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To cite this article: M. Asadzadeh, P. Kowalczyk & C. Standar (2019): Convergence of *hp*-Streamline Diffusion Method for Vlasov–Maxwell System, Journal of Computational and Theoretical Transport

To link to this article: <https://doi.org/10.1080/23324309.2019.1694542>



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Published online: 27 Nov 2019.



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Convergence of hp -Streamline Diffusion Method for Vlasov–Maxwell System

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ABSTRACT

In this paper we study stability and convergence for hp -streamline diffusion (SD) finite element method for the, relativistic, time-dependent Vlasov–Maxwell (VM) system. We consider spatial domain $\Omega_x \subset \mathbb{R}^3$ and velocities $\mathbf{v} \in \Omega_v \subset \mathbb{R}^3$. The objective is to show globally optimal *a priori* error bound of order $\mathcal{O}(h/p)^{s+1/2}$, for the SD approximation of the VM system; where h ($= \max_K h_K$) is the mesh size and p ($= \max_K p_K$) is the spectral order. Our estimates rely on the local version with h_K being the diameter of the *phase-space-time* element K and p_K the spectral order for K . The optimal hp estimates require an exact solution in the Sobolev space $H^{s+1}(\Omega)$. Numerical implementations, performed for examples in one space- and two velocity dimensions, are justifying the robustness of the theoretical results.

KEYWORDS

Vlasov–Maxwell; stability; convergence; hp method

1. Introduction

In the SD method the weak form is modified by adding a multiple of the streaming part in the equation, to the test function. So we obtain a multiple of streaming terms in both test and trial functions. This can be viewed as an extra diffusion in the streaming direction in the original equation. Hence, the name of the method (the streamline diffusion). Such an extra diffusion would improve both the stability and convergence properties of the underlying Galerkin scheme. It is well known that the standard Galerkin method, used for hyperbolic problems, has a suboptimal behavior: Converges as $\mathcal{O}(h^{s-1})$ (versus $\mathcal{O}(h^s)$ for the elliptic and parabolic problems with exact solution in the same space: $H^s(\Omega)$). The SD method improves this drawback by $\mathcal{O}(h^{1/2})$, also, having an unwinding character, enhances the stability. The properties that are achieved by the discontinuous Galerkin as well. The hp -approach is to capture local behavior in the sense

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that: in the vicinity of singularities refined mesh h is combined with the lower spectral order (small p), whereas in more smooth regions higher order polynomials (large p) and non-refined (large h) meshes are used. Hence, the hp -approach may be viewed as a kind of automatic adaptivity.

The Vlasov–Maxwell (VM) system describing the time evolution of collisionless plasma is formulated as

$$\begin{aligned}\partial_t f + \hat{v} \cdot \nabla_x f + q(E + c^{-1} \hat{v} \times B) \cdot \nabla_v f &= 0, \\ \partial_t E &= c \nabla_x \times B - j, \quad \nabla_x \cdot E = \rho, \\ \partial_t B &= -c \nabla_x \times E, \quad \nabla_x \cdot B = 0\end{aligned}\tag{1.1}$$

with, physically relevant, initial data $f(0, x, v) = f^0(x, v) \geq 0, E(0, x) = E^0(x), B(0, x) = B^0(x)$. Here f is the density, in phase space, time of particles with charge q , mass m and velocity

$$\hat{v} = (m^2 + c^{-2}|v|^2)^{-1/2} v \quad (v \text{ is momentum}).$$

Moreover, c is the speed of light and the charge and current densities: ρ and j are

$$\rho(t, x) = 4\pi \int qf \, dv \quad \text{and} \quad j(t, x) = 4\pi \int qf \hat{v} \, dv.\tag{1.2}$$

The Vlasov–Maxwell equations arise in several context such as, e.g., continuum, plasma physics and rarefied gas dynamics, where the main assumption underlying the model is that collisions are rare and therefore negligible. The system (1.1) is modeling the motion of a collisionless plasma, e.g., a high-temperature, low-density, ionized gas. The mathematical challenge in study of the Vlasov–Maxwell system is the presence of the nonlinear term: $(E + \hat{v} \times B) \cdot \nabla_v f$. In the case of divergence free field, this term is written as $\text{div}_v((E + \hat{v} \times B)f)$. A thorough mathematical analysis, with such nonlinearity, is studied by Di Perna and Lions in DiPerna and Lions (1989).

An outline of the remaining part of this paper is as follows: In [Section 2](#) we present the canonical form of the equations and state the numerical approximation strategy. [Section 3](#) describes hp and SD strategies. [Section 4](#) is devoted to the approximation of the projection error. In [Section 5](#) we study the convergence of the constructed schemes. In our concluding [Section 6](#) we implement some numerical examples in a lower dimensional case that justifies our theoretical study.

2. Canonical forms

As a general framework we introduce similar convective forms f or both Vlasov and Maxwell equations. The convection coefficients are *nonlinear vector functions* for the Vlasov equation and *constant sparse multiple*

matrices in the Maxwell case. Despite physical diversities, these coefficients are exhibiting a common property: they both are divergence free, a property that substantially simplifies the analysis. Let $E = (E_1, E_2, E_3)^T, B = (B_1, B_2, B_3)^T, j = (j_1, j_2, j_3)^T$, then the Maxwell's equations can be written as

$$\begin{cases} \partial_t E_1 = \partial_2 B_3 - \partial_3 B_2 - j_1, \\ \partial_t E_2 = \partial_3 B_1 - \partial_1 B_3 - j_2, \\ \partial_t E_3 = \partial_1 B_2 - \partial_2 B_1 - j_3, \end{cases} \quad \text{and} \quad \begin{cases} \partial_t B_1 = -\partial_2 E_3 + \partial_3 E_2, \\ \partial_t B_2 = -\partial_3 E_1 + \partial_1 E_3, \\ \partial_t B_3 = -\partial_1 E_2 + \partial_2 E_1, \end{cases}$$

where ∂_i denotes the derivative with respect to x_i . We also introduce 6×6 *symmetric* matrices M_k , $k = 1, 2, 3$:

$$\begin{cases} (M_1)_{35} = (M_1)_{53} = -1, & \text{else } (M_1)_{ij} = (M_1)_{ji} = 0, \\ (M_2)_{16} = (M_2)_{61} = -1, & (M_2)_{34} = (M_2)_{43} = 1, & \text{else } (M_2)_{ij} = (M_2)_{ji} = 0, \\ (M_3)_{24} = (M_3)_{42} = -1, & (M_3)_{15} = (M_3)_{51} = 1, & \text{else } (M_3)_{ij} = (M_3)_{ji} = 0. \end{cases} \quad (2.1)$$

To write the Maxwell equations in the *matrix-vector* form, we introduce the matrix-vector notation $\mathbf{M} = (M_1, M_2, M_3)$, and set $\mathbf{E} = (E_1, E_2, E_3)$, $\mathbf{B} = (B_1, B_2, B_3)$, $\mathbf{j} = (j_1, j_2, j_3)$, $\mathbf{0} = (0, 0, 0)$, $\mathbf{b} = (\mathbf{j}, \mathbf{0})$. Further let $\mathbf{W} = (\mathbf{E}, \mathbf{B})^T$ and use the notations for the initial values $\mathbf{E}^0, \mathbf{B}^0, \mathbf{W}^0$. Then

$$\begin{cases} \partial_t \mathbf{W} + \mathbf{M} \cdot \nabla \mathbf{W} = \mathbf{b}, \\ \mathbf{W}(0, x) = \mathbf{W}^0(x). \end{cases} \quad (2.2)$$

Note that (2.2) is convective with, matrix-vector form, divergent free ($\text{Div} \mathbf{M} = \mathbf{0}$) convection coefficient.

The Vlasov possesses a similar structure, but with nonlinear coefficient vectors depending on x and v :

$$\begin{cases} \partial_t f + \hat{v} \cdot \nabla_x f + (E + \hat{v} \times B) \cdot \nabla_v f = 0, \\ f(0, x, v) = f^0(x, v) \geq 0, \end{cases} \quad (2.3)$$

which can be rewritten in a compact form as

$$\partial_t f + G(f) \cdot \nabla f = 0. \quad (2.4)$$

where $\nabla f = (\nabla_x f, \nabla_v f)$ is the total gradient and $G(f) = (\hat{v}, E + \hat{v} \times B)$ is divergence free:

$$\text{div } G(f) = \sum_{i=1}^d \frac{\partial \hat{v}}{\partial x_i} + \sum_{i=d+1}^{2d} \frac{\partial (E + \hat{v} \times B)}{\partial v_{i-d}} = \nabla_v (\hat{v} \times B) = 0. \quad (2.5)$$

Throughout the paper C will denote a generic constant, not necessarily the same at each occurrence, unless otherwise explicitly specified.

2.1. The numerical approximation strategy

Consider a function F in an infinite dimensional Hilbert (or Banach) space \mathcal{F} . We want to approximate F with a function \tilde{F} in a finite dimensional subspace $\tilde{\mathcal{F}}$ of \mathcal{F} . To this end, first we assume a *classical form approximation* (an interpolant, or a projection, e.g. L_2 - or Ritz-projection) $\Pi F \in \tilde{\mathcal{F}}$ of F . Then, roughly, the idea is i) to use (or establish) a *theoretical error estimate*, in a certain norm, for the interpolation or projection error $F - \Pi F$, and ii) split the *numerical error* as

$$F - \tilde{F} = (F - \Pi F) + (\Pi F - \tilde{F}) := \eta + \xi,$$

and prove that there is a constant C such that

$$\|\xi\| \leq C\|\eta\|. \quad (2.6)$$

The intermediate (theoretical) part, i.e., control of $\|\eta\|$, is the most crucial step in the *efficiency* of the numerical approximation procedure. The numerical approach gets its reliability from the *stability* of the constructed numerical method and its *accuracy* from the convergence analysis.

3. The SD and hp structures

3.1. Constructing finite element spaces

For numerical studies we need to restrict the variables to bounded domains: $\Omega_x \subset \mathbf{R}^3$ and $\Omega_v \subset \mathbf{R}^3$, as the space and velocity, respectively. We assume that $f(t, x, v)$, $E_i(t, x)$ and $B_i(t, x)$, for $i = 1, 2, 3$, have compact supports in Ω_x and that $f(t, x, v)$ has compact support in both Ω_x and Ω_v . Assume that h ($0 < h < 1$), and let $T_h^x = \{\tau_x\}$ and $T_h^v = \{\tau_v\}$ be finite element subdivisions of Ω_x with elements τ_x and Ω_v with elements τ_v , respectively. Then $T_h = T_h^x \times T_h^v = \{\tau_x \times \tau_v\} = \{\tau\}$ is a subdivision of Ω . (with a, phase-space, mesh size h). Let now $0 = t_0 < t_1 < \dots < t_{N-1} < t_N = T$ be a partition of the time interval $[0, T]$ into sub-intervals $I_n = (t_{n-1}, t_n]$, with $|I_n| := t_n - t_{n-1} =: k_n \approx h$, $n = 1, \dots, N$. Further let \mathcal{C}_h be the corresponding subdivision of $Q_T = [0, T] \times \Omega$ into the prismatic elements $K = I_n \times \tau$, with $h_K = \text{diam } K$ as the mesh parameter. We also define a piecewise constant mesh function $h(t, x, v) := h_K$, $(t, x, v) \in K$. Finally, induced from \mathcal{C}_h , we introduce $\tilde{\mathcal{C}}_h$ as the finite element subdivision of $[0, T] \times \Omega_x$.

Remark 3.1. From now on, the discrete schemes are the finite element approximations of Equations (2.2) and (2.3) with $(x, v, t) \in (0, T] \times \Omega_x \times \Omega_v$, associated with initial and boundary conditions. Assuming also that f has compact support in the velocity space we get homogeneous Dirichlet boundary condition for Ω_v .

To formulate variational (weak) form of the VM system we introduce the product spaces

$$\mathcal{H}_0 = \prod_{n=1}^N \mathcal{H}_0^1(\Omega_n) \quad \text{and} \quad \tilde{\mathcal{H}}_0 = \prod_{n=1}^N \mathcal{H}_0^1(\tilde{\Omega}_n),$$

where $\Omega_n := I_n \times \Omega_x \times \Omega_v$, $\tilde{\Omega}_n := I_n \times \Omega_x$, and for $\Omega := \Omega_x \times \Omega_v$

$$\mathcal{H}_0^1(I_n \times \Omega) := \{w \in H^1(I_n \times \Omega); w = 0 \quad \text{on} \quad \partial\Omega\}.$$

For $k = 0, 1, 2, \dots$, we define the finite element spaces for the Maxwell's equations (resp. Vlasov equation) as being the space of piecewise polynomials that are continuous in x (resp. in x and v) with possible discontinuities at the interior time levels t_n , $n = 1, \dots, N - 1$:

$$\tilde{V}_h = \{g \in [\tilde{\mathcal{H}}_0]^6; g_i|_{\tilde{K}} \in P_{p_{\tilde{K}}}(I_n) \times P_{p_{\tilde{K}}}(\tau_x), \quad \forall \tilde{K} = I_n \times \tau_x \in \tilde{\mathcal{C}}_h, \quad 1 \leq i \leq 6\},$$

with the extension for the Vlasov part and with $\tau = \tau_x \times \tau_v$ as

$$V_h = \{g \in \mathcal{H}_0; g|_K \in P_{p_K}(I_n) \times P_{p_K}(\tau_x) \times P_{p_K}(\tau_v), \quad \forall K = I_n \times \tau \in \mathcal{C}_h\}.$$

where $P_{p_K}(\cdot)$ is the set of polynomial of degree at most p_K on the underlying set. Here we allow the degree of polynomial to vary from elementwise. Therefore we may define the piecewise constant function $p(t, x, v) := p_K$. Finally we shall use the following notation:

$$\begin{aligned} (f, g)_n &= (f, g)_{S_n}, \quad \|g\|_n = (g, g)_n^{1/2} \\ \langle f, g \rangle_n &= (f(t_n, \dots), g(t_n, \dots))_{\Omega}, \quad |g|_n = \langle g, g \rangle_n^{1/2}, \end{aligned}$$

where, for $n = 1, \dots, N$, $S_n := \tilde{\Omega}_n$, for the Maxwell's equations and $S_n := \Omega_n$ in the Vlasov case.

3.2. The SD strategy

The idea of the weak form for a differential equation, e.g., variational formulation and the finite element methods, rely on the basic structure of the Fourier transformation: to multiply the function f by a basis function $e^{-ix\xi}$ and integrate. Then for each $\xi \in \mathbf{R}$ we extract an information, $\hat{f}(\xi)$, from the function f (these are the Fourier coefficients when considering Fourier series). In this way we get an “excessive amount (∞) of information,” which (i) is unrealistic to tackle and (ii) contains a great deal of unnecessary data. The remedy is a clever choice of a finite number of, treatable, multipliers that provides the most crucial information needed. This is basically the goal of the Galerkin approach with finitely many multipliers. However, here we need to extract the information from the PDE itself rather than its solution. We seek an approximate solution for a PDE in a

finite dimensional space (trial space) with the multipliers being in the same or a closely related finite dimensional space known as the test function space. The idea with the SD method is to multiply the underlying PDE by a sum of the test function and a multiple of an expression as the streaming term in the PDE, with the solution is replaced by the test function, say g . Hence, we consider a common form for (2.2) and (2.4), viz.

$$F_t + \mu \cdot \nabla F = b, \quad (3.1)$$

and formulate the SD scheme to find the solution F such that

$$(F_t + \mu \cdot \nabla F, g + \delta \mu \cdot \nabla g) = (b, g + \delta \mu \cdot \nabla g). \quad (3.2)$$

Expanding Equation (3.2) we get a term of the form $\delta(\mu \cdot \nabla F, \mu \cdot \nabla g)$. Setting $g = F$, this term can be interpreted as coming from a diffusion term $-\delta \Delta_\mu F$, of order δ , in the streamline direction μ , hence the name of the method.

3.3. The hp approach

Here, we introduce the main function spaces employed in hp -studies for approximating the projection errors, (see, e.g. Houston et al. 2000): Let \mathcal{P} be a partition of $\Omega_T = (0, T) \times \Omega_x \times \Omega_v$ into open patches P which are images of canonical ‘‘cubes’’: $\hat{P} = (-1, 1)^{2d+1} := (\hat{I})^{2d+1}$, $d = 1, 2, 3$, under smooth bijections F_P :

$$\forall P \in \mathcal{P} : \quad P = F_P(\hat{P}).$$

We construct a mesh $\mathcal{T} := \cup_{P \in \mathcal{P}} \mathcal{T}_P$ on Ω_T by subdividing each $\hat{P} = (-1, 1)^{2d+1}$, into $2d + 1$ -dimensional generalized quadrilateral elements $\hat{\tau}$, or $2d + 1$ -dimensional prisms. We label them as $\hat{\tau}$ which are affine equivalent to \hat{P} , and call the mesh $\hat{\mathcal{T}}_P$ (on \hat{P}). Then, on each $P \in \mathcal{P}$ we define a mesh \mathcal{T}_P by setting

$$\forall P \in \mathcal{P} : \quad \mathcal{T}_P := \{\tau \mid \tau = F_P(\hat{\tau}), \quad \hat{\tau} \in \hat{\mathcal{T}}_P\}.$$

Each $\hat{\tau}(\tau)$ is an image of the domain \hat{P} under an affine mapping $A_{\hat{\tau}} : \hat{P} \rightarrow \hat{\tau}$ ($F_\tau = F_P \circ A_{\hat{\tau}} : \hat{P} \rightarrow \tau$). Next we define the space

$$F_{\mathcal{P}} = \{F_P : P \in \mathcal{P}\},$$

and the polynomial space

$$\mathcal{A}_p = \text{span}\{(\hat{x}, \hat{v}, \hat{t})^\alpha : 0 \leq \alpha_i \leq p, \quad 1 \leq i \leq 2d + 1\},$$

where $(\hat{x}, \hat{v}, \hat{t}) \in \hat{P} := \{(\hat{x}, \hat{v}, \hat{t}) \in \mathbf{R}^d \times \mathbf{R}^d \times \mathbf{R}^+ : |\hat{x}_j| \leq 1 \ \& \ |\hat{v}_j| \leq 1, \ \hat{t} \leq 1 \ j = 1, \dots, d\}$, and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{2d+1})$ is multi-index Moreover let \mathbf{p} be a polynomial degree vector in \mathcal{T} ,

$$\mathbf{p} = \{p_\tau : \tau \in \mathcal{T}\},$$

and define the continuous hp -finite element spaces

$$\mathbb{S}^{\mathbf{p},k}(\Omega_T, \mathcal{T}, F_{\mathcal{P}}) := \{f \in H^k(\Omega_T) : f|_\tau \circ F_\tau \in \mathcal{A}_{p_\tau}, \quad \tau \in \mathcal{T}\}, \quad k = 0, 1, \dots,$$

for polynomials with degree vector \mathbf{p} , and with

$$\mathbb{S}^{\mathbf{p},k}(\Omega_T, \mathcal{T}, F_{\mathcal{P}}) := \{f \in \mathbb{S}^{\mathbf{p},k}(\Omega_T, \mathcal{T}, F_{\mathcal{P}}) : \mathbf{p} = (p, p, \dots, p)\},$$

for the uniform polynomial degree $p_\tau = p$, $\forall \tau$, $p > 1$. Finally we denote by $\|f\|_{k,\hat{I}}$ and $|f|_{k,\hat{I}}$ the $H^k(\hat{I})$ norm and seminorm on \hat{I} , respectively (we shall suppress $k=0$, corresponding to the L_2 -norm). We also denote by $\mathbb{S}^p(\hat{I})$ the set of polynomials of degree p on \hat{I} . In the Maxwell case one needs to replace the domain $\Omega_T = \Omega_x \times \Omega_v \times (0, T)$ by $\tilde{\Omega}_T = \Omega_x \times (0, T)$.

4. Approximation of the projection error

The projection error estimates are of analytic nature and often predictable, however too involved to derive. In this section we provide estimates, without proofs, for $\|\eta\|$ and $\|\mathcal{D}\eta\|$, where $\mathcal{D} := (\nabla_x, \nabla_v, d/dt)$ denotes the total gradient operator. We denote by $\pi_p^i f$ the 1-dimensional H^1 -projection of f onto the polynomials of degree p in the i th coordinate, where $1 \leq i \leq d$ would correspond to x_i 's for the spatial variable, $d+1 \leq i \leq 2d$ to v_i 's for the velocity, and $i = 2d+1$ for the time variable. We shall apply the tensor product in $2d+1$ -dimensions to the following one-dimensional results by Schwab (1998).

Proposition 4.1. *Let $f \in H^{k+1}(\hat{I})$, $\hat{I} = (-1, 1)$ for some $k \geq 0$. Then, for every $p \geq 1$, there exists a projection $\pi_p f \in \mathbb{S}^p(\hat{I})$ such that, for any $0 \leq s \leq \min(p, k)$,*

$$\|f' - (\pi_p f)'\|_{\hat{I}}^2 \leq \frac{(p-s)!}{(p+s)!} |f|_{s+1,\hat{I}}^2, \quad (4.1)$$

$$\|f - \pi_p f\|_{\hat{I}}^2 \leq \frac{1}{p(p+1)} \frac{(p-s)!}{(p+s)!} |f|_{s+1,\hat{I}}^2. \quad (4.2)$$

The theorem below is a generalized version of [Proposition 4.1](#), in $2d+1$ dimensions with $\Pi_p = \prod_{i=1}^{2d+1} \pi_p^i$; the tensor product projectors and binary multi-index $|m|_l \equiv \sum_{n=1}^l m_n$, with $m_n = 0$ or 1 , is derived in Asadzadeh and Sopsakis (2007).

Theorem 4.2. *Let $\mathcal{T}_h = \cup K$ be a partition of the domain $D \subset \mathbf{R}^{\mathcal{N}}$, and $K \in \mathcal{T}_h$ be an image of the \mathcal{N} -dimensional canonical hypercube $\hat{R} := (-1, 1)^{\mathcal{N}}$, with \mathcal{N} -dimensional mesh $\mathcal{T}_{\hat{R}}$, under the bijection $G_K : \tau = G_K(\hat{R})$. For the polynomial degree distribution $\mathbf{r} = \{r_K \mid K \in \mathcal{T}_h\}$ and $\forall K \in \mathcal{T}_h$, let $f|_K \in$*

$H^{k_K+1}(K)$ for some $k_K \geq 1$. Then, for $r_K \geq 1$ and $0 \leq s_K \leq \min(r_K, k_K)$ we have

$$\begin{aligned} \|\eta\|^2 &\leq C \sum_{K \in \mathcal{T}_h} \left(\frac{h_K}{2}\right)^{2s_K+2} \Phi_1(r_K, s_K) \|\hat{f}\|_{s_K+1, \hat{K}}^2, \\ \|\mathcal{D}\eta\|^2 &\leq C \sum_{K \in \mathcal{T}_h} \left(\frac{h_K}{2}\right)^{2s_K} \Phi_2(r_K, s_K) \|\hat{f}\|_{s_K+1, \hat{K}}^2, \end{aligned}$$

with $\hat{f} = f \circ G_\tau$, $K = G_\tau(\hat{K})$, $\|\cdot\|_{s_K+1, \hat{K}}$ is the Sobolev norm in $H^{s_K+1}(\hat{K})$ and

$$\begin{aligned} \Phi_1(p, s) &= \mathcal{N} \sum_{i=1}^{\mathcal{N}} 2^{i-1} \sum_{|m|_{i-1} \leq i-1} \alpha_p^{|m|_{i-1}+1} \beta_{|m|_{i-1}}, \\ \Phi_2(p, s) &= \mathcal{N} \sum_{i=1}^{\mathcal{N}} \sum_{j=1}^{\mathcal{N}} 2^j \sum_{\substack{|m|_{j-1} \leq j-1 \\ m_i=1}} \alpha_p^{|m|_{j-1}} \beta_{|m|_{j-1}}. \end{aligned}$$

where $\partial^{|m|_{i-1}} = \partial_1^{m_1} \partial_2^{m_2} \dots \partial_{i-1}^{m_{i-1}}$, $\alpha_p = \frac{1}{p(p+1)}$, and $\beta_{|m|_k} = \frac{(p-s+|m|_k)!}{(p+s-|m|_k)!}$.

The proof is based on a scaling argument for an affine mapping and a generalization of the [Proposition 4.1](#).

Remark 4.3. The above estimates relying on Sobolev inequalities are dimension dependent and give rise to singularities in higher dimensions. In addition, the bound for s ($0 \leq s \leq \min(p, k)$) and the fact that the spectral order p cannot be chosen very large, limits the value of $|m|_k$ in $\beta_{|m|_k}$ to $|m|_k < p + s$. This yields additional bounds for the indices of the interior sums.

Using Stirling's formula, see Houston et al. (2000), one can show that

$$\max(\Phi_2(p, s), \Phi_2(p, s)) \leq Cp^{-2s}.$$

An estimate that is crucial in deriving p-version of hp estimates.

5. Convergence analysis

Due to the common structure for the Maxwell (2.2) and Vlasov (2.4) equations, and since the hp-SD approach has been considered for (2.4) in Asadzadeh and Sopsakis (2007), below we give the details only in the Maxwell case (2.2).

5.1. hp-method for Maxwell equations

Starting with, sufficiently smooth, f^0 , b^0 and W^0 , let $f^{h,i}$, $b^{h,i}$ and $W^{h,i}$ denote the approximations on the i th iteration step ($i > 0$) for f , b and W ,

respectively. The h version of the SD method on the i th step for the Maxwell's part (with the short operator notation: $M \cdot \nabla := \sum_{\ell=1}^3 M_\ell \partial_\ell$, recall that M_ℓ are the 6×6 matrices in (2.1)) can be formulated as: find $W^{h,i} \in \tilde{V}_h$ such that

$$\tilde{\mathcal{B}}(W^{h,i}, g) = \tilde{\mathcal{L}}(b^{h,i-1}; g) \quad \forall g \in \tilde{V}_h, \quad (5.1)$$

where $g = (g_1, \dots, g_6)^T$, with the bilinear form $\tilde{\mathcal{B}}$ defined as

$$\begin{aligned} \tilde{\mathcal{B}}(W, g) &= \sum_{n=0}^{N-1} (\partial_t W + M \cdot \nabla W, g + \delta(\partial_t g + M \cdot \nabla g))_n + \sum_{n=1}^{N-1} \langle [W], g_+ \rangle_n \\ &\quad + \langle W_+, g_+ \rangle_0. \end{aligned}$$

$[W] = W_+ - W_-$ denotes the jump ($g_\pm(t, x) = \lim_{s \rightarrow 0^\pm} g(t + s, x)$), and the linear form $\tilde{\mathcal{L}}$ is given by

$$\tilde{\mathcal{L}}(b; g) = \sum_{n=0}^{N-1} (b, g + \delta(\partial_t g + M \cdot \nabla g))_n + \langle W^0, g_+ \rangle_0.$$

Let now $(\cdot, \cdot)_K$ denote the L_2 product over K and define a nonnegative piecewise constant function δ by

$$\delta|_K = \delta_K, \quad \text{for } K \in \tilde{\mathcal{C}}_h.$$

We may formulate a finite element method based on the local space-time elements. Then, the problem (5.1) would have an alternative formulation replacing in the definitions for $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{L}}$ the sum of the inner products $(\cdot, \cdot)_n$ involving δ_K by the corresponding sum $\sum_{K \in \tilde{\mathcal{C}}_h} (\cdot, \cdot)_K$ and all δ by δ_K . Then, (5.1) will be rewritten as

$$\begin{aligned} \tilde{\mathcal{B}}(W, g) &= \sum_{K \in \tilde{\mathcal{C}}_h} (\partial_t W + M \cdot \nabla W, g + \delta_K(\partial_t g + M \cdot \nabla g))_K \\ &\quad + \sum_{n=1}^{N-1} \langle [W], g_+ \rangle_n + \langle W_+, g_+ \rangle_0, \end{aligned} \quad (5.2)$$

and the solution W of Equation (2.2) satisfies the variational formulation

$$\tilde{\mathcal{B}}(W, g) = \tilde{\mathcal{L}}(b; g) \quad \forall g \in \tilde{V}_h. \quad (5.3)$$

Subtracting Equation (5.1) from Equation (5.3) we end up with the following relation:

$$\tilde{\mathcal{B}}(W - W^{h,i}, g) = \tilde{\mathcal{L}}(b; g) - \tilde{\mathcal{L}}(b^{h,i-1}; g) \quad \forall g \in \tilde{V}_h. \quad (5.4)$$

Finally, with jumps at the time levels $t = t_n$, $n = 1, \dots, N-1$, we introduce the computational norm

$$|||g|||^2 = \frac{1}{2} \left(|g_+|_0^2 + |g_-|_N^2 + \sum_{n=1}^{N-1} |[g]|_n^2 \right) + \sum_{K \in \tilde{\mathcal{C}}_h} \delta_K \|\partial_t g + M \cdot \nabla g\|_K^2.$$

Lemma 5.1. [Well-posedness] *The bilinear form $\tilde{\mathcal{B}}(\cdot, \cdot)$ satisfies the M -coercivity relation*

$$\tilde{\mathcal{B}}(g, g) = |||g|||^2, \quad \forall g \in \tilde{\mathcal{H}}_0.$$

Proof. We split $\tilde{\mathcal{B}}$, viz.

$$\tilde{\mathcal{B}}(g, g) = \sum_{K \in \tilde{\mathcal{C}}_h} \delta_K \|\partial_t g + M \cdot \nabla g\|_K^2 + A_1 + A_2,$$

with A_i , $i = 1, 2$ defined as

$$\begin{aligned} A_1 &:= \sum_{n=0}^{N-1} (\partial_t g, g) + \sum_{n=1}^{N-1} \langle [g], g_+ \rangle_n + |g_+|_0^2 = \frac{1}{2} \left(\sum_{n=1}^{N-1} |[g]|_n^2 + |g_-|_N^2 + |g_+|_0^2 \right) \\ A_2 &:= \sum_{m=0}^{M-1} (M \cdot \nabla g, g)_m = 0, \end{aligned} \tag{5.5}$$

where we used integration by parts and in the latter also $g(t, x) = 0$ on $I \times \partial\Omega_x$. Then, the proof follows adding all above terms and the definition of $|||g|||_M$. \square

Let now \tilde{W} be an interpolant of W in the finite dimensional space \tilde{V}_h and denote by $W^{h,i}$ a solution to (5.1). Further, set $\tilde{\eta} = (\tilde{\eta}_1, \dots, \tilde{\eta}_6)^T$, $\tilde{\xi} = (\tilde{\xi}_1, \dots, \tilde{\xi}_6)^T$ and introduce a split of the error \tilde{e} , viz.

$$\tilde{e} = W - W^{h,i} = (W - \tilde{W}) - (W^{h,i} - \tilde{W}) = \tilde{\eta} - \tilde{\xi}.$$

Now we can state and prove our main result: the hp -convergence theorem for the Maxwell's equations.

Theorem 5.2. *Assume that $W \in H^{k+1}([0, T] \times \Omega)$, $\delta_K = C_1 \frac{h_K}{p_K}$ for some constant $C_1 > 0$, and $p_K h_K \leq C_2 < 1$ for some constant $C_2 > 0$. Then there exists a constant $C > 0$ such that*

$$|||W - W^{h,i}|||^2 \leq C \sum_{K \in \tilde{\mathcal{C}}_h} h_K^{2s_K+1} p_K^{-1} \Phi_M(p_K, s_K) \|W\|_{s_K+1, K}^2 + C \|f - f^{h,i-1}\|_{Q_T}^2,$$

where $\Phi_M = \max(\Phi_1, \Phi_2)$ with $\mathcal{N} = \dim \Omega_x + 1$ for Φ_1 and Φ_2 (defined in Theorem 4.2) and C is independent of p_K , h_K and s_K .

Proof. By the coercivity [Lemma 5.1](#) and [\(5.4\)](#) we have that

$$\begin{aligned} |||\tilde{\xi}|||^2 &= \tilde{\mathcal{B}}(\tilde{\xi}, \tilde{\xi}) = \tilde{\mathcal{B}}(\tilde{\eta}, \tilde{\xi}) + \tilde{\mathcal{B}}(\tilde{\xi} - \tilde{\eta}, \tilde{\xi}) \\ &= \tilde{\mathcal{B}}(\tilde{\eta}, \tilde{\xi}) - \tilde{\mathcal{L}}(b; \tilde{\xi}) + \tilde{\mathcal{L}}(b^{h,i-1}; \tilde{\xi}). \end{aligned}$$

We start estimating the first term on the right-hand side:

$$\begin{aligned} \tilde{\mathcal{B}}(\tilde{\eta}, \tilde{\xi}) &= \sum_{K \in \tilde{\mathcal{C}}_h} (\partial_t \tilde{\eta} + M \cdot \nabla \tilde{\eta}, \tilde{\xi} + \delta_K (\partial_t \tilde{\xi} + M \cdot \nabla \tilde{\xi}))_K + \sum_{n=1}^{N-1} \langle [\tilde{\eta}], \tilde{\xi}_+ \rangle_n \\ &\quad + \langle \tilde{\eta}_+, \tilde{\xi}_+ \rangle_0. \end{aligned}$$

Integration by parts: $(\partial_t \tilde{\eta}, \tilde{\xi})_n = \langle \tilde{\eta}_-, \tilde{\xi}_- \rangle_{n+1} - \langle \tilde{\eta}_+, \tilde{\xi}_+ \rangle_n - (\tilde{\eta}, \partial_t \tilde{\xi})_n$. That $\tilde{\eta}$ and $\tilde{\xi}$ have compact support in Ω_x yields $(M \cdot \nabla \tilde{\eta}, \tilde{\xi})_n = -(\tilde{\eta}, M \cdot \nabla \tilde{\xi})_n$. Inserting in $\tilde{\mathcal{B}}(\tilde{\eta}, \tilde{\xi})$, by standard inequalities

$$\begin{aligned} |\tilde{\mathcal{B}}(\tilde{\eta}, \tilde{\xi})| &\leq \frac{1}{16} |||\tilde{\xi}|||^2 + 32 \sum_{n=0}^{N-1} |\tilde{\eta}_-|_{n+1}^2 \\ &\quad + \sum_{K \in \tilde{\mathcal{C}}_h} \left(\frac{32}{\delta_K} \|\tilde{\eta}\|_K^2 + 32 \delta_K \|\partial_t \tilde{\eta} + M \cdot \nabla \tilde{\eta}\|_K^2 \right). \end{aligned}$$

With a similar argument we can bound the second term as

$$\begin{aligned} |\tilde{\mathcal{L}}(b^{h,i-1}; \tilde{\xi}) - \tilde{\mathcal{L}}(b; \tilde{\xi})| &\leq \frac{1}{16} |||\tilde{\xi}|||^2 + \sum_{K \in \tilde{\mathcal{C}}_h} \left(\frac{C}{2} + 8\delta_K \right) \|b - b^{h,i-1}\|_K^2 \\ &\quad + \sum_{n=0}^{N-1} \left(\frac{1}{2C} \|\tilde{\eta}\|_n^2 + \frac{1}{2C} \|\tilde{e}\|_n^2 \right). \end{aligned}$$

The above inequalities and a Poincaré inequality, with properly chosen C and bound on $p_K h_K$, yield

$$\begin{aligned} |||\tilde{\xi}|||^2 &\leq \frac{1}{2} |||\tilde{e}|||^2 + C \|f - f^{h,i-1}\|_{Q_T}^2 + C \sum_{n=0}^{N-1} \left(|\tilde{\eta}_-|_{n+1}^2 + h |\tilde{e}_-|_{n+1}^2 \right) \\ &\quad + C \sum_{K \in \tilde{\mathcal{C}}_h} \left((1 + \delta_K^{-1}) \|\tilde{\eta}\|_K^2 + \delta_K \|\partial_t \tilde{\eta} + M \cdot \nabla \tilde{\eta}\|_K^2 \right), \end{aligned} \tag{5.6}$$

$$\begin{aligned} |||\tilde{e}|||^2 &\leq |||\tilde{\eta}|||^2 + |||\tilde{\xi}|||^2 \leq \frac{1}{2} |||\tilde{e}|||^2 + C \|f - f^{h,i-1}\|_{Q_T}^2 \\ &\quad + C \sum_{n=1}^N h |\tilde{e}_-|_n^2 + C(J_1 + J_2), \end{aligned}$$

where in the latter I_1 and I_2 are the interpolation error terms given by:

$$I_1 := \sum_{K \in \tilde{\mathcal{C}}_h} \left(\delta_K^{-1} \|\tilde{\eta}\|_K^2 + \delta_K \|\partial_t \tilde{\eta} + M \cdot \nabla \tilde{\eta}\|_K^2 \right),$$

$$I_2 := |\tilde{\eta}_+|_0^2 + \sum_{n=0}^{N-1} |\tilde{\eta}_-|_{n+1}^2 + \sum_{n=1}^{N-1} \|\tilde{\eta}\|_n^2.$$

To bound I_1 we use the estimates in [Theorem 4.2](#) and get

$$I_1 \leq C \sum_{K \in \tilde{\mathcal{C}}_h} \left(\frac{h_K}{2} \right)^{2s_K} \Phi_M(p_K, s_K) p_K^{-2} (\delta_K^{-1} h_K^2 + \delta_K) \|W\|_{s_K+1, K}^2. \quad (5.7)$$

By trace and inverse inequalities combined with interpolation estimates and [Theorem 4.2](#) we end up with

$$I_2 \leq C \sum_{K \in \tilde{\mathcal{C}}_h} \left[\left(\frac{h_K}{2} \right)^{2s_K+1} \left(\Phi_M^{1/2}(p_K, s_K) p_K^{-1} \Phi_M^{1/2}(p_K, s_K) + p_K^{-2} \Phi_M(p_K, s_K) \right) \right] \|W\|_{s_K+1, K}^2, \quad (5.8)$$

Now, a kick-back argument and the estimates [\(5.7\)](#) and [\(5.8\)](#) gives

$$\begin{aligned} \|\tilde{e}\|^2 &\leq C \sum_{K \in \tilde{\mathcal{C}}_h} h_K^{2s_K+1} p_K^{-1} \Phi_M(p_K, s_K) \|W\|_{s_K+1, K}^2 + C \|f - f^{h, i-1}\|_{Q_T}^2 \\ &\quad + C \sum_{n=1}^N h |\tilde{e}_-|_n^2. \end{aligned} \quad (5.9)$$

Finally it is easy to verify the discrete Grönwall's inequality,

$$|\tilde{e}_-|_\ell^2 \leq C \sum_{K \in \tilde{\mathcal{C}}_h} h_K^{2s_K+1} p_K^{-1} \Phi_M(p_K, s_K) \|W\|_{s_K+1, K}^2 + C \|f - f^{h, i-1}\|_{Q_T}^2$$

for $\ell = 1, \dots, N$. Plugging these inequalities into [\(5.9\)](#) completes the proof. \square

Corollary 5.3. *By the above theorem and the estimate in Schwab (1998), we can derive $\Phi_M(p_K, s_K) \leq P_n^{-2s_K}$ and hence, suppressing the subscripts in final setting get a hp error estimate as*

$$\|\tilde{e}\| \leq C \left(\frac{h}{p} \right)^{s+1/2}, \quad (5.10)$$

which is the most desired form of optimal hp error estimate.

5.1.1. Vlasov–Maxwell equations

As mentioned in the introduction, for the Vlasov part, we omit all details and state the final result:

Theorem 5.4. Let $f^{h,i}$ be the discrete solution to the Vlasov equation at the iteration step i and assume that the exact solution f of (2.3) is in the Sobolev class $H^{K+1}(Q_T)$ and satisfies the bound

$$\nabla f_\infty + G(f)_\infty + \nabla \eta_\infty \leq C. \quad (5.11)$$

Further assume that the parameter δ_K on each K satisfies $\delta_K = C_1 \frac{h_K}{p_K}$ for some positive constant C_1 with $p_K h_K \leq C_2 < 1$ for some constant $C_2 > 0$. Then there exists a generic constant $C > 0$ such that

$$\begin{aligned} \| \|f - f^{h,i}\| \|_{\mathbb{V}}^2 \leq C \left(\sum_{K \in \tilde{\mathcal{C}}_h} h_K^{2s_K+1} p_K^{-1} \Phi_M(p_K, s_K) \|W\|_{s_K+1, K}^2 + ph \|f - f^{h,i-1}\|_{Q_T}^2 \right. \\ \left. + \sum_{K \in \mathcal{C}_h} h_K^{2s_K+1} p_K^{-1} \Phi_V(p_K, s_K) \|f\|_{s_K+1, K}^2 \right). \end{aligned} \quad (5.12)$$

Here, $0 \leq s_K \leq \min(p_K, k)$ and the subscripts V or M in the triple norms as well as Φ :s is to emphasize the quantities for the Vlasov or Maxwell part. Here, $\Phi_V = \max(\Phi_1, \Phi_2)$ with $\mathcal{N} = \dim \Omega_x + \dim \Omega_v + 1$ for Φ_1 and Φ_2 as defined in Theorem 4.2.

We omit the proof which is a lengthy extension of the proof of the main result in Asadzadeh and Sopasakis (2007), replacing the contribution of scalar Fokker-Planck term with that of the Maxwell's system.

5.2. Conservation of charge

In this section we state the conservation of charge (mass conservation) result.

Theorem 5.5. For any $T > 0$ and the initial data f^0 as in the convergence theorem, the discrete solution $f^h \in V_h$ of the Vlasov equation satisfies

$$\int_{\Omega} f^h(T, x, v) \, dx dv = \int_{\Omega} f^0(x, v) \, dx dv.$$

Proof. Let B be the convex open set such that $f^h(t, x, v) = 0$ for all $t \in [0, T]$ and $(x, v) \in \Omega \setminus B$. Let $g \in V_h$ such that $g(t, x, v) = 1$ for all $t \in [0, T]$ and $(x, v) \in B$. Taking this g as a test function in the variational formulation of the Vlasov equation and using the fact that $\operatorname{div} G = 0$ we get the result. \square

6. Numerical results

We present in this section results of some numerical implementations that justify the accuracy of the SD method. The computations are performed for

Table 1. L_1 and L_2 errors for different polynomial degrees and fixed mesh sizes set H_1 .

Error	Degree	f	E_1	E_2	B
L_1	$p=1$	3.801e-1	7.086e-4	1.599e-6	1.645e-5
	$p=2$	1.614e-1	3.248e-9	1.770e-7	9.092e-7
	$p=3$	1.891e-2	2.295e-10	9.753e-9	3.321e-8
L_2	$p=1$	7.302e-1	6.204e-7	3.517e-12	4.303e-10
	$p=2$	1.632e-1	1.498e-17	4.113e-14	1.070e-12
	$p=3$	2.833e-3	6.648e-20	1.185e-16	2.186e-15

the simplified case of one spatial variable and two velocities (cf., Standar 2016), which takes the following form:

$$\partial_t f + v_1 \partial_x f + (E_1 + v_2 B) \partial_{v_1} f + (E_2 - v_1 B) \partial_{v_2} f = 0,$$

$$\partial_t E_1 = - \int v_1 f dv = -j_1(t, x),$$

$$\partial_t E_2 + \partial_x B = - \int v_2 f dv = -j_2(t, x),$$

$$\partial_t B + \partial_x E_2 = 0,$$

where $f = f(t, x, v_1, v_2)$, $E_1 = E_1(t, x)$, $E_2 = E_2(t, x)$, and $B = B(t, x)$ where we take $x \in \Omega_x \subset \mathbf{R}$ and $v = (v_1, v_2) \in \Omega_v \subset \mathbf{R}^2$. Here we consider the non-relativistic case since there is an excessive amount of literature available to compare the numerical results. Our results are also valid in this case. We consider the following, somewhat general, initial conditions:

$$f(0, x, v_1, v_2) = \frac{1}{\pi\beta} e^{-v_1^2/\beta} [\mu e^{-(v_2 - v_0,1)^2/\beta} + (1 - \mu) e^{-(v_2 + v_0,2)^2/\beta}],$$

$$E_1(0, x) = E_2(0, x) = 0, \quad B(0, x) = -b \sin(k_0 x_1).$$

They correspond to the streaming Weibel instability (cf Cheng et al. 2014) with the specific values for the parameters: $\beta = 0.01$ and $b = 0.001$. Our implementations are performed for $x \in [0, L]$, $L = 2\pi/k_0$ and the following two parameter sets:

$$\text{case 1: } \mu = 0.5, v_{0,1} = v_{0,2} = -0.3, k_0 = 0.2,$$

$$\text{case 2: } \mu = 1/6, v_{0,1} = -0.5, v_{0,2} = -0.1, k_0 = 0.2.$$

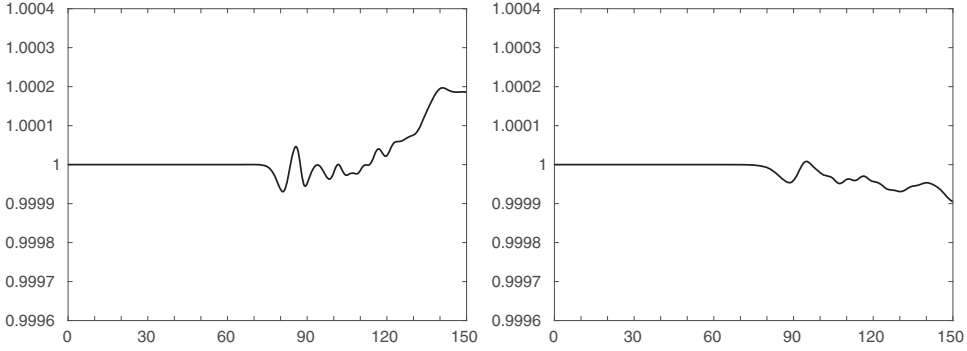
We assume periodic boundary condition in x , which we normalized in the computations taking $x \in [0, 1]$. The accuracy tests are performed for $\Omega_v = [-1, 1]^2$, whereas for the other test we set $\Omega_v = [-1.1, 1.1]^2$.

6.1. Accuracy tests

For the above setting, the Vlasov–Maxwell system is reversible in the time variable. Hence, with the initial condition $f(0, x, v)$, $E(0, x)$, $B(0, x)$, yield, for $t = T$, the solution $f(T, x, v)$, $E(T, x)$, and $B(T, x)$. Taking $f(T, x, -v)$,

Table 2. L_1 and L_2 errors for different mesh sizes and fixed polynomial degree $p = 1$.

Error	Mesh sizes set	f	E_1	E_2	B
L_1	H_1	3.801e-1	7.086e-4	1.599e-6	1.645e-5
	H_2	1.629e-1	8.304e-10	1.791e-7	8.387e-6
	H_3	4.324e-2	2.016e-10	4.750e-8	2.099e-6
L_2	H_1	7.302e-1	6.204e-7	3.517e-12	4.303e-10
	H_2	1.939e-1	8.520e-19	3.956e-14	9.298e-11
	H_3	1.444e-2	5.014e-20	2.784e-15	5.850e-12


Figure 1. Charge vs. time for case 1 (left) and case 2 (right) of the streaming Weibel instability.

$E(T, x)$, $-B(T, x)$ as solution at $t=0$, we recover $f(0, x, -v)$, $E(0, x)$, $-B(0, x)$ at $t=T$.

We run the calculation for $T=5$ and show L_1 and L_2 errors of solutions for several choices of degree of polynomials p and mesh parameters h_t , h_x , and h_v . For all calculations we used the uniform degrees p in all cells of uniform meshes. We present the results for the following choice of mesh sizes sets: H_1 corresponds to $h_t = h_x = 0.1$ and $h_v = \sqrt{2}/6$; H_2 corresponds to $h_t = h_x = 0.05$ and $h_v = \sqrt{2}/12$; H_3 corresponds to $h_t = h_x = 0.025$ and $h_v = \sqrt{2}/24$.

Table 1 lists the errors for the fixed mesh set H_1 and increasing degree of finite elements polynomial approximation, whereas in Table 2 we list the errors for the fixed degree $p = 1$ and different mesh sizes.

The tables show the convergence of our hp -SD algorithm for all considered cases. We present the results for one value of the stability parameter $\delta = 0.05$, since its choice does not influence importantly the accuracy, but only the stability of the method.

6.2. Conservation properties

To verify the conservation properties of our scheme we run the tests for the streaming Weibel instability. The results are shown in Figure 1 for the mass and Figure 2 for the total energy. Both moments are very well conserved up to time $t \approx 70$ for case 1 and $t \approx 75$ for case 2, when the

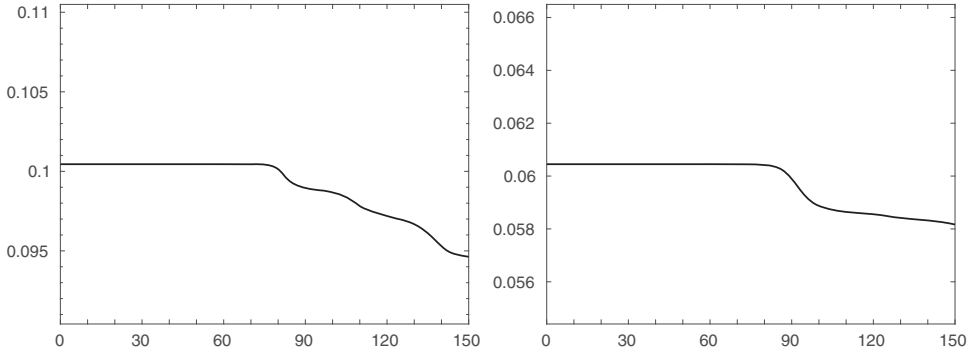


Figure 2. Total energy vs. time for case 1 (left) and case 2 (right) of the streaming Weibel instability.

instabilities occur. After that the error for the mass is less than $2 \cdot 10^{-4}$, however we can observe the large decay in the total energy.

7. Conclusions

This study is devoted to the hp streamline diffusion for the relativistic Vlasov–Maxwell system in 3D. Our objective is to present space-time (for Maxwell’s) and phase-space-time (for VM system) discretization schemes that have optimal order of convergence $\mathcal{O}\left(\left(h/p\right)^{s+1/2}\right)$ for solutions in $H^{s+1}(\Omega)$, where h is the mesh size and p the spectral order. The adaptivity in a priori regime is based on refining in the vicinity of singularities combined with lower order approximating polynomials and nonrefined mesh with a higher spectral order in smooth regions. To our knowledge, except in some work in convection–diffusion problems, e.g. Asadzadeh and Sopsakis (2007), Houston et al. (2000), such approach is not considered elsewhere.

The results are justified, in lower dimensional cases, through the presented accuracy and the streaming Weibel instability tests. In all our simulations we used uniformly spaced grids in all variables and uniform degree of polynomials, and in the future paper we plan to examine the non-uniform grids and degree cases.

The full-dimensions are too complex and expensive to experiment. However, the theoretical analysis and numerical justifications in lower dimensions are indicating the robustness of the considered schemes.

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