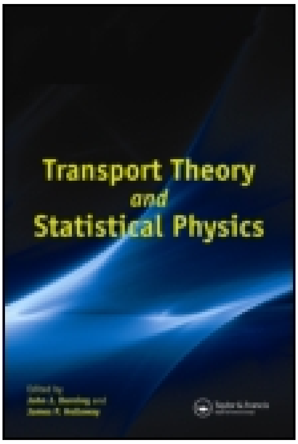

Transport Theory and Statistical Physics
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/ltty20

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Tobias Gebäck [a] [b] & Mohammad Asadzadeh [a]

[a] Department of Mathematics, Chalmers University of Technology and University of Gothenburg, Göteborg, Sweden

[b] Department of Oncology-Pathology, Karolinska Institute, Stockholm, Sweden

Version of record first published: 16 Oct 2012.

To cite this article: Tobias Gebäck & Mohammad Asadzadeh (2012): Analytical Solutions for the Pencil-Beam Equation with Energy Loss and Straggling, Transport Theory and Statistical Physics, 41:5-6, 325-336

To link to this article: http://dx.doi.org/10.1080/00411450.2012.671207

Transport Theory and Statistical Physics, 41:325–336, 2012
Copyright © Taylor & Francis Group, LLC
ISSN: 0041-1450 print / 1532-2424 online
DOI: 10.1080/00411450.2012.671207

Analytical Solutions for the Pencil-Beam Equation with Energy Loss and Straggling

Tobias Gebäck[1] and Mohammad Asadzadeh[2]

[1]Department of Mathematics, Chalmers University of Technology and University of Gothenburg, Göteborg, Sweden; Department of Oncology-Pathology, Karolinska Institute, Stockholm, Sweden
[2]Department of Mathematics, Chalmers University of Technology and University of Gothenburg, Göteborg, Sweden

In this article, we derive equations approximating the Boltzmann equation for charged particle transport under the continuous slowing down assumption. The objective is to obtain analytical expressions that approximate the solution to the Boltzmann equation. The analytical expressions found are based on the Fermi-Eyges solution, but include correction factors to account for energy loss and spread. Numerical tests are also performed to investigate the validity of the approximations.

Keywords Fermi-Eyges equation; energy loss straggling; analytical solution

1. INTRODUCTION

Since its introduction by Fermi in 1941 (quoted by Rossi and Greisen, 1941) and generalization by Eyges (1948), the Fermi-Eyges equation for charged particle transport has attracted much attention. The main advantage of the equation is the existence of an analytical solution, both for the original Dirac delta-function boundary condition and for Gaussian boundary conditions (ICRU, 1984). In many cases, the Fermi-Eyges solution gives good enough estimates of the fluence of charged particles to be of use in dose calculations, both for electrons (ICRU, 1984; Hogstrom et al., 1981) and for light ions (Luo and Brahme, 1993; Carlsson et al., 1997). Still, there is a need for more accurate analytical expressions to get quick estimates of fluence and dose distribution. Therefore,

This research is partially supported by the *Swedish Foundation of Strategic Research* (SSF) in *Gothenburg Mathematical Modeling Center* (GMMC) and *Swedish Research Counsil* (VR).
Address correspondence to Tobias Gebäck, Department of Mathematics, Chalmers University of Technology and University of Gothenburg, SE-412 96 Göteborg, Sweden. E-mail: tobias.geback@chalmers.se

there have been attempts at deriving correction factors for the Fermi-Eyges solution to increase the accuracy of predictions. One such factor was derived by Kempe and Brahme (2010) to include absorption and energy loss in the model. However, the derivation by Kempe and Brahme was somewhat unclear mathematically and the underlying assumptions were not apparent. Therefore, the first part of this article is devoted to derive the same correction factor as Kempe and Brahme, with clearly stated assumptions and a mathematical treatment that answers some of the remaining questions.

In the second part, the analysis is extended to include energy loss straggling; and an analytical correction factor for this case is also derived. The resulting analytical expression gives an estimate of the energy loss and spread in the particle beam, which is not included in the original Fermi-Eyges solution. In Section 3, we also compare the expression with numerical solutions to check the validity of the underlying assumptions.

2. DERIVATION OF ENERGY-DEPENDENT CORRECTION FACTORS

2.1. Basic Setup and Assumptions

We assume that a charged particle beam is normally incident in the z-direction on the surface of a homogeneous semi-infinite medium at $z = 0$. This situation is described by the following charged particle transport equation for the fluence $f(\mathbf{r}, \Omega, E)$ of particles at position $r = (x, y, z)$, moving in direction $\Omega \in S^2$ with kinetic energy $E \geq 0$. The continuous slowing down assumption (CSDA) with energy-loss straggling (ELS) yields derivatives in energy so that the transport equation takes the form

$$\begin{aligned}\Omega \cdot \nabla f(r, \Omega, E) + \sigma_a(E) f(r, \Omega, E) - \frac{\partial}{\partial E}(S(E) f(r, \Omega, E)) - \frac{1}{2}\frac{\partial^2}{\partial E^2}(\omega(E) f(r, \Omega, E)) \\ = \int_{S^2} \sigma_s(E, \Omega' \cdot \Omega)(f(\mathbf{r}, \Omega', E) - f(\mathbf{r}, \Omega, E)) d\Omega', \qquad (2.1)\end{aligned}$$

where $S(E)$ is the stopping power, and $\omega(E)$ is the energy-straggling coefficient (corresponding to CSDA and ELS, respectively, cf. Luo and Brahme, 1992). Furthermore, σ_s is the scattering cross-section, and σ_a is the absorption cross-section, which may for example model the fragmentation of light ions or thermalization of particles below a certain cut-off energy.

We wish to apply the boundary condition

$$f(\mathbf{r}, \Omega, E)\big|_{z=0} = G_1(x, y, \Omega) G_2(E), \qquad (2.2)$$

where $G_1(x, y, \Omega)$ is a Gaussian as for the original Fermi-Eyges equation (ICRU, 1984) to be specified next, and $G_2(E)$ will be either a Dirac function or a Gaussian.

A general closed form analytical solution for (2.1)–(2.2) is not available. Our aim here is, under certain assumptions, to derive approximations for (2.1) that we can solve analytically. Then, we show that such obtained analytical expressions yield a good approximate solution for the Equation (2.1)–(2.2). To this end, we will make the following approximations:

A1. The collision integral will be approximated using the Fermi-Eyges operator.

A2. In the scattering cross-section, $\sigma_s(E, \Omega' \cdot \Omega)$, we will replace the energy E by the depth-dependent mean energy $E_a(z)$ and assume an explicit dependence on the depth z.

A3. The straggling term, $\frac{\partial^2(\omega f)}{\partial E^2}$, will be ignored in Section 2.2. In Sections 2.3 and 3, however, it will be included in order to approximate the spread in energies.

A4. The narrow energy spectrum approximation (NESA) will be applied in Section 2.3 in order to obtain an analytical solution. In Section 3, its validity will be investigated using numerical computations.

Using the approximations A1–A3, Equation (2.1) turns into the following equation for the fluence $f = f(\mathbf{r}, \theta_x, \theta_y, E)$, where (θ_x, θ_y) is the projection of the particle movement direction, Ω, onto the (x, y)-plane.

$$\frac{\partial f}{\partial z} + \theta_x \frac{\partial f}{\partial x} + \theta_y \frac{\partial f}{\partial y} + \sigma_a(E) f - \frac{\partial}{\partial E}(S(E) f) = T(z) \left(\frac{\partial^2 f}{\partial \theta_x^2} + \frac{\partial^2 f}{\partial \theta_y^2} \right). \quad (2.3)$$

Here,

$$T(z) = \int_{-1}^{1} \sigma_s(E_a(z), \mu)(1 - \mu) d\mu, \qquad \mu = \cos(\Omega' \cdot \Omega)$$

2.2. Ignoring Energy-Loss Straggling

In order to solve Equation (2.3), we note that it is in fact a separable equation, where we may single out the energy dependence and seek a solution of the form

$$f(\mathbf{r}, \theta_x, \theta_y, E) = g(E) h(\mathbf{r}, \theta_x, \theta_y). \quad (2.4)$$

Assuming further that $g(E) \neq 0$ and $h(\mathbf{r}, \theta_x, \theta_y) \neq 0$, we may now rewrite Equation (2.3) as

$$\frac{\frac{\partial h}{\partial z} + \theta_x \frac{\partial h}{\partial x} + \theta_y \frac{\partial h}{\partial y} - T(z) \left(\frac{\partial^2 h}{\partial \theta_x^2} + \frac{\partial^2 h}{\partial \theta_y^2} \right)}{h(\mathbf{r}, \theta_x, \theta_y)} = \frac{\frac{\partial}{\partial E} \left(S(E) g(E) \right) - \sigma_a(E) g(E)}{g(E)}. \quad (2.5)$$

Now, as the left-hand side depends on $\mathbf{r}$, θ_y, θ_z, and the right-hand side depends on E only, they must both be equal to a constant, λ. Thus, we end up with the two equations

$$\frac{\partial}{\partial E}\big(S(E)g(E)\big) - \sigma_a(E)g(E) = \lambda g(E) \tag{2.6}$$

$$\frac{\partial h}{\partial z} + \theta_x \frac{\partial h}{\partial x} + \theta_y \frac{\partial h}{\partial y} - T(z)\left(\frac{\partial^2 h}{\partial \theta_x^2} + \frac{\partial^2 h}{\partial \theta_y^2}\right) = \lambda h(\mathbf{r}, \theta_x, \theta_y), \tag{2.7}$$

which we will now solve separately.

To solve Equation (2.6), we set $\tilde{g}(E) = S(E)g(E)$, giving us the equation

$$\frac{\partial \tilde{g}}{\partial E}(E) - \frac{\lambda + \sigma_a(E)}{S(E)}\tilde{g}(E) = 0. \tag{2.8}$$

This equation is easily solved by the method of integrating factor, giving the solution

$$\tilde{g}(E) = \tilde{g}(E_0)\exp\left(-\int_E^{E_0} \frac{\lambda + \sigma_a(E')}{S(E')}dE'\right). \tag{2.9}$$

Returning to $g(E)$, and introducing the CSDA range,

$$R(E) = \int_0^E \frac{dE'}{S(E')}, \tag{2.10}$$

we get

$$g(E) = g(E_0)\frac{S(E_0)}{S(E)}\exp\left(\lambda\,(R(E) - R(E_0)) - \int_E^{E_0} \frac{\sigma_a(E')}{S(E')}dE'\right). \tag{2.11}$$

Now we turn to Equation (2.7). This equation is almost the Fermi-Eyges equation, which may be solved by separating the x and y directions and applying the Fourier transform in both x and θ_x (Eyges, 1948).

Performing the same steps here, we look for solutions $h(\mathbf{r}, \theta_x, \theta_y) = H(z, x, \theta_x)H(z, y, \theta_y)$, where $H(z, \xi, \theta)$ satisfies

$$\frac{\partial H}{\partial z}(z, \xi, \theta) + \theta\frac{\partial H}{\partial \xi}(z, \xi, \theta) - T(z)\frac{\partial^2 H}{\partial \theta^2}(z, \xi, \theta) = \frac{\lambda}{2}H(z, \xi, \theta), \tag{2.12}$$

with a Gaussian initial condition

$$H(0, \xi, \theta) = C\exp\big(-(a_1\xi^2 + a_2\xi\theta + a_3\theta^2)\big) \tag{2.13}$$

with the same coefficients $a_i \in \mathbb{R}$ and $C > 0$ in both x- and y-directions. Both the spacial distribution and the angular distribution are thus circular symmetric about the z-axis, although there may be a correlation between radial offset and motion angle, if $a_2 \neq 0$.

By applying the Fourier-transform in both y and θ_y, and proceeding as in Eyges original paper (Eyges, 1948), with the addition of the Gaussian initial condition (see Brahme, 1975), we get the solution

$$h(z,\rho,\boldsymbol{\Theta}) = \frac{A^2}{4\pi^2} e^{\lambda z} \frac{\exp\left(-\frac{|\rho|^2}{2\overline{\xi^2}(z)}\right)}{\overline{\xi^2}(z)} \frac{\exp\left(-\frac{1}{2\tilde{B}(z)}\left|\boldsymbol{\Theta} - \frac{\overline{\theta\xi}(z)}{\overline{\xi^2}(z)}\rho\right|^2\right)}{\tilde{B}(z)}, \tag{2.14}$$

where $\tilde{B}(z) = \overline{\theta^2}(z) - (\overline{\theta\xi}(z))^2/\overline{\xi^2}(z)$, and we have introduced cylindrical coordinates $\rho = (x,y) = |\rho|(\cos\phi, \sin\phi)$, $\boldsymbol{\Theta} = (\theta_x, \theta_y) = \boldsymbol{\Theta}(\cos\psi, \sin\psi)$. Furthermore, $\overline{\theta^2}(z)$, $\overline{\theta\xi}(z)$, and $\overline{\xi^2}(z)$ are the second moments of the separated solution $H(z,\xi,\theta)$, which are related to the coefficients of the initial condition through $\overline{\xi^2}(0) = 2a_1/D$, $\overline{\theta\xi}(0) = a_2/D$, $\overline{\theta^2}(0) = 2a_3/D$, where $D = 4a_1a_3 - a_2^2$ is the discriminant. $A = 2\pi C/D$ is the integral of $H(0,\xi,\theta)$, that is

$$A = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(0,\xi,\theta)\, d\theta\, d\xi.$$

The z-dependence of the second moments is explicitly given by

$$\overline{\theta^2}(z) = \overline{\theta^2}(0) + \int_0^z T(z')\, dz' \tag{2.15}$$

$$\overline{\theta\xi}(z) = \overline{\theta\xi}(0) + \overline{\theta^2}(0)z + \int_0^z (z-z')T(z')\, dz' \tag{2.16}$$

$$\overline{\xi^2}(z) = \overline{\xi^2}(0) + 2\overline{\theta\xi}(0)z + \overline{\theta^2}(0)z^2 + \int_0^z (z-z')^2 T(z')\, dz'. \tag{2.17}$$

Equation (2.14) is the ordinary Fermi-Eyges solution for a Gaussian initial condition, multiplied by a factor $e^{\lambda z}$. Therefore, we write

$$h(z,\rho,\boldsymbol{\Theta}) = e^{\lambda z} h_{\mathrm{FE}}(z,\rho,\boldsymbol{\Theta}), \tag{2.18}$$

where h_{FE} is the Fermi-Eyges solution.

Combining Equations (2.11) and (2.14) as in (2.4), we get the solution to (2.3) as

$$f(z,\rho,\boldsymbol{\Theta},E) = g(E_0)\frac{S(E_0)}{S(E)} e^{\lambda(z+R(E)-R(E_0))} \exp\left(-\int_E^{E_0} \frac{\sigma_a(E')}{S(E')} dE'\right) h_{\mathrm{FE}}(z,\rho,\boldsymbol{\Theta}). \tag{2.19}$$

However, in order to fulfill the boundary condition (2.2), with $G_2(E) = \delta(E - E_0)$ for some initial energy E_0, we introduce a relationship between z and E through the CSDA range $R(E)$ defined by (2.10), as

$$z(E) = R(E_0) - R(E). \tag{2.20}$$

This corresponds to studying the path of a monoenergetic particle beam, which remains monoenergetic for all z as there is no mixing of energies in (2.3). Using (2.20) will make the first exponential factor containing λ in (2.19) disappear, so that the value of λ is irrelevant. Furthermore, we may set $g(E_0) = 1$, since the initial value at $z = 0$ (where $E = E_0$) is now determined by A in (2.14).

Thus, our approximate solution to (2.3) and (2.2) is

$$f(\rho, \boldsymbol{\Theta}, E) = \frac{S(E_0)}{S(E)} \exp\left(- \int_E^{E_0} \frac{\sigma_a(E')}{S(E')} dE' \right) h_{\mathrm{FE}}(R(E_0) - R(E), \rho, \boldsymbol{\Theta}). \quad (2.21)$$

Remark 2.1. *(The classical Fermi-Eyges problem)*

When $T(z) = T$ *is constant,* (2.7) *would correspond to the Fermi equation in its classical setting, viz*

$$\frac{\partial h_F}{\partial z} + \theta_x \frac{\partial h_F}{\partial x} + \theta_y \frac{\partial h_F}{\partial y} = \frac{\sigma_{tr}}{2} \left(\frac{\partial^2 h_F}{\partial \theta_x^2} + \frac{\partial^2 h_F}{\partial \theta_y^2} \right), \quad (2.22)$$

with $\sigma_{tr} = 2T$ *being the* transport cross-section. *The main virtue of the Fermi approximation is that by artificially extending the range of* θ_x *and* θ_y *to the entire real line and by Fourier transforming with respect to* x, y, θ_x*, and* θ_y*, the Equation* (2.22) *associated with the Dirac boundary data*

$$h_F(0, x, y, \theta_x, \theta_y) = \delta(x)\delta(y)\delta(\theta_x)\delta(\theta_y) \quad (2.23)$$

admits the exact solution, (cf. Rossi and Greisen, 1941; see also Börgers and Larsen, 1996),

$$h_F(x, y, z, \theta_x, \theta_y) = \frac{3}{\pi^2 \sigma_{tr}^2 z^4} \exp\left[- \frac{2}{\sigma_{tr}} \left(\frac{\theta_x^2 + \theta_y^2}{z} - 3\frac{x\theta_x + y\theta_y}{z^2} + 3\frac{x^2 + y^2}{z^3} \right) \right]. \quad (2.24)$$

Thus, for a constant T, h_{FE} *may be presented as the rather simpler form* (2.24) *(with* $\sigma_{tr} = 2T$ *= constant) than Equation* (2.14). *Solution* (2.24) *is Gaussian in* x *and* y *for fixed* z, θ_x*, and* θ_y*, and Gaussian in* θ_x *and* θ_y *for fixed* x, y, *and* z. *Eyges (1948) obtained a closed-form solution for the case of* $\sigma_{tr} = \sigma_{tr}(z)$*, which corresponds to* (2.14). *The only discrepancy is in the shape of boundary data* (2.23) *and the one arising from* (2.13).

It is not generally possible to derive an exact closed-form solution for a more general case of $\sigma_{tr} = \sigma_{tr}(x, y, z)$*, i.e., when the assumption A2 is replaced by a more general one, assuming that the energy is a function of the spatial variable* $\mathbf{r} = (x, y, z)$.

2.3. Including the Energy-Loss Straggling Term

In this section we include the energy loss straggling term: $-\frac{1}{2}\frac{\partial^2(\omega(E)f)}{\partial E^2}$ in Equation (2.3) and write the basic equation for charged particle beams under assumptions A1–A2

$$\begin{aligned} &\frac{\partial f}{\partial z} + \theta_x \frac{\partial f}{\partial x} + \theta_y \frac{\partial f}{\partial y} - \frac{\partial}{\partial E}\big(S(E)f\big) - \frac{1}{2}\frac{\partial^2(\omega(E)f)}{\partial E^2} \\ &\quad = T(z)\left(\frac{\partial^2 f}{\partial \theta_x^2} + \frac{\partial^2 f}{\partial \theta_y^2}\right) - \sigma_a(E) f. \end{aligned} \tag{2.25}$$

Since this is a second-order equation in E, we now need a boundary condition at $E = 0$, so in addition to (2.2) we assume

$$f(\mathbf{r}, \mathbf{\Theta}, 0) = 0 \tag{2.26}$$

and $f \to 0$ as $E \to \infty$.

Now following the previous procedure for the derivation of the equation for $h(\mathbf{r}, \theta_y, \theta_z)$, we make a more general ansatz:

$$f(x, \theta_x, y, \theta_y, z, E) = h_{\mathrm{FE}}(z, \rho, \mathbf{\Theta}) \cdot Z(z, E) \neq 0, \tag{2.27}$$

where h_{FE} is the Fermi-Eyges solution. Here differentiating in z yields

$$\frac{\partial f}{\partial z} = \frac{\partial h_{\mathrm{FE}}}{\partial z} Z + \frac{\partial Z}{\partial z} h_{\mathrm{FE}}. \tag{2.28}$$

Inserting (2.27)–(2.28) in (2.25), we end up with the equation

$$\Upsilon[h_{\mathrm{FE}}] \cdot Z + h_{\mathrm{FE}}\Big(\frac{\partial Z}{\partial z} - \frac{\partial(S(E)Z)}{\partial E} - \frac{1}{2}\frac{\partial^2(\omega(E)Z)}{\partial E^2} + \sigma_a(E)Z\Big) = 0, \tag{2.29}$$

where $\Upsilon[h_{\mathrm{FE}}]$ stands for the Fermi-Eyges Equation ((2.7) with $\lambda = 0$) and is identically zero. Hence, recalling $h_{\mathrm{FE}} \neq 0$ we have the partial differential equation for Z given by:

$$\begin{cases} \dfrac{\partial Z}{\partial z} - \dfrac{\partial(SZ)}{\partial E} - \dfrac{1}{2}\dfrac{\partial^2(\omega Z)}{\partial E^2} + \sigma_a(E)Z = 0, & (z, E) \in [0, \infty) \times [0, \infty), \\ Z(0, E) = G_2(E), & E \geq 0, \\ Z(z, 0) = 0, & z \geq 0. \end{cases} \tag{2.30}$$

We wish to apply the Fourier transform, and therefore need to extend the problem to $E \in \mathbb{R}$. In order to retain the boundary condition $Z(z, 0) = 0$, we look for solutions Z that are odd in E, so that $Z(z, -E) = -Z(z, E)$. Applying the Fourier transform, i.e., letting $\mathcal{F}$ in $E(\widehat{Z}(z, \xi) := \mathcal{F}Z(z, E))$, we get

$$\frac{\partial \widehat{Z}}{\partial z} - i\xi\mathcal{F}(S \cdot Z) + \frac{\xi^2}{2}\mathcal{F}(\omega \cdot Z) + \mathcal{F}(\sigma_a \cdot Z) = 0. \tag{2.31}$$

Now, we invoke the *NESA*, (an approximate version of the fundamental theorem of calculus cf., e.g., Asadzadeh et al., 2010),

$$\mathcal{F}\Big(w(E)\cdot Z(E)\Big) \approx w(E_a(z))\cdot \widehat{Z}(z,\xi), \tag{2.32}$$

where $E_a(z)$ is the average energy at depth z given by (2.37) next, and write an approximation for Equation (2.31) in the form of

$$\frac{\partial \widehat{Z}}{\partial z} + \Big(-i\xi S(E_a(z)) + \frac{\xi^2}{2}\cdot \omega(E_a(z)) + \sigma_a(E_a(z))\Big)\widehat{Z} = 0. \tag{2.33}$$

Equation (2.33) admits the analytical solution

$$\widehat{Z}(z,\xi) = C_0 \cdot \exp(-\Phi(z,\xi)), \tag{2.34}$$

with $\Phi(z,\xi)$given by

$$\begin{aligned}\Phi(z,\xi) &= -i\xi\int_0^z S(E_a(z))\,dz' + \frac{\xi^2}{2}\int_0^z \omega(E_a(z'))\,dz' \\ &\quad + \int_0^z \sigma_a(E_a(z'))\,dz' + C(\xi) \equiv i\xi E_a(z) + \frac{\xi^2}{2}\Omega(z) + \Sigma_a(z)\end{aligned} \tag{2.35}$$

where $C(\xi)$ has been chosen according to (2.37) next. Finally, using the inverse Fourier transform, and remembering that we look for solutions $Z(z,E)$, which are odd in E, we get the approximate solution for (2.30) under the NESA, as

$$\begin{aligned}Z(z,E) = \frac{C_0}{\sqrt{2\pi\Omega(z)}}\exp(-\Sigma_a(z))\Bigg(&\exp\Big(-\frac{1}{2}\frac{(E-E_a(z))^2}{\Omega(z)}\Big) \\ &-\exp\Big(-\frac{1}{2}\frac{(E+E_a(z))^2}{\Omega(z)}\Big)\Bigg),\end{aligned} \tag{2.36}$$

where

$$\begin{aligned}\Sigma_a(z) = \int_0^z \sigma_a(E_a(z'))\,dz', \quad E_a(z) = E_0 - \int_0^z S(E_a(z'))\,dz', \\ \Omega(z) = \Omega_0 + \int_0^z \omega(E_a(z'))\,dz'.\end{aligned} \tag{2.37}$$

Thus, recalling the ansatz (2.27), we have obtained the final *approximate* solution as the product of the right-hand sides of (2.14) (with $\lambda = 0$) and (2.36). Note that the boundary condition (2.2) is satisfied with

$$G_2(E) = \frac{C_0}{\sqrt{2\pi\Omega_0}}\left(\exp\left(-\frac{1}{2}\frac{(E-E_0)^2}{\Omega_0}\right) - \exp\left(-\frac{1}{2}\frac{(E+E_0)^2}{\Omega_0}\right)\right). \tag{2.38}$$

If Ω_0 is small enough compared to E_0^2, so that the second term has negligible influence for $E > 0$, this is a Gaussian energy distribution centered around $E = E_0$, with width determined by Ω_0 and with integral C_0.

3. NUMERICAL RESULTS

In order to assess the accuracy of the NESA in (2.33), we have performed numerical computations for Equation (2.31) using the finite element method. The equation was solved treating z as a time variable with a backward Euler scheme, and with piece-wise linear (CG1) elements in E (see, e.g., Eriksson et al., 1996). A nonuniform grid was used, with smaller mesh size for small E and z. The number of grid points was 800×800. The cross-sections, stopping power and straggling coefficients used in the computations model a beam of electrons incident into water, and were taken from Luo and Brahme (1992). The parameter values for the initial condition (2.38) were chosen as

$$\text{(Figure 1):} \qquad E_0 = 20 \text{ MeV}, \qquad \Omega_0 = 0.7 \text{ MeV}^2, \qquad C_0 = \sqrt{2\pi\,\Omega_0},$$

$$\text{(Figure 2):} \qquad E_0 = 50 \text{ MeV}, \qquad \Omega_0 = 4.4 \text{ MeV}^2, \qquad C_0 = \sqrt{2\pi\,\Omega_0},$$

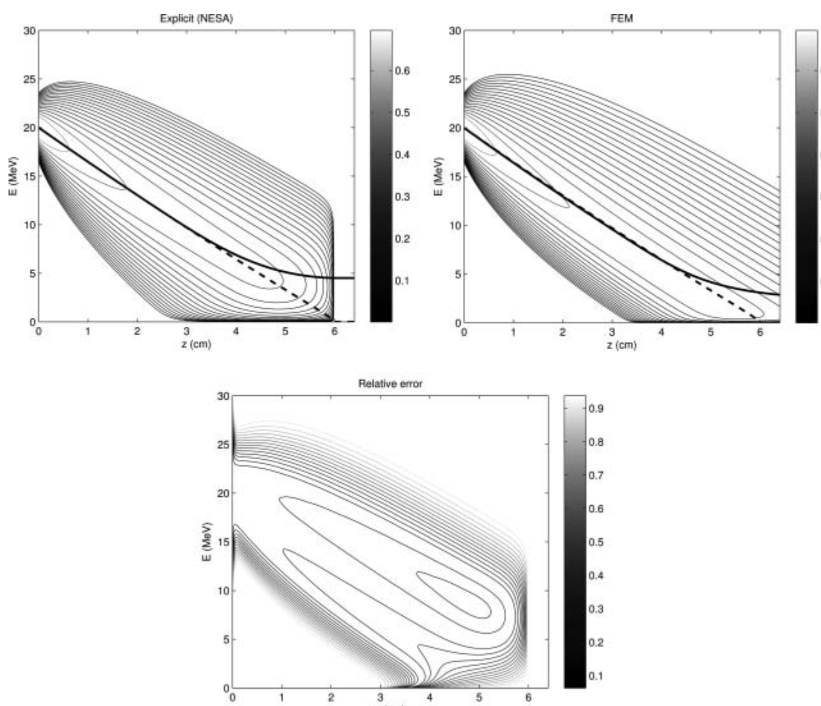

Figure 1: Level curves for the analytical solution Z_{anal} given by (2.36) under the narrow energy spectrum approximation (NESA) (top left), and the corresponding numerical solution Z_{FEM} to equation (2.30) (top right). The initial energy was $E_0 = 20$ MeV. The cross-sections for electrons in water were used with the absorption cross-section? $_a = 0$. The dashed thick lines are the curves $E = E_a(z)$, and the solid thick lines are the average energies for the respective solutions. The figure at the bottom shows level curves for the relative error $e = |Z_{FEM}\text{-}Z_{anal}| / (Z_{FEM}+Z_{anal})$.

where C_0 was chosen to get the maximum function value of 1. The other boundary values were set to $Z|_{E=0} = Z|_{E=E_{\max}} = 0$, where $E_{\max} > E_0$ was chosen large enough not to influence the solution appreciably.

Figures 1 and 2 show level curves for the numerical solution and the analytical (approximate) solution (2.36) for initial energies 20 MeV and 50 MeV, respectively. The solutions agree quite well overall, especially for the lower energy. However, the numerical solution is not symmetric about $E_a(z)$, while the analytical solution is in the beginning, before the boundary condition at $E = 0$ has any impact. Also, the trajectories for the average energy (red curves) differ slightly. Finally, the numerical solution is affected by the different cross-sections for low energies, while the symmetric energy distribution of the analytical solution ignores these effects as the cross-sections are evaluated only at $E_a(z)$.

The figures also show the relative error, defined as $e = |f_{\text{FEM}} - f_{\text{exact}}|/(f_{\text{FEM}} + f_{\text{exact}})$ in order to avoid extremely large values where one of the

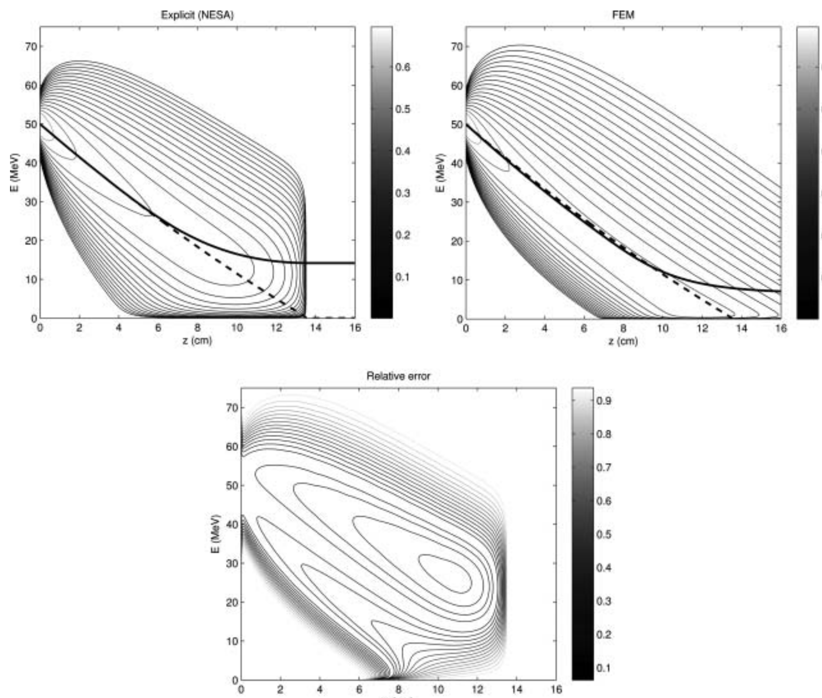

Figure 2: Level curves for the analytical solution Z_{anal} given by (2.36) under the narrow energy spectrum approximation (NESA) (top left), and the corresponding numerical solution Z_{FEM} to equation (2.30) (top right). The initial energy was $E_0 = 50$ MeV. The cross-sections for electrons in water were used with the absorption cross-section? $_a = 0$. The dashed thick lines are the curves $E = E_a(z)$, and the solid thick lines are the average energies for the respective solutions. The figure at the bottom shows level curves for the relative error $e = |Z_{FEM}\text{-}Z_{anal}|/(Z_{FEM}+Z_{anal})$.

solutions (or both) is close to zero. The relative error is about 10% or less in the central area, but becomes large where the solution approaches zero due to the fact that the analytical and numerical solutions decay differently.

4. CONCLUSIONS

We have derived approximate analytical solutions to the charged particle transport equation under the continuous slowing down assumption, with and without energy loss straggling. These solutions may be used to get more accurate results for fluence and dose, while retaining the simplicity of the Fermi-Eyges solution.

The analysis performed in Section 2.2 shows that the results by Kempe and Brahme (2010) are essentially valid. They introduced a factor $\varepsilon(E)$, which was then assumed to be zero without further motivation. Our analysis shows that this assumption gives correct results when the energy-loss straggling term is ignored, and there is no need to introduce $\varepsilon(E)$ in the first place.

By including the energy-loss straggling in Section 2.3, we have also investigated a more general case and have derived a more general energy-dependent correction factor. Numerical investigations show that although several approximations are used in order to achieve a closed form solution, this solution gives a generally good approximation, at least for the energies and cross-sections used here.

Kempe and Brahme (2010) compared the analytical solution to results of Monte Carlo calculations for light ion beams and showed good agreement for the absorbed dose. The solution including energy loss straggling presented here has yet to be compared to Monte Carlo data, but should improve the accuracy of the solution and make it applicable to cases where straggling is important. Such comparisons are the subject of future work.

ACKNOWLEDGMENTS

The ideas in Section 2.3 were inspired by the work of Luo Zhengming. The authors thank Prof. Anders Brahme for inspiring discussions.

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