, the operator T_μ^h is positive on $L^2(\Omega)$.

Proof. For $\mu\in$ D, let $u=T_\mu^hg$, where $g\in L^2(\Omega)$ is nonnegative. We write $u=u^+-u^-$ with $u^-=\max(0,-u)$ and $u^+=\max(0,u)$. Choosing u^- as a test function in (8), and using the fact that the supports of u^+ and u^- are disconnected, we get

$$\int_\Omega u^-gdx=-\sum_{K\in\mathcal{C}_h}\Big[\big(\mu\cdot\nabla u^-+\Sigma u^-,u^-\big)_K+\int_{\partial K_-}[u^-]u_+^-|\mu\cdot n|d\sigma\Big]. \tag{13}$$

We assume that u^- has a non-empty support and proceed as in the proof of lemma 3 we can prove using the Green's formula that (we omit the details),

$$-\sum_{K\in\mathcal{C}_h}\Big[\big(\mu\cdot\nabla u^-+\Sigma u^-\,,\,u^-\big)_K+\int_{\partial K_-}[u^-]u_+^-|\mu\cdot n|d\Gamma<0.$$

But $\int_\Omega u^-gdx\geq0$, therefore, Equality (13) implies that $u^-\equiv0$.

We are now in position to solve the spectral problem (10).

Theorem 4.1. There exists a real and positive eigenvalue λ_n^h associated with a unique normalized nonnegative eigenfunction $\varphi_n^h\in l_w^2\big(\Delta_n;L^2(\Omega)\big)$ such that

$$\varphi_n^h-T_n^hK_s^n\varphi_n^h=\frac{1}{\lambda_n^h}T_n^hK_f^n\varphi_n^h.$$

Proof. To simplify the notation let $B := \big(Id - T_n^h K_s^n\big)^{-1} T_n^h K_f^n$. By Lemma 3 we have

$$B = \big(Id - T_n^h K_s^n\big)^{-1} T_n^h K_f^n = \sum_{m\geq 0} \big(T_n^h K_s^n\big)^m T_n^h K_f^n.$$

By Lemma 4 the operator B is positive. Since T_μ^h and $\big(T_\mu^h\big)^\star$ have finite dimensional ranges, and $\big(Id - T_n^h K_s^n\big)^{-1}$ is bounded, we deduce that B and its adjoint $B^\star$ have also finite dimensional ranges. Consequently, $\big(\ker B\big)^\perp = R\big(B^\star\big)$ has finite dimensional range, and the operator B acting from $\big(\ker B\big)^\perp$ into $R\big(B\big)$ is a bijective positive matrix. Then the spectral radius of B is a positive eigenvalue, not necessary simple, associated with a unique normalized nonnegative eigenfunction, *i.e.* $\varphi_n^h \in l_w^2\big(\Delta_n; L^2(\Omega)\big)$, (see also [5]-[7]).

Theorem 4.2. Let u and u_h be the solutions of (2) and (9), respectively. Then we have

$$||u - u^h|| \leq C h^{1-\varepsilon} ||u||_{H^{3/2-\varepsilon}(\Omega)}, \qquad \forall \text{ small } \varepsilon > 0. \tag{14}$$

Proof [sketchy]. The geometry of Ω (convex polygonal) yields to a solution u with the first partial derivatives depending on the outward unit normal n to $\partial\Omega$, *i.e.* a linear combination of Heaviside functions. Thus, by a trace estimate, the maximal available regularity of u is $u \in H^{3/2-\varepsilon}(\Omega)$ and hence the optimal convergence order for DG in this case is $\mathcal{O}(h^{1-\varepsilon})$. To deal with such fractional derivatives, we need embeddings between Sobolev and Besov spaces. We skip these technical details and refer the reader to the methodology strategy developed in [3].

5. – Eigenvalue estimates

Below we show that the largest eigenvalue λ^{-1} of the transport operator T, (which makes $(I - \lambda T)^{-1}$ singular), can be found more accurately than the pointwise scalar flux. Observe that, cf [2] the kernel of the integral operator T is symmetric and positive. Hence T is self-adjoint (on $L_2(\Omega)$), and thus has only real eigenvalues. Furthermore, by the Krien-Rutman theory, its largest eigenvalue is positive and simple. We prove that

Lemma 5.1. Let κ, κ_n and κ_n^h be the largest eigenvalues of the operators T, T_n and T_n^h, respectively. Then for any $\varepsilon > 0$ and $\varepsilon_1 > 0$, and any arbitrary quadrature set Q, there are constants $C = C(\varepsilon_1, \kappa)$ and $C(Q) = C(\varepsilon, \kappa, Q)$ such that for sufficiently large N and M (even) and sufficiently small h,

$$||\kappa - \kappa_n|| \leq C\left(\frac{1}{N^4} + \frac{1}{M^{2-\varepsilon_1}}\right), \tag{15a}$$

$$||\kappa - \kappa_n^h|| \leq C\Big(\frac{1}{N^4} + \frac{1}{M^{2-\varepsilon_1}}\Big) + C(Q)h^{3-\varepsilon}. \tag{15b}$$

These estimates, are rather involved, and follow from the results in [1]- [3]. The assumption on the number of angular quadrature points M (even), makes the quadrature set Δ symmetric, so that, $\mu \in \Delta$ implies $-\mu \in \Delta$. Then it follows that T_n is self-adjoint and thus its eigenvalues are real, which is crucial in the proof of (15b).

Remark 5.1. In the two-dimensional case with $\Omega \subset \mathbf{R}^2$ and the velocities in the unit circle S. Using a quadrature rule with N discrete points on a quadrature set $Q \subset S$, and by a duality argument for spatial discretization, it is possible to show that for the largest eigenvalue κ_N, of the corresponding semidiscrete operator T_N, there exists an eigenvalue κ_N^h, for the fully discrete operator T_N^h such that

$$\|\kappa_N - \kappa_N^h\| \le C(Q)h^{3-\varepsilon}. \tag{16}$$

Whereas, for discrete ordinates with N uniformly distributed points on S, the optimal result, based on error analysis for DG for scalar flux estimates in [4], yields

$$\|\kappa_N - \kappa_N^h\| \le Ch^{1-\varepsilon}. \tag{17}$$

Similarly, in the case of infinite cylindrical domains using Theorem 4. 2 and cf [4], we get

$$\|\kappa_n - \kappa_n^h\| \le Ch^{1-\varepsilon}, \tag{18}$$

here κ_n and κ_n^h are the largest eigenvalues of T_n and T_n^h respectively, n is the number of discrete points on the unit disc and the constant C is independent of the quadrature set. Combining $(15a)$ and a duality argument applied to (18) gives $(15b)$.

Concluding remarks. We present a mathematical framework that picks up the maximal available regularity of the exact solution and yields an optimal convergence for the DG method for transport equation. In eigenvalue estimates $(15b)$, if the quadrature set $Q \subset \mathbf{D}$ is properly chosen, so that $C(Q)$ does not cause a decay on efficiency of the scheme, then to obtain sharper fully discrete eigenvalue estimates, a change of the compatibility concept; *i.e.* the condition $h \sim \frac{1}{N}$, is necessary. The optimal relations $h = h(N)$, as well as $M = M(N)$, in the above estimates should be chosen in such a way that the contributions of the spatial and angular errors, to the global error, be of the same order of magnitude. Omitting ε and ε_1 powers of h and M, respectively, we conclude that in the case of the infinite cylindrical domains, with the duality argument $h \sim N^{-4/3} \sim M^{-2/3}$ is optimal whereas the corresponding condition without using the duality is $h \sim N^{-4} \sim M^{-2}$.

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