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On a Canonical Form for Maxwell Equations and Convergence of Finite Element Schemes for a Vlasov-Maxwell System

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This work is a swift introduction to the nature of governing laws involved in the Maxwell equations. We then approximate a "one and one-half" dimensional relativistic Vlasov-Maxwell (VM) system using streamline diffusion finite element method. In this geometry d'Alembert representation for the fields functions guarantees the existence of a unique solution of the Maxwell equations. The VM system is then approximated using the streamline diffusion finite element method. In this part we derive some stability inequalities and optimal *a priori* error estimates due to the maximal available regularity of the exact solution.

Keywords Vlasov-Maxwell; canonical form; finite element; stability; convergence

1. INTRODUCTION

In this article we introduce some basic properties of the equations in the Maxwell's system and point out the special roles played by the field functions. We then consider a one and one-and-half dimensional, relativistic, Vlasov-Maxwell (VM) system. In this geometry we derive closed form solutions for the field equations represented in d'Alembert's formulas. This is the simplest possible to derive the existence of a unique solution for the field functions in Maxwell's equation. There is no d'Alembert representation in higher dimensions. Our main concern will be approximation of the VM system by a semi-classical finite element approach: the streamline diffusion (SD) method. Both

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the geometry, and the current combination of the equations in the system require a particular, and seemingly detailed, investigation. Here, we highlight only novel aspects and leave the common features to reference literature. It is well-known that the standard finite element method for hyperbolic equations (e.g., VM), with the exact solution in the Sobolev space H^{r+1}, has an L_2-optimal convergence rate of order $\mathcal{O}(h^r)$. Whereas, with the same regularity (H^{r+1}) the corresponding optimal convergence rate for the elliptic and parabolic problems is $\mathcal{O}(h^{r+1})$. SD method is constructed, roughly, based on a variational formulation with the test functions possessing a multiple of the convection term in the equation. This eventually corresponds to addition of extra diffusion to the continuous problem, which enhances the regularity in the streamline (characteristic) directions. That is why the method is called the streamline diffusion method. Using the SD strategy would improve the convergence rate of the corresponding finite element scheme for the hyperbolic problems by an order of 1/2:—$\mathcal{O}(h^{r+1/2})$. Then, by interpolation space techniques, one can show that for the hyperbolic problems this rate is optimal.

An outline of this article is as follows. First we present a discussion on the role of each equation in Maxwell's system. Then, we introduce a continuous model problem for the one and one-half dimensional, relativistic, Vlasov-Maxwell system. In the subsequent sections, we discretize this system using the SD method and prove the stability of the scheme and derive convergence rates.

Throughout this article C will denote a generic constant, not necessarily the same at each occurrence, and independent of the parameters in the equations, unless otherwise explicitly specified.

2. VLASOV-MAXWELL SYSTEM IN VECTOR ANALYSIS FORM

The VM system describes time evolution of collosionless plasma of particles with mass m and charge q, formulated as

$$\begin{aligned}
&\partial_t f + \hat{v}\cdot\nabla_x f + q(E + c^{-1} v\times B)\cdot\nabla_v f = 0,\\
&\text{(Ampere's Law)}\quad \partial_t E = c\nabla\times B - j, \qquad \nabla\cdot E = \rho,\\
&\text{(Faraday's Law)}\quad \partial_t B = -c\nabla\times E, \qquad \nabla\cdot B = 0. \qquad (2.1)
\end{aligned}$$

Here f is density in phase space, c is the speed of light, v is momentum, and the velocity, $\hat{v}$, is given by

$$\hat{v} = (m^2 + c^{-2}|v|^2)^{-1/2}.$$

Further, the charge and current densities are given by

$$\rho(t,x) = 4\pi\int qf\,dv, \quad j(t,x) = 4\pi\int qf\hat{v}\,dv. \qquad (2.2)$$

A mathematical proof for the existence (and uniqueness) of the solution to VM system can be obtained using the Schauder fixed point theorem: Insert an assumed and given g for f in (2.2). Compute ρ_g, j_g and insert the results in Maxwell equations to get E_g, B_g. Then insert such obtained E_g and B_g in the Vlasov equation to get f_g via an operator Λ: $f_g = \Lambda g$. A fixed point of Λ is the solution of the VM system. For the discretized version one should, instead, use the Brouwer fixed point theorem. (For a survey on fixed point theory and proofs for Schauder and Brouwer fixed point theorems see Dugundji and Granas, 2003, and Kellogg, Li, and Yorke, 1976, respectively.) Besides the fact that both these proofs are rather involved and nontrivial, they are not attractive for physicists in the sense that, for example, the quantities f, B, E, j, and ρ are physically related to each others by the Vlasov-Maxwell system of equations and again, physically, it is not the case that some of them are given to determine the others: they act in concert and do not follow any ordering. Whereas, for example, in Vlasov-Poisson and Vlasov-Poisson-Fokker-Planck systems, the physical connection is simpler and also the Poisson equation is treated easily. To conclude: for some physical problems (including Vlasov-Maxwell system) the use of mathematically relevant, fixed point approach is vulnerable. Therefore, next we shall employ a different approach to investigating the existence of a unique solution for our VM system.

2.1. Particular Manner that Various Quantities Enter Maxwell's Equations

Why can we not have a plus sign entering the right hand side of Faraday's law?

If one does consider such "revised" equations, it is easy to show that one obtains, in the Lorentz gauge, perfectly respectable mathematical relations

$$\nabla^2 A + (1/c^2)\ddot{A} = -(4\pi/c)j, \qquad B = \nabla \times A$$
$$\nabla^2 \varphi + (1/c^2)\ddot{\varphi} = -4\pi\rho, \qquad E = -\nabla\varphi + (1/c)A.$$

However, they do not have the properties that Maxwell's customary solutions do, for example,

$$A(x,t) = A(x)e^{i\omega t}, \qquad (\rho = 0 = j)$$

do not yield customary wave solution for $A(x, t)$ and yield an $A(x)$ that decreases by an exponential factor everywhere within any closed surface.

If we, instead, change the sign on the right hand side in Ampere's law, we get a mathematically permissible set of equations containing wave solutions, but denying charge conservation.

Question. What makes the fields E and B satisfy, in particular, Maxwell's equations, rather than some other mathematically permissible scheme? Such questions on physical relevance of these quantities are behind the doubt on suitability of using a Scheuder-type approach, to the existence and uniqueness mentioned previously.

2.2. Relativistic Mode in One and One-half Dimensional Geometry

The most convenient VM system to discretize is the relativistic Vlasov-Maxwell (RVM) model, in one and one-half dimensional geometry, where $x \in \mathbb{R}$, $v \in \mathbb{R}^2$, and $E = (E_1, E_2)$, which then can be generalized to higher dimensions

$$\left.\begin{array}{ll} l@ \quad c\partial_t f + \hat{v}_1 \cdot \partial_x f + q(E + BM_0\hat{v})\cdot & \nabla_v f = 0, \\ \partial_t E_1 = -4\pi j_1, & \partial_x E_1 = 4\pi\rho, \\ \partial_t E_2 = -\partial_x B - 4\pi j_2, & \partial_t B = -\partial_x E_2. \end{array}\right\} \quad M_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

To carry out discrete analysis, we need global existence of classical solution. Due to the physical concerns about the fixed here we employ another approach that requires some regularity assumptions as follows:

Assumption A1: The background density $n(x)$ is smooth, has compact support, and is neutralizing. This yields, for

$$\rho(t, x) = q \int f\, dv - n(x); \qquad \text{that} \int_{-\infty}^{\infty} \rho(0, x)\, dx = 0. \tag{2.3}$$

Assumption A2: We also assume that $f^0(x, v) := f(0, x, v) \geq 0$. Then choosing

$$E_1(0, x) = 4\pi \Big(\int_{-\infty}^{x} f^0(y, v)\, dv - n(y) \Big)\, dy, \tag{2.4}$$

$\partial_x E_1 = 4\pi\rho$is the only possibility that lead to finite energy solution.

Theorem 2.1 *(Glassey and Schaeffer, 1990). Assume that* n *is naturalizing, and let $r \geq 1$ be an integer. Further*

$$(i)\ 0 \leq f^0(x, v) \in C_0^r(\mathbb{R}^3), \qquad (ii)\ E_2^0,\ \ B^0 \in C_0^{r+1}(\mathbb{R}^1).$$

Then, there exists global *C^r solutions of RVM: (f, E, B) of class C^r over $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2$.*

The theorem is an existence result. For $r = 2$ we differentiate with respect to x and t to get

$$\begin{cases} \partial_x \partial_t E_2 = -\partial^2_{xx} B - \partial_x 4\pi j_2, \\ \partial^2_{tt} E_2 = -\partial_t \partial_x B - \partial_t 4\pi j_2, \end{cases} \quad \text{and} \quad \begin{cases} \partial^2_{tt} E_2 = -\partial_t \partial_x B - \partial_t 4\pi j_2, \\ \partial_x \partial_t B = -\partial^2_{xx} E_2, \end{cases}$$

respectively. Subtracting either of the resulting equations we see that both E_2 and B, satisfying the wave equation, admit closed form solution of *d' Alembert* type. The closed form solution for E_1 is yet simpler. Hence (by uniqueness of the solution for the wave equation), we have now both existence and uniqueness. Our first attempt is to use simple algebraic manipulation and derive existence and uniqueness for the Maxwell's equations system, without requiring the higher regularity ($r = 2$) than what is present in the equations, that is, with $r = 1$ only.

2.3. Reduced Regularity Requirements and d'Alembert Form

To show the uniqueness with lower regularity assumption, let $F = E_2 + B$ and $G = E_2 - B$. From the equations for E_2 and B, we get:

$$\begin{cases} \partial_t F + \partial_x F = j_2(t, x), \; F(0, x) = E^0_2(x) + B^0(x) \\ \partial_t G - \partial_x G = j_2(t, x), \; G(0, x) = E^0_2(x) - B^0(x). \end{cases}$$

Then $E_2 = \frac{1}{2}(F + G)$ and $B = \frac{1}{2}(F - G)$, will have closed form solutions of *d'Alembert forms:*

$$\begin{aligned} E_2(t, x) = &\frac{1}{2}\left(E^0_2(x - t) + E^0_2(x + t) + B^0(x - t) - B^0(x + t)\right) + \\ &+\frac{1}{2}\int_0^t j_2(\tau, x + \tau - t) + j_2(\tau, x + t - \tau)d\tau, \end{aligned}$$

$$\begin{aligned} B(t, x) = &\frac{1}{2}\left(E^0_2(x - t) - E^0_2(x + t) + B^0(x - t) + B^0(x + t)\right) + \\ &+\frac{1}{2}\int_0^t j_2(\tau, x + \tau - t) - j_2(\tau, x + t - \tau)\, d\tau. \end{aligned}$$

Here, the equations for E_1 yield locally, that is, for $x \in [x_0, x_1]$:

$$E_1(t, x) = \int_{x_0}^x \left(\int f(t, y, v) dv - n(y)\right) dy.$$

This is not a crucial restriction in our study, since our discretizations concerns, spatially, bounded domains.

3. THE STREAMLINE DIFFUSION METHOD

Despite the representations for the field functions of the Maxwell's equations, the realistic investigation of the solution for the Vlasov-Maxwell system is through numerical approaches. In this regard, we shall consider the study of a finite element method based on a weak form that enhance the regularity of the hyperbolic type problems, through introducing diffusion corresponding terms in the equation, whence having a regularizing effect. To this end, we shall assume that $(x, v) \in \Omega_x \times \Omega_v \subset \mathbb{R} \times \mathbb{R}^2$ (when only Maxwell's equations are considered, we just drop all v components and Ω_v), that f, E_2, B, and n have compact support in Ω_x and that f has compact support in Ω_v. Since $\int \rho(0, x)dx = 0$, it follows that also E_1 has compact support in Ω_x.

Let $\Omega := \Omega_x \times \Omega_v$ be the phase-space domain and consider a triangulation $\mathcal{T}_h := \{\tau = \tau_x \times \tau_v\}$ of Ω into shape regular elements $\tau = \tau_x \times \tau_v$, with τ_x and τ_v being shape regular triangulation of Ω_x and Ω_v, respectively. Further for $t \in [0, T]$ we let $0 = t_0 < t_1 < \ldots < t_{M-1} < t_M = T$ be a partition of $[0, T]$. For $m = 1, \ldots, M$, we denote the subinterval $(t_{m-1}, t_m]$ by I_m. Now we define the finite element subdivision of the slab $S_m := \Omega \times I_m$ by

$$\mathcal{C}_h := \{K | K := \tau \times I_m,\ \ \tau \in \mathcal{T}_h\}.$$

For $k = 0, 1, \ldots,$ we define the finite element space over each S_m as

$$V_h = \{w \in \mathcal{H} : w|_K \in P_k(\tau) \times P_k(I_m);\ \forall K = \tau \times I_m \in \mathcal{C}_h\},$$

where $\mathcal{H} := \Pi_{m=1}^M H^1(S_m)$, or $\prod_{m=1}^M L_2(S_m)$ with $S_m = I_m \times \Omega = (t_{m-1}, t_m) \times \Omega$. Finally we define

$$\tilde{V}_h := \{f \in [\mathcal{H}]^3 : f_i|_K \in \mathcal{P}_k(\tau) \times \mathcal{P}_k(I_m);\ \ \forall K = \tau \times I_m \in \mathcal{C}_h;\ i = 1, 2, 3\},$$

to be the vector version of the finite element space corresponding to the discrete versions of unknowns (E_1, E_2, B) or (f, E_2, B). We shall also use the following notation

$$\begin{array}{ll}
(f, g)_m = (f, g)_{I_m}; & \|g\|_m = (g, g)_m^{1/2} \\
< f, g >_m = \Big(f(\cdot, \cdot, t_m), g(\cdot, \cdot, t_m)\Big); & |g|_m = < g, g >_m^{1/2} \\
[g] := g^+ - g^-; & g^\pm = \lim_{s \to 0\pm} g(x, v, t + s) \\
< f^\mp, g^\mp >_{\Gamma^\pm} = \int_{\Gamma^\pm} f^\mp g^\mp |G^h \cdot n| dv\ (*) & < f^\mp, g^\mp >_{\lambda^\pm} = \int_{I_m} (*)\, dt.
\end{array}$$

Note that $(f, g)_D := \int_D fg$ (D any domain). In this way we have a discretization environment, suitable for the SD method in which we employ a modified version of the test functions. We have studied the SD method in Vlasov-Poisson and Vlasov-Poisson-Fokker-Planck settings in Asadzadeh (1990) and Asadzadeh and Kowalczyk (2005). We have tried to avoid overlappings and presented only the novelty aspects required in the VM system. For a thorough study of the SD method, we refer to Asadzadeh and Asadzadeh and Kowalczyk. The SD method was introduced by Brooks and Hughes (1982) for the fluid problems. A more elaborated approach is given by Hughes, Franca, and Hulbert (1989) in a monograph volume. A rigorous mathematical analysis of the method for the two-dimensional Euler and Navier-Stokes equations was given by Johnson and Saranen (1986). The choice of the stabilization parameter in SD-modification in a simple geometry can found in for example, Hansbo (1994). Other relevant studies in this direction, for example convergence of the SD method for conservation laws is given by Szepessy (1991). Linear nonconforming finite element method for Maxwell's equations in two dimensions is considered in Hansbo and Rylander (2010), where the focus has been on numerical tests with a brief discussion on theoretical aspects and no convergence analysis included. Further related studies can be found in Asadzadeh and Sopasakis (2007) as well as some recent studies by this author and Asadzadeh, Rostamy, and Zabihi (2011) and with Asadzadeh and Kazemi (2013) and the references therein.

3.1. Streamline Diffusion for Maxwell Equations in Vector Analysis Form

In this section, $\Omega = \Omega_x$, we state the basic SD results for the Maxwell's equations in the one-half dimensional case and give some of the main ideas in the proofs. The detailed proofs, although following the path of analysis in Asadzadeh (1990); Asadzadeh and Kowalczyk (2005); Asadzadeh and Sopasakis (2007); Asadzadeh et al. (2011); and Asadzadeh and Kazemi (2013), are too lengthy to be included in this note and are the subject of a forthcoming paper. To proceed we set

$$M_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

and let $W = (E_1, E_2, B)^T$, $W^0 = (E_1^0, E_2^0, B^0)^T$, and $\mathbf{b} = (\rho, j_1, j_2, 0)^T$. Then, the Maxwell equations can then be written, in the concise form, as

$$\begin{cases} M_1 W_t + M_2 W_x = \mathbf{b} \\ W(0, x) = W^0(x). \end{cases} \tag{3.1}$$

The SD method for the Maxwell equations can now be formulated as follows: Find $W^h \in \tilde{V}_h$ such that for $m = 1, \ldots, M$,

$$(M_1 W_t^h + M_2 W_x^h, \hat{U} + \delta(M_1 U_t + M_2 U_x))_m + \langle W_+^h, U_+ \rangle_m$$

$$= (\mathbf{b}, \hat{U} + \delta(M_1U_t + M_2U_x))_m + \langle W_-^h, U_+\rangle, \quad \forall\, U \in \tilde{V}_h,$$

where, $\hat{U} = (U_1, U_1, U_2, U_3)^T$ and δ is a multiple of the mesh parameter h.

Remark 3.1. The choice of δ is, however, a more delicate task and being an h-dependent stabilization factor has an impact in the convergence estimates as well. For our purpose, with Vlasov- or transport-type equations (also for the case of Euler and Navier-Stokes problems as in Johnson and Saranen (1986), it suffices to choose $\delta \sim h$. For Elastoplast-plasticity and some other models, arising, for example, in mechanical engineering problems, the term involving the parameter δ will have a more involved and different form. For details about the stabilization parameter see Principe and Codina (2010) and the references therein.

Thus, we have a variational formulation with the test functions of the form $\hat{g} + \delta(M_1g_t + M_2g_x)$. To simplify the presentation we introduce the bilinear form

$$\begin{aligned}\tilde{\mathcal{B}}(W, U) := &\sum_{m=0}^{M-1}(M_1W_t + M_2W_x, \hat{U} + \delta(M_1U_t + M_2U_x))_m\\ &+ \sum_{m=1}^{M-1}\langle [W], U_+\rangle_m + \langle W_+, U_+\rangle_0,\end{aligned} \tag{3.2}$$

and define the linear form

$$\tilde{\mathcal{L}}(\mathbf{b}, U) := \sum_{m=0}^{M-1}(\mathbf{b}, \hat{U} + \delta(M_1U_t + M_2U_x))_m + \langle W^0, U_+\rangle_0. \tag{3.3}$$

Thus, in short, we reformulate the SD problem. Find $W^h \in \tilde{V}_h$ such that

$$\tilde{\mathcal{B}}(W^h, U) = \tilde{\mathcal{L}}(\mathbf{b}, U), \qquad \forall\, U \in \tilde{V}_h. \tag{3.4}$$

Then, the triplenorm will be an adequate measuring instrument:

$$|||U|||^2 = \frac{1}{2}\left(|U_+|_0^2 + |U_-|_M^2 + \sum_{m=1}^{M-1}|[U]|_m^2 + 2\delta\sum_{m=0}^{M-1}\|M_1U_t + M_2U_x\|_m^2\right),$$

where $[U] = U_+ - U_-$ is the jump with $U_\pm = \lim_{s\to 0+} U(x \pm s)$.

It is the spirit of Reisz representation and Lax-Milgram theorems to guarantee existence of a unique solution for the discretized equation (3.4) via the following lemma:

Lemma 3.1. *For any constant* C *we have for* $U \in \mathcal{H}$,

$$\|U\|_m^2 \le \left(|U_-|_{m+1}^2 + \frac{1}{C}\|M_1U_t + M_2U_x\|_m^2\right) he^{2Ch}. \tag{3.5}$$

Further, the following coercivity relation holds true

$$\tilde{\mathcal{B}}(U,U) = |||U|||^2 \quad \forall U \in \mathcal{H}, \qquad \mathcal{H} := \Pi_{m=1}^M H^1(S_m). \tag{3.6}$$

Proof. The estimate (3.5) is a generalized version of the Poincare inequality stated for the slab S_m. To derive Poincare inequality one needs to restrict the constant C to depend on the size of Ω and take sufficiently small h. Then, the exponential factor is a bounded constant. The proof of (3.5) follows by the same calculus an in the next section, where we derive the corresponding inequality for the Vlasov-Maxwell system. As for the coercivity estimate (3.6), by the definition of $\tilde{\mathcal{B}}$,

$$\begin{aligned}\tilde{\mathcal{B}}(U,U) = &\sum_{m=0}^{M-1}\left((M_1U_t + M_2U_x, \hat{U})_m + \delta\|M_1U_t + M_2U_x\|_m^2\right)\\ &+\sum_{m=1}^{M-1}\langle [U], U_+\rangle_m + |U_+|_0^2.\end{aligned}$$

Using partial integration we get

$$\sum_{m=0}^{M-1}(M_1U_t, \hat{U})_m + \sum_{m=1}^{M-1}\langle [U], U_+\rangle_m + |U_+|_0^2 = \frac{1}{2}\left(\sum_{m=0}^{M-1}|[U]|_m^2 + |U_-|_M^2 + |U_+|_0^2\right). \tag{3.7}$$

On the other hand, since $U(t,x) = 0$ on $I \times \partial\Omega_x$, we have that

$$\sum_{m=0}^{M-1}(M_2U_x, \hat{U})_m = 0. \tag{3.8}$$

Now, (3.6) follows adding the relations (3.7) and (3.8). □

Remark 3.2. The conventional coercivity is usually an inequality of the form $\tilde{\mathcal{B}}(\xi,\xi) \ge |||\xi|||^2$, rather than the equality (3.6). Then, one may employ the corresponding Galerkin orthogonality and estimate the triple norm of the error. In the previous setting, we are just on the borderline of being able to carry out the error estimate procedure. Some other approaches are given in Asadzadeh and Kazemi (2013). Note that the triple norm does not contain L_2-norm. The relation (3.5) guarantees a type of L_2-control. This, however, is on a price of the exponential coefficient there. A more clean L_2-estimate, h independent coefficient and with no exponential factor, can be obtained using the usual Poincare inequality. This is adequate due to the fact that all involved functions (fields) and data are assumed to have compact supports. Therefore we may assume larger bounded domain and vanishing boundary data.

Now using the Lemma 3.1 and the Lax-Milgram theorem we can show that:

Lemma 3.2. *For any $h > 0$ the problem (3.4) has a solution and if* h *is small enough the solution is unique.*

Remark 3.3. Note that by introducing the streamline diffusion term, the bilinear form $\tilde{\mathcal{B}}(\cdot,\cdot)$, is no longer symmetric. Therefore using the Riesz representation theorem (a scalar product Based approach for the existence of unit solution) does not work.

For the error analysis of Maxwell equations, in addition to the Lemma 3.1, following stability bounds are also needed: We set $Q_T = [0,T] \times \Omega_x$, by some standard inequalities (e.g., Cauchy-Schwarz and triangle inequalities) we can derive

$$\|E_1\|^2_{Q_T} \leq C\left(\|f\|^2_{Q_T} + T\int_{\Omega_x} |n(x)|^2 dx\right),$$

and

$$\|E_2\|^2_{Q_T} \leq CT\left(\int_{\Omega_x} |E_2^0(x)|^2 dx + \int_{\Omega_x} |B^0(x)|^2 dx + \|\hat{v}_2 f\|^2_{Q_T}\right).$$

In a similar way we get, for $i = 1,2$, that

$$\|\hat{v}_i B\|^2_{Q_T} \leq CT\left(\int_{\Omega_x} |B^0(x)|^2 dx + \int_{\Omega_x} |E_2^0(x)|^2 dx + \|\hat{v}_2 f\|^2_{Q_T}\right).$$

These inequalities are the main ingredients to perform error analysis. Then, the proof of the following convergence theorem rely on the interpolation (η) and projection (ξ) error estimates in the following split: let W^h be the SD solution for (3.4) and $\tilde{W}$ an interpolant of W, set

$$W - W^h = (W - \tilde{W}) - (W^h - \tilde{W}) := \eta - \xi.$$

Theorem 3.1. *If W^h is a solution to (3.4) and the exact solution* W *of (3.1) satisfies $\|W\|_{k+1} \leq C$, then there exists a constant* C *such that*

$$|||W - W^h||| \leq Ch^{k+1/2}, \qquad W \in H^{k+1}(\Omega_x). \tag{3.9}$$

Proof. We may choose the interpolant $\tilde{W} \in \tilde{V}_h$, then it follows that the interpolation error satisfies the previous bound, that is, $|||\eta||| \le Ch^{k+1/2}$, see, for example, Ciarlet (1980; p. 124). Hence it suffices to show that ξ (the error of a the approximate solution and the interpolant) also satisfies the same bound. To this end, we show that the $|||\xi|||$ is dominated by $|||\eta|||$. First we note that, for $U \in [\mathcal{H}]^3$, (3.2) and (3.3) are the two sides in

$$\tilde{\mathcal{B}}(W, U) = \tilde{\mathcal{L}}(W, U), \tag{3.10}$$

so that restricting U to $\tilde{V}_h$ and subtracting (3.4) from it, we end up with the Galerkin orthogonality relation

$$\tilde{\mathcal{B}}(W - W^h, U) \equiv \tilde{\mathcal{B}}(e, U) = 0, \qquad \forall\, U \in \tilde{V}_h, \tag{3.11}$$

where, $e = W - W^h$ is the finite element error. Using this relation with $U = W^h - \tilde{W}(= \xi)$ we have that $\tilde{\mathcal{B}}(\xi, \xi) = \tilde{\mathcal{B}}(\eta - e, \xi) = \tilde{\mathcal{B}}(\eta, \xi)$. Thus by (3.6) $|||\xi||| = \tilde{\mathcal{B}}(\eta, \xi)$ and hence

$$\begin{aligned} |||\xi|||^2 = \tilde{\mathcal{B}}(\eta, \xi) = & \sum_{m=0}^{M-1} (M_1\eta_t + M_2\eta_x, \hat{\xi} + \delta(M_1\xi_t + M_2\xi_x))_m \\ & + \sum_{m=0}^{M-1} \langle [\eta], \xi_+\rangle_m + \langle \eta_+, \xi_+\rangle_0. \end{aligned} \tag{3.12}$$

Further, integrating by parts

$$(M_1\eta_t, \hat{\xi})_m = \langle \eta_-, \xi_-\rangle_{m+1}\langle \eta_+, \xi_+\rangle_m - (\eta, \xi)_m, \tag{3.13}$$

and since both η and ξ are compactly supported in Ω_x,

$$(M_2\eta_x, \hat{\xi})_m = (\eta_x, \xi)_m = -(\eta, \xi_x)_m. \tag{3.14}$$

Inserting in (3.12) and rearranging the terms we get that

$$\begin{aligned} \tilde{\mathcal{B}}(\eta, \xi) \le \Big| \langle \eta_-, \xi_-\rangle_M - \sum_{m=0}^{M-1} \langle \eta_-, [\xi]\rangle_m \\ + \sum_{m=0}^{M-1} (\hat{\eta}, M_1\xi_t + M_2\xi_x)_m + \delta((M_1\eta_t + M_2\eta_x, M_1\xi_t + M_2\xi_x)_m \Big|. \end{aligned}$$

From this, by some standard inequality, we can derive

$$|||\xi|||^2 \le \frac{1}{4}|||\xi|||^2 + \sum_{m=0}^{M-1}\left(4|\eta|^2_{m+1} + \frac{8}{\delta}\|\eta\|^2_m + 4\delta\|M_1\eta_t + M_2\eta_x\|^2\right).$$

Now, by a kick-back argument and the fact that η-terms on the right hand side, as interpolation error, all satisfy the error bound (3.9) and are included in $|||\eta|||^2$, we get the desired result. $\square$

4. STREAMLINE DIFFUSION FOR THE VLASOV-MAXWELL SYSTEM

We have already studied the Maxwell equations in the Vlasov-Maxwell system. Hence, it remains to consider the Vlasov equation:

$$\begin{cases} f_t + \hat{v}_1 f_x + (E_1 + \hat{v}_2 B) f_{v_1} + (E_2 - \hat{v}_1 B) f_{v_2} = 0, & \text{in } \Omega_T := (0,T) \times \Omega_x \times \Omega_v, \\ f(0,x,v) = f_0(x,v) \geq 0, & \text{in } \Omega_0 := \Omega \times \{0\}, \\ f(t,x,v) = f_b(t,x,v), & \text{on } (0,T) \times \partial\Omega. \end{cases} \tag{4.1}$$

Introducing a phase-space characteristic direction $G := (\hat{v}_1, E_1 + \hat{v}_2 B, E_2 - \hat{v}_1 B)$ and the total derivative $\nabla f := (\nabla_x f, \nabla_v f)$ or, more specifically, in our case $\nabla f := (f_x, f_{v_1}, f_{v_2})$, we end up with a concise form of the first equation in (4.1):

$$f_t + G \cdot \nabla f = 0, \text{ in } \Omega_T := \Omega \times (0,T) := \Omega_x \times \Omega_v \times (0,T).$$

Now the streamline diffusion method for the Vlasov part can be formulated. Find $f^h \in V_h$ such that for $m = 1, \dots, M$,

$$\big(f_t^h + G(f^h) \cdot \nabla f^h, g + \delta(g_t + G(f^h) \cdot \nabla g)\big)_m + \langle f_+^h, g_+ \rangle_m = \langle f_-^h, g_+ \rangle_m, \qquad \forall g \in V_h,$$

with $G(f^h) := (\hat{v}_1, E_1^h + \hat{v}_2 B^h, E_2^h - \hat{v}_1 B^h)$. Now in a similar way as for the Maxwell equations we introduce the bilinear form

$$\begin{aligned} \mathcal{B}(G; f, g) := & \sum_{m=1}^{M} (f_t + G \cdot \nabla f, g + \delta(g_t + G(f^h) \cdot \nabla g))_m \\ & + \sum_{m=1}^{M} \langle [f], g_+ \rangle_m + \langle f_+, g_+ \rangle_0 \end{aligned} \tag{4.2}$$

and the linear form

$$\mathcal{L}(g) := \langle f^0, g_+ \rangle_0. \tag{4.3}$$

Hence, in short we have the weak formulation. Find $f \in V (= \mathcal{H})$, such that

$$\mathcal{B}(G(f); f, g) = \mathcal{L}(g), \qquad \forall g \in V. \tag{4.4}$$

The streamline diffusion method can now be formulated in the following concise form. Find $f^h \in V_h$ such that

$$\mathcal{B}(G(f^h); f^h, g) = \mathcal{L}(g), \qquad \forall g \in V_h. \tag{4.5}$$

In this part a natural measuring environment is given as

$$|||g|||_V^2 := \frac{1}{2}\left(|g_+|_0^2 + |g_-|_M^2 + \sum_{m=1}^{M-1} |[g]|_m^2 \right.$$
$$\left. +\ 2\delta \sum_{m=1}^{M} ||g_t + G(f^h)\cdot\nabla g||_m^2 + \int_{\partial\Omega\times I} g^2 |G^h\cdot\mathbf{n}|\, d\nu ds\right),$$

where $\mathbf{n}$ is the outward unit normal to the boundary. The outline of the convergence estimates rely again on existence of interpolation/projection of the exact solution $\Pi f \in \mathcal{F}$ in a certain finite dimensional function space, $\mathcal{F}$. Our finite dimensional space will be piecewise polynomial space V_h and for a finite element approximation, $f_h \in V_h$, of f we have that the error can be split as

$$f - \tilde{f} = (f - \Pi f) - (f_h - \Pi f-) \equiv \theta + \zeta; \quad \zeta \in V_h.$$

We discretize Ω_T using streamline diffusion method with test functions of the form $u + \delta(u_t + G(\tilde{f})\cdot\nabla u)$, and $\delta \sim h$, the mesh size. Then the error analysis consists of the following two steps:

(i) Use approximation theory to derive sharp error bounds for the interpolant θ: $|||\theta|||_V \le |||data|||_V$.

(ii) Establish the ζ-term by the projection error: $|||\zeta|||_V \le C|||\theta|||_V$.

Observe that (i) and (ii) work only if u_t is included inside test function. A multiplier without the u_t-term (i.e., a test function of the form $u + \delta(G(\tilde{f})\cdot\nabla u)$) sought for some time iteration, which, generating accumulative error, deteriorates the optimality of the convergence.

4.1. Stability and Convergence

In the stationary problems (with no f_t), the modification test function $u + \delta(G^h\cdot\nabla u)$ together with $G^h\cdot\nabla f_h$ introduces a term of the form $\delta(G^h\cdot\nabla f_h, G^h\cdot\nabla u) := \delta(f_\gamma^h, u_\gamma)$, $(\gamma := G^h, \zeta_\gamma := \gamma\cdot\nabla\zeta)$, interpreted as resulting from a diffusion $-\delta f_{\gamma\gamma}$ acting only in the streamline direction γ: Motivation for the use of the name streamline diffusion.

As a result, the SD test function adds numerical dissipation in the vicinity of large gradients improving convergence rates. The corresponding stability and coercivity results in this case will take the following form (see also Asadzadeh and Kowalczyk, 2005, and Asadzadeh and Kazemi, 2013):

Lemma 4.1. (Stability and Coercivity) *For any constant* C *we have for* $u \in \mathcal{H}$ *that*

$$||u||_{\Omega_T}^2 \le \left[\frac{1}{C_1}||u_t + G(f^h)\cdot\nabla u||^2 + \sum_{m=1}^{M-1} |[u]|_m^2 + \int_{\partial\Omega\times I} u^2|G^h\cdot\mathbf{n}|\right]\delta e^{C_1\delta},$$
$$\forall C_1 \ge 0. \tag{4.6}$$

We can also prove the following coercivity:

$$\mathcal{B}(G(f^h); u, u) \geq \frac{1}{2}|||u|||_V^2, \qquad \forall u \in \mathcal{H} \quad \forall C_1 \geq 0. \tag{4.7}$$

Proof. Here, the proof of the coercivity estimate follows the same path as that of the (3.6) in the previous section. As for the stability estimate (4.6), the following layout would work both for Poincare inequality (3.5) and with a general constant as in here. To this end for each slab S_m, that is, for $t_m < t < t_{m+1}$, we may write

$$\begin{aligned}
\|u(t)\|_\Omega^2 &= |u|_{m+1}^2 - \int_t^{t_{m+1}} \frac{\partial}{\partial s}\|u(s)\|_\Omega^2\, ds \\
&= |u|_{m+1}^2 - 2\int_t^{t_{m+1}} [(u_t + G(f)\cdot\nabla u, u)_m - \frac{1}{2}\int_{\partial\Omega} u^2|G(f)\cdot\mathbf{n}|\, d\sigma - \langle u_+, u_+\rangle_\Gamma^-]\, ds \\
&\leq |u|_{m+1}^2 + \frac{1}{C}\|u_t + +G(f^h)\cdot\nabla u\|_m^2 + C\int_t^{t_{m+1}} \|u(s)\|\, ds.
\end{aligned} \tag{4.8}$$

Now, a Gronwall inequality applied for $t \in I_m = (t_m, t_{m+1})$, integration over I_m and summing over $m = 0, 1, \ldots, M$ yields the desired result. □

As in the case of Maxwell's equations, the existence of a unique solution is again a consequence of the Lax-Milgram theorem and the error analysis, although technical, rely on similar estimates as in those of the previous section (we leave the reader to work out the details, see also some relevant estimates in Asadzadeh and Kowalczyk (2005); Asadzadeh and Sopasakis (2007); Asadzadeh et al. (2011); and Asadzadeh and Kazemi (2013). Finally, similar procedure as in the proof of Theorem 3.1 yields the main result of this section:

Theorem 4.1 *Let f^h be the SD approximate solution satisfying (4.5) and assume that there is a constant* C *such the exact solution* f *for (4.4) and the projection / interpolation error satisfies the bound*

$$\|\nabla f\|_\infty + \|G(f)\|_\infty + \|\nabla\eta\|_\infty \leq C.$$

then, for sufficiently small h*, we have that*

$$|||f - f^h|||_V \leq Ch^{k+1/2}||f||_{H^{k+1}(\Omega_T)}.$$

5. CONCLUSION

We have considered approximate solutions for a one and one-half dimensional Vlasov-Maxwell system of equations using the streamline-diffusion finite element method. This geometry was chosen in order to have a simpler approach to the existence of a unique solution for Maxwell's equations with a low regularity requirement. We have discussed the particular role played by the field equations in Maxwell's system. Their physical interpretation was the reason for avoiding the conventional fixed-point approaches used for the Vlasov-type systems. To circumvent such uncertainty we considered instead a simpler geometry. To have an insight on examining the method, we studied the SD procedure for Maxwell's equation and the Vlasov-Maxwell system separately. We have proved coercivity and stability estimates required to prove optimal convergence rates. In a forthcoming paper, concerning a posteriori error estimates, we shall consider other geometries, include the discontinuous Galerkin approach and introduced results of some implementations.

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REFERENCES

Asadzadeh, M. (1990). Streamline diffusion methods for The Vlasov-Poisson equation. *Math. Model. Numer. Anal.* 24(2):177–196.

Asadzadeh, M., Kazemi, E. (2013). On convergence of the streamline diffusion and discontinuous Galerkin methods for the multi-dimensional Fermi pencil beam equation. *Int. J. Numer. Anal. Model.* 10(4):860–875.

Asadzadeh, M., Kowalczyk, P. (2005). Convergence of streamline diffusion methods for the Vlasov-Poisson-Fokker-Planck system. *Numer. Methods Partial Differential Eqs.* 21:472–495.

Asadzadeh, M., Rostamy, D., Zabihi, F. (2011). A posteriori error estimates for a coupled wave system with a local damping. Problems in mathematical analysis. *J. Math. Sci.* 56(3):228–248.

Asadzadeh, M., Sopasakis, A. (2007). Convergence of a hp streamline diffusion scheme for Vlasov-Fokker-Planck system. *Math. Mod. Meth. Appl. Sci.* 17:1159–1182.

Brooks, A., Hughes, T. (1982). Streamline upwind/Petrov-Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier-Stokes equations. FENOMECH' 81, Part I. *Comput. Methods Appl. Mech. Engrg.* 32(1–3):199–259.

Ciarlet, P. G. (1980). *The Finite Element Method for Elliptic Problems*. Amsterdam, New York, Oxford: North-Holland Publishing.

Dugundji, J., Granas, A. (2003). *Fixed Point Theory*. New York: Springer-Verlag. ISBN 0-387-00173-5.

Fezoui, L., Lanteri, S., Lohrengel, S., Serge, S. (2005). emphConvergence and stability of a discontinuous Galerkin time-domain method for the 3D heterogeneous Maxwell equations on unstructured meshes. *M2AN Math. Model. Numer. Anal.* 39(6):1149–1176.

Glassey, R., Schaeffer, J. (1990). The "one and one-half-dimensional" relativistic Vlasov Maxwell system. *Math. Methods Appl. Sci.* 13(2):169–179.

Glassey, R., Schaeffer, J. (1997). The "two and one-half-dimensional" relativistic Vlasov Maxwell system. *Comm. Math. Phys.* 185(2):257–284.

Hansbo, P. (1994). Space-time oriented streamline diffusion methods for nonlinear conservation laws in one dimension. *Comm. Numer. Methods Engrg.* 10(3):203–215.

Hansbo, P., Rylander, T. (2010). A linear nonconforming finite element method for Maxwell's equations in two dimensions. Part I: frequency domain. *J. Comput. Phys.* 229(18):6534–6547.

Hughes, T., Franca, L., Hulbert, G. (1989). A new finite element formulation for computational fluid dynamics. VIII. The Galerkin/least-squares method for advective-diffusive equations. *Comput. Methods Appl. Mech. Engrg.* 73(2):173–189.

Johnson, C., Saranen, J. (1986). Streamline diffusion methods for the incompressible Euler and Navier-stokes equations. *Mathematics of Computation* 47:1–18.

Kellogg, R. B., Li, T. Y., Yorke, J. (1976). A constructive proof of the Brouwer fixed-point theorem and computational results. *SIAM J. Numer. Anal.* 13(4):473–483.

Principe, J., Codina, R. (2010). On the stabilization parameter in the subgrid scale approximation of scalar convection-diffusion-reaction equations on distorted meshes. *Comput. Methods Appl. Mech. Engrg.* 199(2–22):1386–1402.

Szepessy, A. (1991). Convergence of a streamline diffusion finite element method for scalar conservation laws with boundary conditions. *RAIRO Modl. Math. Anal. Numr.* 25(6):749–782.