Convergence Analysis of the Streamline Diffusion and Discontinuous Galerkin Methods for the Vlasov-Fokker-Planck System

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We prove stability estimates and derive optimal convergence rates for the streamline diffusion and discontinuous Galerkin finite element methods for discretization of the multi-dimensional Vlasov-Fokker-Planck system. The focus is on the theoretical aspects, where we deal with construction and convergence analysis of the discretization schemes. Some related special cases are implemented in M. Asadzadeh [Appl Comput Meth 1(2) (2002), 158–175] and M. Asadzadeh and A. Sopasakis [Comput Meth Appl Mech Eng 191(41–42) (2002), 4641–4661]. Numer Methods Partial Differential Eq 21: 472–495, 2005

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1. INTRODUCTION

In this article we study the approximate solution for the deterministic, multidimensional Vlasov-Fokker-Planck (VFP) system using the streamline diffusion and discontinuous Galerkin finite element methods. We prove stability estimates and derive optimal convergence rates for the regularized VFP system. A similar approach for the Vlasov-Poisson equation was considered by the first author in [1]. This work extends the results introduced in [1] to the case of the multidimensional Vlasov-Fokker-Planck system. Here we (i) include the Fokker-Planck diffusion term (as a right-hand side), and (ii) add a viscosity term to the convection part.

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The Vlasov-Poisson-Fokker-Planck (VPFP) system arising in the kinetic description of a plasma of Coulomb particles under the influence of a self-consistent internal field and an external force can be formulated as follows. Given the initial distribution of particles $f_0(x, v) \geq 0$, in the phase-space variable $(x, v) \in R^d \times R^d$, $d = 1, 2, 3$, and the physical parameters $\beta > 0$ and $\sigma > 0$, find the distribution function $f(x, v, t)$ for $t > 0$, satisfying the nonlinear system of evolution equations

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \operatorname{div}_v[(E - \beta v)f] = \sigma \Delta_v f, & \text{in } R^{2d} \times (0, \infty), \\ f(x, v, 0) = f_0(x, v), & \text{for } (x, v) \in R^{2d}, \\ E(x, t) = \dfrac{\theta}{|\mathcal{S}|^{d-1}} \dfrac{x}{|x|^d} *_x \rho(x, t), & \text{for } (x, t) \in R^d \times (0, \infty), \\ \rho(x, t) = \displaystyle\int_{R^d} f(x, v, t)\, dv, & \theta = \pm 1, \end{cases} \tag{1.1}$$

where $x \in R^d$ is the position, $v \in R^d$ is the velocity, and $t > 0$ is the time, $\nabla_x = (\partial/\partial x_1, \ldots, \partial/\partial x_d)$, $\nabla_v = (\partial/\partial v_1, \ldots, \partial/\partial v_d)$, and $\cdot$ is the inner product in R^d. The parameters β and σ are assumed to be the viscosity and the thermal diffusivity coefficients, respectively, which are related by $\sigma = \beta \kappa T_0/m$, with κ being the Boltzmann's constant, T_0 the temperature of the surrounding medium and m the mass of a particle (thus, for "normal" temperatures the physical parameter σ is very small). In our studies the parameter σ is, basically, of the order of mesh size or smaller, and decoupled from $\beta = \mathcal{O}(1)$. $|\mathcal{S}|^{d-1} \sim 1/\omega_d$ is the surface area of the unit disc in R^d. Finally $\rho(x, t)$ is the spatial density, and $*_x$ denotes the convolution in x. E and ρ can be interpreted as the electrical field and charge, respectively. The macroscopic force field E can also be assumed to be of the form

$$E(x, t) = -\nabla_x(\psi(x) + \phi(x, t)), \tag{1.2}$$

with $\psi(x) \geq 0$ being an external potential force, and $\phi(x, t)$ the internal potential field. Then, for $\theta = 1$ the VPFP system models a gas of charged particles, with an external potential ψ, interacting through a mean electrostatic field $-\nabla_x \phi$, generated by their spatial density ρ. Whereas $\theta = -1$ corresponds to a VPFP system modeling particles under the effect of the gravitational potential ψ.

For a gradient field, when E is divergence free and with no viscosity, i.e., for $\beta = 0$, the first equation in (1.1), would become

$$\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = \sigma \Delta_v f, \tag{1.3}$$

which, together, with the rest of equations in (1.1), gives rise to a simplified VPFP system. When E is given (known), we refer to this system as the VFP system. For $\sigma = 0$, and with a zero external force, i.e., $\psi(x) \equiv 0$ and, hence, $E(x, t) = -\nabla_x \phi(x, t)$, we obtain the classical Vlasov-Poisson equation with an internal potential field $\phi(x, t)$ satisfying the Poisson equation

$$\Delta_x \phi(x, t) = -\theta \int_{R^d} f(x, v, t) dv = -\theta \rho(x, t), \tag{1.4}$$

with the asymptotic boundary condition

$$\begin{cases} \phi(x,t) \to o, & \text{for } d > 2, \quad \text{as } |x| \to \infty, \\ \phi(x,t) = \mathcal{O}(\log|x|), & \text{for } d = 2, \quad \text{as } |x| \to \infty. \end{cases} \tag{1.5}$$

For $\beta \neq 0$ (and $\psi(x) = 0$) we have the following (modified) version of the VPFP equation

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f = \nabla_v \cdot (\beta v f + \sigma \nabla_v f), \tag{1.6}$$

where ϕ is assumed to be the exact solution for the Poisson Equation (1.4) given by

$$\phi(x,t) = \theta \int_{R^{2d}} \mathcal{G}(x-y) f(y, v', t) dy dv', \tag{1.7}$$

with $\mathcal{G}$ being the Green's function associated with the fundamental solution of the Laplace operator $-\Delta_x$.

The mathematical study of the VPFP system has been considered by several authors in various settings. The main approach is based on controlling the behavior of the trajectories, i.e., the solutions of the ordinary differential equations underlying the Vlasov-Poisson equation, (see, e.g., [2–4]). Here are some articles concerning the properties of the analytic solution for the VPFP system: Asymptotic behavior, parabolic limit, and stability properties have been carried out in, e.g., [5, 6]. Existence of local in time, classical, smooth solution is given in, e.g., [7], and sufficient condition for the global existence of classical solution in three dimensions can be found in [8]. Existence and uniqueness of smooth global in time solution for large class of initial data are given in [9]. Large time behavior and steady state are considered in [10]. In a recent work [11], the time-discrete variational formulations are studied by certain Kantorovich type functionals.

Compared to the analytical studies the numerical analysis of, purely, multidimensional VPFP/VFP system, both in theory and implementations, is much less developed. However, certain related problems are widely considered in the literature. In this setting, classically, the dominant part of the deterministic numerical studies have been based on the method of characteristics, i.e., mostly the well-known particle methods developed for the Vlasov-Poisson equation in, e.g., [12–14], and more recently in [15] with equally spaced initial data points. Some recent studies are focused on the numerical analysis of Fokker-Planck-Landua (FPL) models by the spectral methods, see, e.g., [16, 17], or conservative and entropy schemes studied, e.g., in [18] for a nonlinear FPL model in three-dimensional (3D) velocity space with some implementations, and in [19] for a space homogeneous problem with a convergence proof (to a discrete Maxwellian) in two dimensions. As for some other related approaches: [20] discusses the efficiency of some numerical algorithms for FPL in cylindrical geometry and presents some test results. In [21] the authors study finite difference schemes for an ion/electron collision operator of the Fokker-Planck type, where a semidiscrete scheme in velocity is combined with, explicit and semi-implicit, time discretizations. Numerical implementations indicate the advantage of the semi-implicit time discretization versus the explicit one. [22] considers a finite volume scheme for the 1D Vlasov-Poisson system and, assuming $W^{1,\infty}$ regularity, derives a convergence rate of order $\mathcal{O}(\Delta t^{1/2} + h^{1/2})$.

Our goal in this article is, however, *to construct and analyze finite element schemes for the multidimensional VFP system, discretizing in space-time-velocity variables, and give optimal convergence rates:* an approach that in our knowledge is not considered elsewhere. Our motivation is to use, as far as possible, the existing numerical strategies for the fluid problems: The VFP system possess, formally, a similar structure as that of fluid equations and, therefore, may be treated using finite element methods, which are more developed for the fluid problems than the equations of gas dynamics, as, e.g., for incompressible Euler and Navier-Stokes equations, conservation laws, or convection-diffusion equations, in [23–27], respectively.

In our studies, assuming a continuous Poisson solver of type (1.7) for the Equation (1.4), we focus on the numerical convergence analysis of a deterministic model problem for the VFP system in a bounded phase-space-time domain. This is a convection dominated convection-diffusion problem of degenerate type (full convection, but only small diffusion in v) for which we study the *streamline-diffusion* (SD) and *discontinuous Galerkin* (DG) finite element methods.

To give a *general motivation* (see also the *final comments* concluding the article) for the use of SD and DG methods and see different aspects, to begin with, let us consider a fully nondegenerate convection-diffusion-absorption problem: a modified version of (1.3), by adding (I) an absorption term αf on the left hand side (this may be extracted from the β term in (1.6) to the price of a somewhat modified ϕ), and (II) a diffusion term of order σ in x. The modified problem is considered in a *bounded space-time-velocity region* $Q_T := \Omega \times (0, T] := \Omega_x \times \Omega_v \times (0, T]$, with both $\Omega_x \subset R^d$ and $\Omega_v \subset R^d$ being bounded domains, and associated with initial and boundary conditions:

$$\begin{cases} \partial_t f + G \cdot \nabla f + \alpha f - \sigma \Delta f = S, & \text{in } Q_T, \\ f(x, v, 0) = f_0(x, v), & \text{in } \Omega, \\ f(x, v, t) = w(x, v, t), & \text{for } (x, v, t) \in \Gamma \times (0, T], \end{cases} \tag{1.8}$$

where $G := (v, -\nabla_x \phi)$, $\nabla := (\nabla_x, \nabla_v)$, $\Gamma := \partial\Omega$, and α and S are assumed to be smooth functions of (x, v, t). Note that, even for smooth data S and w, the solution f to this problem is in general not globally smooth: For $\sigma \neq 0$, the solution f will oscillate in a layer of width $\mathcal{O}(\sigma)$ at the outflow boundary of the phase-space domain Ω: $\Gamma_G^+ = \{(x, v) \in \partial\Omega : \mathbf{n}(x, v) \cdot G \geq 0\}$, where $\mathbf{n}(x, v) := (\mathbf{n}(x), \mathbf{n}(v))$, with $\mathbf{n}(x)$ being the outward unit normals to $\partial\Omega_x$ at the point $x \in \partial\Omega_x$ and $\mathbf{n}(v)$ the outward unit normal to $\partial\Omega_v$ at the point $v \in \partial\Omega_v$. On the other hand in the limit case when $\sigma = 0$ the boundary data w is only prescribed on the *inflow boundary* $\Gamma_G^- = \{(x, v) \in \partial\Omega : \mathbf{n}(x, v) \cdot G < 0\}$. In this case and if w is discontinuous at $p_0 := (x_0, v_0) \in \Gamma_G^-$, then the solution f will be discontinuous along the characteristic curve through p_0, and for all $t \in [0, T]$. If $\sigma > 0$ then such discontinuity is spread out over a layer around the characteristic of width $\mathcal{O}(\sqrt{\sigma})$ (see [28] for the details).

The idea is now to construct general finite element schemes, i.e., with a mesh not oriented along the characteristics, for problems of type (1.8) that (i) are higher order accurate and (ii) have good stability properties without requiring the mesh size h to be smaller than σ. Conventional finite difference or finite element schemes for (1.8), usually, would satisfy only one of these two conditions. Whereas the *streamline diffusion method* introduced by Hughes and Brooks [29] satisfies both the conditions (i) and (ii). This is a Petrov-Galerkin type method, where artificial diffusion in the, *full-characteristic, streamline direction,* $\omega = (1, G)$ is introduced by modifying the test functions from g to $g + \delta\omega \cdot \mathfrak{D}g$, where $\delta \sim h$, or ($<h$), and $\mathfrak{D} = (\partial_t, \nabla_x, \nabla_v)$ is the total gradient. Then a *global streamline diffusion error estimate,* in approximating

with piecewise polynomials of degree k and for $f \in H^{k+1}(Q_T)$, (for simplicity $\alpha = 1$) would read as

$$\sqrt{\sigma}\|\nabla e_h\|_{L_2(Q_T)} + \sqrt{\delta}\|\omega \cdot \mathcal{D} e_h\|_{L_2(Q_T)} + \|e_h\|_{L_2(Q_T)} \leq Ch^{k+1/2}\|f\|_{H^{k+1}(Q_T)}, \tag{1.9}$$

where $e_h := f - f_h$, f_h is the approximate solution, and $H^{k+1}(Q_T)$ denotes the usual, $L_2(Q_T)$-based, Sobolev space. The corresponding error estimate for the standard Galerkin methods is

$$\sqrt{\sigma}\|\nabla e_h\|_{L_2(Q_T)} + \|e_h\|_{L_2(Q_T)} \leq Ch^{k}\|f\|_{H^{k+1}(Q_T)}. \tag{1.10}$$

Comparing (1.9) and (1.10), we conclude that the streamline diffusion method improves the convergence rate by $h^{1/2}$ and has an improved stability due to the presence of the term $\sqrt{\delta}\|\omega \cdot \mathcal{D}e_h\|_{L_2(Q_T)}$ on the left-hand side. This is a consequence of the extra diffusion resulted from the δ-term in the test function: just consider, in a variational formulation, the contribution from the convection term in (1.8) and the δ-term in the test function, when $g = f$.

In the SD method the space of trial and test functions are different and the trial functions are assumed to be continuous in x and v but may be discontinuous in time [however, for simplicity, this discontinuity is not included in the estimate (1.9)].

In this way we have a global estimate viz (1.9). However, most problems have local behavior: they are locally smooth our locally singular. The *discontinuous Galerkin* method allows jump discontinuities across interelement boundaries in x, v, and t in order to count for the local effects. These jump terms are interpreted as derivatives, contributing to additional diffusion terms that is of different type than the one we obtain in the SD method. More specifically, in the simplified models, if we consider the characteristic streamline diffusion method with $k = 1$, i.e., approximation with piecewise linear polynomials, then we can easily show that the actual extra diffusion is of order $\mathcal{O}(h^{3/2})$, whereas the corresponding diffusion added using the discontinuous Galerkin method DG(0), corresponding to $k = 0$, being of order $\mathcal{O}(h)$ is much higher, whereas the one added by DG(1), for $k = 1$, is of order $\mathcal{O}(h^3)$ and, therefore, lower. Also in the DG method the test and trial spaces are the same, which turns out to be advantageous in the error analysis and which also gives improved stability properties for parabolic type problems in comparison with the continuous Galerkin method. For more details see, e.g., [25, 26, 28].

Back to our Equations (1.3) or (1.6): Because of the lack of diffusion in x the boundary conditions are imposed appropriately on $\Gamma_v^- \times \Omega_v \times (0, T]$, where $\Gamma_v^- = \{x \in \partial\Omega_x : \mathbf{n}(x) \cdot v < 0\}$, rather than on the *purely inflow boundary* $\Gamma_G^- \times (0, T]$. Also the estimations corresponding to (1.9) yields a control of $\sqrt{\sigma}\|\nabla_v e_h\|_{L_2(Q_T)}$ rather than $\sqrt{\sigma}\|\nabla e_h\|_{L_2(Q_T)}$. Further, because we do not have an absorption term in (1.6), we cannot, a priori, include the term $\|e_h\|_{L_2(Q_T)}$ in (1.9), controlling the L_2-norm of the error, in our estimates. This, however, may be done simply by imposing a boundary condition that vanishes on a part of the boundary with a positive measure and then using a Poincare-type estimate. We omit carrying out this trivial step. We shall focus on (1.6), where the β term needs somewhat involved estimations. Equation (1.6) can be viewed as a forward (backward) problem assuming $\beta = \beta(v)$ and an orientation on v, e.g., for $v_3 > 0$ (<0); $v = (v_1, v_2, v_3)$.

Both for the streamline-diffusion and the discontinuous Galerkin methods we develop stability estimates and derive optimal convergence rates of order $\mathcal{O}(h^{k+1/2})$ for the piecewise polynomial approximations of degree k, the mesh parameter h, and with the exact solution $f \in H^{k+1}(\Omega)$.

As for the numerical implementations, we have studied two related simplified models: a characteristic method (exact transport + projection), as well as a "semi-streamline diffusion method" (where the time variable is discretized by Backward Euler, Crank-Nicolson, or DG method), for a 2D Fokker-Planck model (as its asymptotic limit in the form of a Fermi pencil beam equation), in [30] and [31], respectively. Further, numerical results, considering less simplified models, are currently under development and will be described elsewhere.

A. The Continuous Problem

Below we summarize a common theoretical framework involving stability estimates in the deterministic case. These results are due to Lions [32] and Degond [4] and are stated for version (1.3) (the corresponding studies for version (1.6) are similar but somewhat lengthy): given the electric field $E^n(x, t)$ and the initial data f_0, with certain regularities, find f^{n+1}, the solution of the Vlasov-Fokker-Planck system, satisfying

$$\begin{cases} \partial_t f^{n+1} + v \cdot \nabla_x f^{n+1} + E^n \cdot \nabla_v f^{n+1} - \sigma \Delta_v f^{n+1} = 0, \\ f^{n+1}(x, v, 0) = f_0(x, v), \end{cases} \tag{1.11}$$

and then compute the charge density ρ^{n+1} and electrical field E^{n+1} according to

$$\rho^{n+1}(x, t) = \int_{R^d} f^{n+1}(x, v, t)\, dv, \qquad E^{n+1}(x, t) = C_d \int_{R^d} \frac{x - y}{|x - y|^d} \rho^{n+1}(y, t)\, dy.$$

Problem (1.11) has a unique solution f^{n+1} satisfying, positivity, L_1 and L_∞ stability estimates:

$$f^{n+1} \geq 0, \qquad \|\rho^{n+1}(t)\|_1 \leq \|f^{n+1}(t)\|_1 \leq \|f_0\|_1, \qquad \|f^{n+1}(t)\|_\infty \leq \|f_0\|_\infty. \tag{1.12}$$

Note that for $\sigma = 0$, Equation (1.11) becomes the classical linear transport equation, which can be solved, e.g., by the method of characteristics, and the stability properties (1.12) are evident.

For the linear Fokker-Planck equation:

$$f_t + v \cdot \nabla_x f + \mathbf{E} \cdot \nabla_v f - \sigma \Delta_v f = S, \qquad f(x, v, 0) = f_0(x, v), \tag{1.13}$$

where $\mathbf{E} = (\mathbf{E}_i(x, v, t))_{i=1}^d$ is a given vector field and $f_0(x, v)$ and $S(x, v, t)$ are given functions; existence, uniqueness, stability, and regularity properties of the solution are straightforward generalizations of the 1D classical results due to Baouendi and Grisvard [33] for the degenerate type equations. These generalizations as well as coupling to the nonlinear problem are due to Lions [32] and require some regularity assumptions on the data: f_0, S, and $\mathbf{E}$.

Remark. We point out that the Fokker-Planck term, as a diffusive term, has smoothing effects on the solution of the system (1.1), which for instance can not be maintained for the Vlasov-Poisson equation lacking this diffusive part. In other words the Fokker-Planck operator $-\sigma\Delta_v f$ although degenerate, provides a smoothing effect related to its hypoellipticity in the sense that it averages in v. However, for optimal convergence rate analysis of the numerical schemes, the desired phase-space regularity is achieved having a diffusive term in x as well. The SD scheme generates automatically the required diffusion for our purpose.

An outline of this article is as follows. In Section 2 we present some notations used throughout the article. Section 3 is devoted to the study of stability estimates and proof of the convergence rates for the streamline diffusion approximation of the VFP system. Our concluding Section 4 is the discontinuous Galerkin counterpart of Section 3.

II. NOTATIONS AND PRELIMINARIES

The continuous problem (1.1), formulated in fully unbounded phase-space-time domain, is not appropriate for numerical considerations. Below we restate the problem (1.1) for $\sigma > 0$ and bounded domains $\Omega_x \subset R^d$ and $\Omega_v \subset R^d$, $d = 1, 2, 3$, as in (1.8), and associated with some boundary conditions. With these assumptions we consider the VPFP problem of finding (f, ϕ) satisfying a VFP system, with an arbitrary outflow and homogeneous inflow boundary conditions:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f = \nabla_v(\beta v f + \sigma \nabla_v f), & \text{in } Q_T, \\ f(x, v, 0) = f_0(x, v), \quad \text{in } \Omega, \\ f(x, v, t) = 0, \quad \text{for } (x, v, t) \in ([\Gamma_v^- \times \Omega_v] \cup [\Omega_x \times \partial\Omega_v]) \times (0, T], \end{cases} \tag{2.1}$$

where $\Omega = \Omega_x \times \Omega_v$ and $Q_T = \Omega \times (0, T]$, associated with the Poisson equation

$$-\Delta_x \phi(x, t) = \int_{\Omega_v} f(x, v, t)\, dv, \qquad (x, t) \in \Omega_x \times (0, T] := \Omega_T, \tag{2.2}$$

where $\nabla_x \phi$ is uniformly bounded and

$$\begin{cases} |\nabla_x \phi| \to 0, & \text{as } |x| \to \partial\Omega_x, \\ \Gamma_v^- = \{x \in \partial\Omega_x : \mathbf{n}(x) \cdot v < 0\}, & \text{for } v \in \Omega_v, \end{cases} \tag{2.3}$$

where $\mathbf{n}(x)$ is the outward unit normal to $\partial\Omega_x$ at the point $x \in \partial\Omega_x$. We can show that the system of Equations (2.1)–(2.3) admits a sufficiently regular unique solution, using a similar argument as in the appendix in [26].

Remark. The considered model problem, studied for $\sigma \neq 0$, would have mixed elliptic-hyperbolic form. For the mixed type problems it is necessary to supply the boundary condition on the whole boundary. However, the ellipticity in our model is only in v variable, which we treat by taking zero boundary condition in Ω_v. Therefore, the specified boundary condition is on the inflow boundary corresponding to the hyperbolic nature of the remaining terms in the equation.

Because we assume a continuous Poisson solver for (2.2)–(2.3), our numerical investigations concerns the VFP system (2.1).

We recall the notation:

$$\nabla f := (\nabla_x f, \nabla_v f) = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_d}, \frac{\partial f}{\partial v_1}, \ldots, \frac{\partial f}{\partial v_d}\right), \qquad d = 1, 2, 3,$$

$$G(f) := (v, -\nabla_x \phi) = \left(v_1, \ldots, v_d, -\frac{\partial \phi}{\partial x_1}, \ldots, -\frac{\partial \phi}{\partial x_d}\right) = (G_1, \ldots, G_{2d}).$$

Note that G is divergent free:

$$\operatorname{div} G(f) = \sum_{i=1}^{d} \frac{\partial G_i}{\partial x_i} + \sum_{i=d+1}^{2d} \frac{\partial G_i}{\partial v_{i-d}} = 0, \qquad d = 1, 2, 3. \tag{2.4}$$

Now we introduce a finite element structure on $\Omega_x \times \Omega_v$: Let $T_h^x = \{\tau_x\}$ and $T_h^v = \{\tau_v\}$ denote finite element subdivisions of Ω_x and Ω_v, with elements τ_x and τ_v, respectively. Note that, for nonpolygonal domains Ω_x and Ω_v, with this discretization we create discrete space and velocity domains, which we denote by Ω_x^h and Ω_v^h, respectively. Thus, $T_h = T_h^x \times T_h^v = \{\tau_x \times \tau_v\} = \{\tau\}$ will be a subdivision of $\Omega^h := \Omega_x^h \times \Omega_v^h$ with elements $\tau = \tau_x \times \tau_v$. We also let $0 = t_0 < t_1 < \cdots < t_M = T$ be a partition of time interval $I = [0, T]$ into subintervals $I_m = (t_m, t_{m+1})$, $m = 0, 1, \ldots, M-1$. Moreover, let $\mathscr{C}_h$ be the corresponding subdivision of $Q_T := \Omega \times [0, T]$ into elements $K := \tau \times I_m$, with the mesh parameter $h = \operatorname{diam} K$ and $P_k(K) = P_k(\tau_x) \times P_k(\tau_v) \times P_k(I_m)$ the set of polynomials in x, v, and t of degree at most k on K. For notational simplicity we suppress the superscript h in Ω_x^h and Ω^h and, in the sequel, use Ω_x and Ω for the discretized domains. But, to remind, we shall keep the discrete velocity domain as is: Ω_v^h. Furthermore, for piecewise polynomials w_i defined on the triangulation $\mathscr{C}_h' = \{K\}$ with $\mathscr{C}_h' \subset \mathscr{C}_h$ and for D_i being some differential operators, we use the notation

$$(D_1 w_1, D_2 w_2)_{Q'} = \sum_{K \in \mathscr{C}_h'} (D_1 w_1, D_2 w_2)_K, \qquad Q' = \bigcup_{K \in G_h'} K,$$

where $(\cdot, \cdot)_Q$ is the usual $L_2(Q)$ scalar product and $\|\cdot\|_Q$ is the corresponding $L_2(Q)$-norm.

In the sequel estimations C will denote a general constant independent of the involved parameters unless otherwise explicitly specified.

Finally, the procedure in the error estimates can be summarized as follows: Let $\tilde{f}$ denote an approximate solution for (2.1) and decompose the error $e = f - \tilde{f}$ according to

$$f - \tilde{f} = (f - \Pi f) - (\tilde{f} - \Pi f) \equiv \eta - \xi,$$

where Π is an appropriate projection/interpolation operator from the space of the continuous solution f into the (finite dimensional) space of approximate solution $\tilde{f}$. Considering a suitable norm, denoted by $|\!|\!|\cdot|\!|\!|$, the estimation is carried out in two steps: (a) first we use approximation theory results to derive sharp error bounds for the interpolation error: $|\!|\!|\eta|\!|\!|$, and then (b) establish

$$|\!|\!|\xi|\!|\!| \leq C|\!|\!|\eta|\!|\!|, \tag{2.5}$$

which rely on the stability estimates of bounding $|\!|\!|\tilde{f}|\!|\!|$ by the $|\!|\!|\text{data}|\!|\!|$. The first step has theoretical nature and is related to the character of the projection operator Π, whereas the second one depending on the structure of the constructed numerical scheme varies in the order of its difficulty.

III. THE STREAMLINE DIFFUSION METHOD

The streamline diffusion (SD) method is a finite element method constructed for convection dominated convection-diffusion problems that (i) is higher order accurate and (ii) has good stability properties. The (SD) method was introduced by Hughes and Brooks [29] for the stationary problems. The mathematical analysis for this method, as well as for the discontinuous Galerkin method, are developed for, e.g., 2D incompressible Euler and Navier-Stokes equations by Johnson and Saranen [26], for multidimensional Vlasov-Poisson equation by Asadzadeh [1], for hyperbolic conservation laws by Szepessy [27], and Jaffre et al. [25], for advection-diffusion problems by Brezzi et al. [23], and also recently, in adaptive setting, by Houston and Süli [24].

In this section we study the SD-method for the VFP system given by (2.1) with the trial functions being continuous in the x and v variables but may have jump discontinuities in t. We let

$$V_h = \{g \in \mathcal{H}_0 : g|_K \in P_k(\tau) \times P_k(I_m);\ \forall K = \tau \times I_m \in \mathcal{C}_n\}, \qquad k = 0, 1, \dots,$$

to be the finite element space, where

$$\mathcal{H}_0 = \prod_{m=0}^{M-1} H_0^1(S_m), \qquad S_m = \Omega \times I_m, \qquad m = 0, 1, \dots, M-1.$$

and

$$H_0^1 = \{g \in H^1 : g \equiv 0 \text{ on } \partial\Omega_v^h\}.$$

Further, for convenience, we write

$$(f, g)_m = (f, g)_{S_m}, \qquad \|g\|_m = (g, g)_m^{1/2},$$

and

$$\langle f, g\rangle_m = (f(\cdot, \cdot, t_m), (g(\cdot, \cdot, t_m))_\Omega, \qquad |g|_m = \langle g, g\rangle_m^{1/2}.$$

Also we present the jump

$$[g] = g_+ - g_-,$$

where

$$g_\pm = \lim_{s \to 0\pm} g(x, v, t+s), \qquad \text{for } (x, v) \in \text{Int}\,\Omega_x \times \Omega_v^h, \qquad t \in I,$$

$$g_\pm = \lim_{s \to 0\pm} g(x+sv, v, t+s), \qquad \text{for } (x, v) \in \partial\Omega_x \times \Omega_v^h, \qquad t \in I,$$

and the boundary integrals

$$\langle f_+, g_+\rangle_{\Gamma^-} = \int_{\Gamma^-} f_+ g_+ (G^h \cdot \mathbf{n})\, d\nu, \qquad \langle f_+, g_+\rangle_{\Gamma_m^-} = \int_{I_m} \langle f_+, g_+\rangle_{\Gamma^-}\, d\nu,$$

and

$$\langle f_+, g_+ \rangle_{\Gamma_I^-} = \int_I \langle f_+, g_+ \rangle_{\Gamma^-} \, d\nu,$$

with $G^h := G(f^h)$ defined above [see also (3.1)] and

$$\Gamma^- = \{(x, v) \in \Gamma = \partial(\Omega_x \times \Omega_v^h) : G^h \cdot \mathbf{n} < 0\}, \qquad \partial\Omega_+ := \partial\Omega \backslash \Gamma^-,$$

where $\mathbf{n} = (\mathbf{n}_x, \mathbf{n}_v)$ with $\mathbf{n}_x$ and $\mathbf{n}_v$ being outward unit normals to $\partial\Omega_x$ and $\partial\Omega_v^h$, respectively. Note that we can split the boundary of Q_T as

$$\Gamma = (\partial\Omega_x \times \Omega_v) \cup (\Omega_x \times \partial\Omega_v) \cup (\partial\Omega_x \times \partial\Omega_v) := \Gamma_1 \cup \Gamma_2 \cup \Gamma_3,$$

with the obvious notations for Γ_i, $i = 1, 2, 3$. Now since Γ^- is characterized by $G^h \cdot \mathbf{n} = \mathbf{n}_x \cdot v + \mathbf{n}_v \cdot \nabla_x\phi < 0$ and $|\nabla_x\phi| \to 0$, as $|x| \to \partial\Omega_x^h$, thus, ignoring Γ_3 (the singular part of the boundary) of measure zero, Γ^- would actually coincide with Γ_v^- defined in (2.3). Below, for the sake of generality, we shall carry out the estimates keeping all the boundary terms as long as possible.

Note further that for two vectors $\mathcal{V}_1$ and $\mathcal{V}_2$ we shall use $\mathcal{V}_1\nabla\mathcal{V}_2 := \mathcal{V}_1 \cdot \nabla\mathcal{V}_2$.

A. Stability

The discrete variational formulation for problem (2.1) reads:

find $f^h \in V_h$ *such that for* $m = 0, 1, \ldots, M-1$, *and for all* $g \in V_h$,

$$\begin{aligned}(f_t^h + G(f^h)\nabla f^h - \nabla_v(\beta v f^h), g + h(g_t + G(f^h)\nabla g))_m + \sigma(\nabla_v f^h, \nabla_v g)_m \\ - h\sigma(\Delta_v f^h, g_t + G(f^h)\nabla g)_m + \langle f_+^h, g_+\rangle_m - \langle f_+^h, g_+\rangle_{\Gamma_m^-} = \langle f_-^h, g_+\rangle_m. \quad (3.1)\end{aligned}$$

We use the discrete version of (2.4):

$$\operatorname{div} G(f^h) = 0, \qquad (3.2)$$

and, for a given appropriate function $\tilde{f}$, define the trilinear form B by

$$\begin{aligned}B(G(\tilde{f}); f, g) = (f_t + G(\tilde{f})\nabla f, g + h(g_t + G(f^h)\nabla g))_{Q_T} + \sigma(\nabla_v f, \nabla_v g)_{Q_T} \\ - h\sigma(\Delta_v f, g_t + G(f^h)\nabla g)_{Q_T} + \sum_{m=1}^{M-1} \langle [f], g_+\rangle_m + \langle f_+, g_+\rangle_0 - \langle f_+, g_+\rangle_{\Gamma_I^-}\end{aligned}$$

and the blinear form K by

$$K(f, g) = (\nabla_v(\beta v f), g + h(g_t + G(f^h)\nabla g))_{Q_T}.$$

Note that both B and K depend implicitly on f^h (hence, also on h) through the term $G(f^h)$. Moreover, we define the linear form L as

$$L(g) = \langle f_0, g_+\rangle_0.$$

Using this notation, we can formulate the problem (3.1) in the concise form:

$$\textit{find } f^h \in V_h \textit{ such that} \qquad B(G(f^h); f^h, g) - K(f^h, g) = L(g) \qquad \forall\, g \in V_h. \tag{3.3}$$

We shall derive stability and convergence estimates for (3.3) in the triple norm:

$$|||g|||^2 = \frac{1}{2}\left[2\sigma\|\nabla_v g\|^2_{Q_T} + |g|^2_M + |g|^2_0 + \sum_{m=1}^{M-1} |[g]|^2_m + 2h\|g_t + G(f^h)\nabla g\|^2_{Q_T} + \int_{\partial\Omega\times I} g^2|G^h \cdot \mathbf{n}|\, dv\, ds\right].$$

Lemma 3.1. *We have that*

$$\forall\, g \in \mathcal{H}_0 \qquad B(G(f^h); g, g) \geq \tfrac{1}{2}\, |||g|||^2.$$

Proof. Using the definition of B we have the identity

$$B(G(f^h); g, g) = (g_t, g)_{Q_T} + \sum_{m=1}^{M-1} \langle [g], g_+\rangle_m + \langle g_+, g_+\rangle_0 - \langle g_+, g_+\rangle_{\Gamma_I^-} + (G(f^h)\cdot\nabla g, g)_{Q_T}$$
$$- h\sigma(\Delta_v g, g_t + G(f^h)\nabla g)_{Q_T} + \sigma\|\nabla_v g\|^2_{Q_T} + h\|g_t + G(f^h)\nabla g\|^2_{Q_T}. \tag{3.4}$$

Integration by parts gives that

$$(g_t, g)_{Q_T} + \sum_{m=1}^{M-1} \langle [g], g_+\rangle_m + \langle g_+, g_+\rangle_0 = \frac{1}{2}\left[|g|^2_M + |g|^2_0 + \sum_{m=1}^{M-1} |[g]|^2_m\right]. \tag{3.5}$$

Using Green's formula and (3.2) we have also

$$(G(f^h)\cdot\nabla g, g)_\Omega - \langle g_+, g_+\rangle_{\Gamma^-} = \frac{1}{2}\int_{\partial\Omega} g^2(G^h\cdot\mathbf{n})\, dv - \int_{\Gamma^-} g^2(G^h\cdot\mathbf{n})\, dv = \frac{1}{2}\int_{\partial\Omega} g^2|G^h\cdot\mathbf{n}|\, dv. \tag{3.6}$$

By the inverse inequality and assumption on σ we get

$$h\sigma|(\Delta_v g, g_t + G(f^h)\nabla g)_{Q_T}| \leq \tfrac{1}{2}\left(\sigma\|\nabla_v g\|^2_{Q_T} + h\|g_t + G(f^h)\nabla g\|^2_{Q_T}\right) \leq \tfrac{1}{2}\, |||g|||^2. \tag{3.7}$$

Now the proof follows from (3.4)–(3.7). ∎

Lemma 3.2. *For any constant $C_1 > 0$ we have for any $g \in \mathcal{H}_0$, that*

$$\|g\|_{Q_T}^2 \leq \left[\frac{1}{C_1} \|g_t + G(f^h)\nabla g\|_{Q_T}^2 + \sum_{m=1}^{M} |g_-|_m^2 + \int_{\partial\Omega\times I} g^2 |G^h \cdot \mathbf{n}| \, dv \, ds \right] h e^{C_1 h}. \tag{3.8}$$

The proof is the same as that of Lemma 3.2 in [1].

B. Error Estimates

Let $\tilde{f}^h \in V_h$ be an interpolant of f with the interpolation error denoted by $\eta = f - \tilde{f}^h$ and set $\xi = f^h - \tilde{f}^h$, so we have

$$e = f - f^h = \eta - \xi. \tag{3.9}$$

The objective in the error estimates is to dominate $|||\xi|||$ by the known interpolation estimates for $|||\eta|||$. Our main result in this section is as follows.

Theorem 3.3. *Assume that $f^h \in V_h$ and $f \in H^{k+1}(Q_T)$, with $k \geq 1$, are the solutions of* (3.3) *and* (2.1), *respectively, such that*

$$\|\nabla f\|_\infty + \|G(f)\|_\infty + \|\nabla \eta\|_\infty \leq C. \tag{3.10}$$

Then there exists a constant C such that

$$|||f - f^h||| \leq C h^{k+1/2} \|f\|_{k+1,Q_T}. \tag{3.11}$$

In the proof of Theorem 3.3 we shall use the following two results estimating the forms B and K.

Lemma 3.4. *Under the assumption of Theorem* 3.3 *and with $\tilde{f}^h$, ξ, and η defined as above we have that*

$$|B(G(f); f, \xi) - B(G(f^h); \tilde{f}^h, \xi)| \leq \frac{1}{8} |||\xi|||^2 + C\Bigg[\int_{\partial\Omega\times I} \eta^2 |G^h \cdot \mathbf{n}| \, dv \, ds + h^{-1} \|\eta\|_{Q_T}^2$$

$$+ \sum_{m=1}^{M} |\eta_-|_m^2 + h(\|\eta_t\|_{Q_T} + \|\nabla \eta\|_{Q_T})^2 \Bigg] + C(\|\xi\|_{Q_T} + \|\eta\|_{Q_T})\|\xi\|_{Q_T} + Ch(\|\xi\|_{Q_T} + \|\eta\|_{Q_T})^2. \tag{3.12}$$

Proof. Using the definition of η we may write

$$B(G(f); f, \xi) - B(G(f^h); \tilde{f}^h, \xi) = B(G(f^h); \eta, \xi) + B(G(f); f, \xi)$$
$$- B(G(f^h); f, \xi) := T_1 + T_2 - T_3.$$

Now we estimate the terms T_1 and $T_2 - T_3$, separately. For the term T_1 we use the inverse inequality and assumption on σ to obtain

$$\sigma|(\nabla_v\eta, \nabla_v\xi)_{Q_T}| \leq \sigma\|\nabla_v\eta\|_{Q_T}\|\nabla_v\xi\|_{Q_T} \leq Ch^{-1}\|\eta\|_{Q_T}^2 + \frac{\sigma}{8}\|\nabla_v\xi\|_{Q_T}^2 \tag{3.13}$$

and

$$h\sigma|(\Delta_v\eta, \xi_t + G(f^h)\nabla\xi)_{Q_T}| \leq h\sigma\|\Delta_v\eta\|_{Q_T}\|\xi_t + G(f^h)\nabla\xi\|_{Q_T} \leq C\|\eta\|_{Q_T}\|\xi_t + G(f^h)\nabla\xi\|_{Q_T}$$
$$\leq Ch^{-1}\|\eta\|_{Q_T}^2 + \frac{h}{8}\|\xi_t + G(f^h)\nabla\xi\|_{Q_T}^2. \tag{3.14}$$

Then integrating by parts, using (3.2), a similar argument as in the proof of Lemma 3.1 and the fact that Ω_v^h is bounded with zero boundary condition we get

$$(\eta_t + G(f^h)\nabla\eta, \xi + h(\xi_t + G(f^h)\nabla\xi))_{Q_T} + \sum_{m=1}^{M-1}\langle[\eta], \xi\rangle_m + \langle\eta_+, \xi_+\rangle_0 - \langle\eta_+, \xi_+\rangle_{\Gamma_I^-}$$
$$= -(\eta, \xi_t + G(f^h)\nabla\xi)_{Q_T} + \langle\eta_-, \xi_-\rangle_M - \sum_{m=1}^{M-1}\langle\eta_-, [\xi]\rangle_m + \int_{\partial\Omega\times I}\eta\xi|G^h\cdot\mathbf{n}|\,dv\,ds$$
$$+ h(\eta_t + G(f^h)\nabla\eta, \xi_t + G(f^h)\nabla\xi)_{Q_T}, \tag{3.15}$$

which together with (3.13) and (3.14) gives

$$|T_1| \leq \frac{1}{8}|||\xi|||^2 + C\left[\int_{\partial\Omega\times I}\eta^2|G^h\cdot\mathbf{n}|\,dv\,ds + h^{-1}\|\eta\|_{Q_T}^2 + \sum_{m=1}^{M}|\eta_-|_m^2 + h\|\eta_t + G(f^h)\nabla\eta\|_{Q_T}^2\right]. \tag{3.16}$$

To bound the last term on the right-hand side of (3.16), we use some basic properties of the solution of the Poisson equation together with the definition of G and derive the estimate

$$\|G(f^h) - G(f)\|_{\Omega_T} \leq C\|f - f^h\|_{Q_T} \leq C(\|\xi\|_{Q_T} + \|\eta\|_{Q_T}),$$

which gives

$$\|\eta_t + G(f^h)\nabla\eta\|_{Q_T} \leq \|\eta_t\|_{Q_T} + \|G(f)\|_\infty\|\nabla\eta\|_{Q_T} + C\|\nabla\eta\|_\infty(\|\xi\|_{Q_T} + \|\eta\|_{Q_T}). \tag{3.17}$$

To estimate the term $T_2 - T_3$, we follow a similar argument as in [1] and get

$$|T_2 - T_3| \leq C(\|\xi\|_{Q_T} + \|\eta\|_{Q_T})\|\nabla f\|_\infty\|\xi\|_{Q_T} + Ch(\|\xi\|_{Q_T} + \|\eta\|_{Q_T})^2\|\nabla f\|_\infty^2 + \tfrac{1}{8}h\|\xi_t + G(f^h)\nabla\xi\|_{Q_T}^2. \tag{3.18}$$

Now combining the estimates (3.16)–(3.18), using assumptions of Theorem 3.3 and hiding the term $\frac{1}{8}h\|\xi_t + G(f^h)\nabla\xi\|_{Q_T}^2$ in the triple norm the proof is complete. ∎

Lemma 3.5. *Under the assumptions of Theorem* 3.3, *we have*

$$|K(f^h, \xi) - K(f, \xi)| \le \tfrac{1}{8}\, |||\xi|||^2 + C\|\xi\|_{Q_T}^2 + Ch^{-1}\|\eta\|_{Q_T}^2.$$

Proof. Using the definition of ξ and η, we have the identity

$$K(f^h, \xi) - K(f, \xi) = K(\xi, \xi) - K(\eta, \xi) := K_1 - K_2.$$

Below we bound the terms K_1 and K_2, separately. For the first term using the vanishing boundary condition on $\partial\Omega_v^h$ we have

$$\begin{aligned}
|K_1| &= |(\nabla_v \cdot (\beta v\xi),\ \xi + h(\xi_t + G(f^h)\nabla\xi))_{Q_T}| \\
&= |(\beta d\xi + \beta v\cdot\nabla_v\xi,\ \xi + h(\xi_t + G(f^h)\nabla\xi))_{Q_T}| \\
&= \left|\beta\frac{d}{2}\|\xi\|_{Q_T}^2 + \beta dh(\xi,\ \xi_t + G(f^h)\nabla\xi)_{Q_T} + \beta h(v\nabla_v\xi,\ \xi_t + G(f^h)\nabla\xi)_{Q_T}\right| \\
&\le C\beta\|\xi\|_{Q_T}^2 + \frac{\beta h}{16}\|\xi_t + G(f^h)\nabla\xi\|_{Q_T}^2 + C_v\beta h\|\nabla_v\xi\|_{Q_T}^2. \qquad (3.19)
\end{aligned}$$

The term K_2 is estimated using the integration by parts, inverse inequality and boundedness of Ω_v^h, according to

$$\begin{aligned}
|K_2| &= |(\nabla_v\cdot(\beta v\eta),\ \xi + h(\xi_t + G(f^h)\nabla\xi))_{Q_T}| \\
&= \beta|(\nabla_v\cdot(v\eta),\ \xi)_{Q_T} + h(\nabla_v\cdot(v\eta),\ \xi_t + G(f^h)\nabla\xi)_{Q_T}| \\
&\le \beta|(v\eta,\ \nabla_v\xi)_{Q_T}| + \beta h\left(4\|\nabla_v\cdot(v\eta)\|_{Q_T}^2 + \frac{1}{16}\|\xi_t + G(f^h)\nabla\xi\|_{Q_T}^2\right) \\
&\le \beta\left((h^{-1/2}\|v\eta\|_{Q_T})(h^{1/2}\|\nabla_v\xi\|_{Q_T}) + C_v h^{-1}\|\eta\|_{Q_T}^2 + \frac{h}{16}\|\xi_t + G(f^h)\nabla\xi\|_{Q_T}^2\right) \\
&\le \beta\left(\frac{h}{16}\|\nabla_v\xi\|_{Q_T}^2 + C_v h^{-1}\|\eta\|_{Q_T}^2 + \frac{h}{16}\|\xi_t + G(f^h)\nabla\xi\|_{Q_T}^2\right). \qquad (3.20)
\end{aligned}$$

Combining the estimates for K_1 and K_2, recalling the assumption on β and hiding the terms of the form $h\|\xi_t + G(f^h)\nabla\xi\|_{Q_T}^2$ and $h\|\nabla_v\xi\|_{Q_T}^2$ in $|||\xi|||^2$, the proof is complete. ∎

Now using Lemmas 3.3 and 3.4 the proof of Theorem 3.3 is straightforward.

Proof of Theorem 3.3. The exact solution f satisfies

$$B(G(f); f, g) - K(f, g) = L(g) \qquad \forall g \in V_h,$$

so that by Lemma 3.1 and some algebraic labor we get

$$\tfrac{1}{2}\,|||\xi|||^2 \le B(G(f^h); f^h - \tilde{f}^h, \xi) = L(\xi) + K(f^h, \xi) - B(G(f^h); \tilde{f}^h, \xi)$$
$$= B(G(f); f, \xi) - B(G(f^h); \tilde{f}^h, \xi) + K(f^h, \xi) - K(f, \xi) := \Delta B + \Delta K. \quad (3.21)$$

Now we use Lemmas 3.4 and 3.5 to bound the terms ΔB and ΔK, respectively. Further, estimating $\|\xi\|^2_{Q_T}$ and $\|\eta\|^2_{Q_T}$ by Lemma 3.2 with sufficiently large C_1, and also using (3.21), we obtain

$$|||\xi|||^2 \le C\left[\int_{\partial\Omega\times I} \eta^2 |G^h \cdot \mathbf{n}|\, dv\, ds + h^{-1}\|\eta\|^2_{Q_T} + \sum_{m=1}^{M} |\eta_-|^2_m + h\|\eta\|^2_{1,Q_T} + \sum_{m=1}^{M} |\xi_-|^2_m h\right].$$

Finally, by a Grönwalls type estimate, proceeding as in [1] the proof is complete. ■

IV. THE DISCONTINUOUS GALERKIN METHOD

A. Stability

In this section we use trial functions that are polynomials of degree $k \ge 1$ on each element K of the triangulation and may be discontinuous across inter-element boundaries in time, space, and velocity variables.

To define a finite element method based on discontinuous trial functions, we introduce the following notation: if $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_{2d})$, $d = 1, 2, 3$, is a given smooth vector field on Ω we define for $K \in \mathscr{C}_h$ the inflow (outflow) boundary with respect to ζ as

$$\partial K_{-(+)}(\zeta) = \{(x, v, t) \in \partial K : \mathbf{n}_t(x, v, t) + \mathbf{n}(x, v, t)\cdot\zeta(x, v, t) < 0(>0)\}, \qquad (4.1)$$

where $(\mathbf{n}, \mathbf{n}_t) := (\mathbf{n}_x, \mathbf{n}_v, \mathbf{n}_t)$ denotes the outward unit normal to $\partial K \subset Q_T$. Further, for $k \ge 0$ we define, for $d = 1, 2, 3$, the function spaces

$$W_h = \{g \in L_2(Q_T) : g|_K \in P_k(K);\ \forall K \in \mathscr{C}_h\},$$
$$\mathbf{W}^d_h = \{\mathbf{w} \in [L_2(Q_T)]^d : \mathbf{w}|_K \in [P_k(K)]^d;\ \forall K \in \mathscr{C}_h\}.$$

To derive a variational formulation, for the diffusive part of (1.1), based on discontinuous trial functions, we introduce the operator R: $W_h \to \mathbf{W}^d_h$ defined in, e.g., [23]. More precisely, given $g \in W_h$ we define R by the following relation

$$(R(g), \mathbf{w})_{Q_T} = -\sum_{\tau_x\times I_m} \int_{\tau_x\times I_m} \sum_{e\in\mathscr{E}_v} \int_e [\![g]\!]\mathbf{n}_v \cdot (\mathbf{w})^0\, dv \qquad \forall \mathbf{w} \in \mathbf{W}^d_h,$$

where we denote by $\mathscr{E}_v$ the set of all interior edges of the triangulation T^v_h of the discrete velocity domain Ω^h_v. Moreover, for an appropriately chosen function χ, we define

$$(\chi)^0 = \frac{\chi + \chi^{ext}}{2}, \qquad \text{and} \qquad [\![\chi]\!] = \chi - \chi^{ext},$$

where χ^{ext} denotes the value of χ in the element τ_v^{ext} having $e \in \mathscr{E}_v$ as the common edge with τ_v. Hence, roughly speaking, $[\![\chi]\!]$ corresponds to the jump and $(\chi)^0$ is the average value of χ in the velocity variable.

Next for $e \in \mathscr{E}_v$ we define the operator r_e: $W_h \to \mathbf{W}_h^d$ to be the restriction of R to the elements sharing the edge $e \in \mathscr{E}_v$, i.e.,

$$(r_e(g), \mathbf{w})_{Q_T} = -\sum_{\tau_x \times I_m} \int_{\tau_x \times I_m} \int_e [\![g]\!] \mathbf{n}_v \cdot (\mathbf{w})^0 \, d\nu, \qquad \forall \mathbf{w} \in \mathbf{W}_h^d.$$

One can easily verify that

$$\sum_{e \subset \partial\tau_v \cap \mathscr{E}_v} r_e = R \qquad \text{on } \tau_v, \tag{4.2}$$

for any element τ_v of the triangulation of Ω_v^h. As a consequence of this, we have the following estimate:

$$\|R(g)\|_K^2 \leq \gamma \sum_{e \subset \partial\tau_v \cap \mathscr{E}_v} \|r_e(g)\|_K^2, \tag{4.3}$$

where τ_v corresponds to the element K and $\gamma > 0$ is a constant depending on d.

Now, since the support of each r_e is the union of elements sharing the edge e, we can evidently deduce

$$\sum_{e \in \mathscr{E}_v} \|r_e(g)\|_{Q_T}^2 = \sum_{K \in \mathscr{C}_h} \sum_{e \subset \partial\tau_v \cap \mathscr{E}_v} \|r_e(g)\|_K^2. \tag{4.4}$$

Using these notations, we are now ready to formulate the variational formulation for the discontinuous Galerkin approximation of (2.1) as follows:

find $f^h \in W_h$ such that for $m = 0, 1, \ldots, M-1$ and for all $g \in W_h$

$$\begin{aligned}(f_t^h + G(f^h)\nabla f^h - \nabla_v(\beta v f^h),\, g + h(g_t + G(f^h)\nabla g))_{Q_T} + \sum_{K \in \mathscr{C}_h} \int_{\partial K_-(G)} [f^h] g_+ |\mathbf{n}_t + G^h \cdot \mathbf{n}| \, d\nu \\ + \sigma(\nabla_v f^h, \nabla_v g)_{Q_T} + \sigma(\nabla_v f^h, R(g))_{Q_T} + \sigma(R(f^h), \nabla_v g)_{Q_T} + \lambda\sigma \sum_{e \in \mathscr{E}_v} (r_e(f^h), r_e(g))_{Q_T} \\ - h\sigma(\Delta_v f^h, g_t + G(f^h)\nabla g)_{Q_T} = 0,\end{aligned} \tag{4.5}$$

where $[u] = u_+ - u_-$, with $u_\pm = \lim_{s \to 0\pm} u((x, v) + G(f^h)s, t + s)$, $\lambda > 0$ is a given constant, $f_-^h(x, v, 0) = f_0(x, v)$, and in $\partial K_-(G)$; $G := G(f^h)$.

To proceed we define the discontinuous Galerkin trilinear form B_{DG} by

$$B_{DG}(G(\tilde{f});f,g) = (f_t + G(\tilde{f})\nabla f, g + h(g_t + G(f^h)\nabla g))_{Q_T} + \sigma(\nabla_v f, \nabla_v g)_{Q_T}$$
$$+ \sum_{K\in\mathscr{C}_h} \int_{\partial K_-(G)'} [f]g_+|n_t + G^h\cdot\mathbf{n}|\,d\nu - h\sigma(\Delta_v f, g_t + G(f^h)\nabla g)_{Q_T} + \langle f_+, g_+\rangle_0$$
$$+ \lambda\sigma \sum_{e\in\mathscr{E}_v} (r_e(f), r_e(g))_{Q_T} + \sigma(\nabla_v f, R(g))_{Q_T} + \sigma(R(f), \nabla_v g)_{Q_T}$$

and the bilinear form K as in the streamline diffusion method, i.e.,

$$K(f, g) = (\nabla_v(\beta v f), g + h(g_t + G(f^h)\nabla g))_{Q_T}.$$

Note that again both B_{DG} and K depend implicitly on f^h (hence, also on h) through the term $G(f^h)$. Moreover, we define the linear form L as before

$$L(g) = \langle f_0, g_+\rangle_0.$$

Now we can formulate the problem (4.5) in the following concise form:

$$\textit{find } f^h \in W_h \textit{ such that} \qquad B_{DG}(G(f^h); f^h, g) - K(f^h, g) = L(g) \qquad \forall\, g \in W_h. \tag{4.6}$$

We shall refer to (4.5) or (4.6) as the *DG-scheme.*

We derive our stability estimate and prove convergence rates for the DG-scheme (4.6) in the triple norm

$$|||g|||^2 = \frac{1}{2}\Bigg[2\sigma\|\nabla_v g\|^2_{Q_T} + 2\sigma \sum_{e\in\mathscr{E}_v} \|r_e(g)\|^2_{Q_T} + 2h\|g_t + G(f^h)\nabla g\|^2_{Q_T} + |g|^2_M + |g|^2_0$$
$$+ \sum_{K\in\mathscr{C}_h} \int_{\partial K_-(G)'} [g]^2|\mathbf{n}_t + G^h\cdot\mathbf{n}|\,d\nu + \int_{\partial\Omega_+\times I} g^2|G^h\cdot\mathbf{n}|\,d\nu\,ds\Bigg],$$

where $\partial K_-(G)' = \partial K_-(G)\backslash\Omega \times \{0\}$.

Lemma 4.1. *There exists a constant $\alpha > 0$ independent of h such that*

$$\forall g \in W_h \qquad B_{DG}(G(f^h); g, g) \geq \alpha|||g|||^2.$$

Proof. Using the definition of B_{DG} and (4.4), we have that

$$B_{DG}(G(f^h); g, g) = |g|^2_0 + \sum_{K\in\mathscr{C}_h}\Bigg[(g_t, g)_K + (G(f^h)\nabla g, g)_K + \int_{\partial K_-(G)'} [g]g_+|\mathbf{n}_t + G^h\cdot\mathbf{n}|\,d\nu$$
$$+ h\|g_t + G(f^h)\nabla g\|^2_K - h\sigma(\Delta_v g, g_t + G(f^h)\nabla g)_K + \sigma\|\nabla_v g\|^2_K + 2\sigma(\nabla_v g, R(g))_K$$
$$+ \lambda\sigma \sum_{e\subset\partial\tau_v\cap\mathscr{E}_v} \|r_e(g)\|^2_K\Bigg] := \sum_{i=1}^{9} T_i.$$

Now we estimate the terms $T_1, \ldots, T_9$, separately. Integrating by parts we get

$$\sum_{i=1}^{4} T_i = \frac{1}{2}\left[|g|_M^2 + |g|_0^2 + \sum_{K\in\mathscr{C}_h} \int_{\partial K_-(G)'} [g]^2 |\mathbf{n}_t + G^h \cdot \mathbf{n}|\, d\nu + \int_{\partial\Omega_+\times I} g^2 |G^h \cdot \mathbf{n}|\, d\nu \right].$$

Using (4.2) and (4.3) we deduce for some $\varepsilon > 0$, that

$$\sum_{i=7}^{9} T_i \geq \sigma \sum_{K\in\mathscr{C}_h} \left[(1-\varepsilon)\|\nabla_v g\|_K^2 - \frac{1}{\varepsilon}\|R(g)\|_K^2 + \lambda \sum_{e\subset\partial\tau_v\cap\mathscr{E}_v} \|r_e(g)\|_K^2 \right]$$

$$\geq \sigma \sum_{K\in\mathscr{C}_h} \left[(1-\varepsilon)\|\nabla_v g\|_K^2 + \left(\lambda - \frac{\gamma}{\varepsilon}\right) \sum_{e\subset\partial\tau_v\cap\mathscr{E}_v} \|r_e(g)_K^2 \right].$$

As for the term T_6, we use an estimate similar to (3.7) to obtain

$$h\sigma|(\Delta_v g, g_t + G(f^h)\nabla g)_{Q_T}| \leq \delta |||g|||^2,$$

where $0 < \delta < 1 - \varepsilon$. Finally T_5 is estimated in a similar way as in (3.17). Combining the estimates for T_i, $i = 1, \ldots, 9$, and taking $\alpha = \min[1 - \varepsilon - \delta, \lambda - (\gamma/\varepsilon)]$ [which is positive for $(\gamma/\lambda) < \varepsilon < 1$ and $0 < \delta < 1 - \varepsilon$], the proof is complete. ∎

Lemma 4.2. *For any constant $C_1 > 0$, we have for $g \in W_h$*

$$\|g\|_{Q_T}^2 \leq \left[\frac{1}{C_1}\|g_t + G(f^h)\nabla g\|_{Q_T}^2 + \sum_{m=1}^{M} |g_-|_m^2 + \sum_{K\in\mathscr{C}_h} \int_{\partial K_-(G)''} [g]^2 |G^h\cdot\mathbf{n}|\, d\nu + \int_{\partial\Omega_+\times I} g^2 |G^h\cdot\mathbf{n}|\, d\nu\, ds \right] h e^{C_1 h},$$

where

$$\partial K_-(G)'' = \{(x, v, t) \in \partial K_-(G)' : \mathbf{n}_t(x, v, t) = 0\}.$$

The proof is similar to that of Lemma 4.2 in [1], and, therefore, is omitted.

B. Error Estimates

We use the same notation as in the SD-method with $\tilde{f}^h \in H_0^1(Q_T)$ denoting the interpolant of the exact solution f.

The main result of this section is the following error estimate.

Theorem 4.3. *Assume $f^h \in W_h$ and $f \in H^{k+1}(Q_T) \cap W^{k+1,\infty}(Q_T)$, with $k \geq 1$, are the solutions of* (4.6) *and* (2.1), *respectively, such that for $\eta = f - \tilde{f}^h$,*

$$\|\nabla f\|_\infty + \|G(f)\|_\infty + \|\nabla \eta\|_\infty \leq C. \tag{4.7}$$

Then there exists a constant C such that

$$|||f - f^h||| \leq Ch^{K+1/2}.$$

To prove this convergence rate, we shall need the following results.

Lemma 4.4. *Let $u \in L^2(\Omega_x \times I, H^1(\Omega_v))$ with $\Delta_v u \in L^2(Q_T)$, and let $w \in W_h$. Then*

$$\sum_{K\in\mathscr{C}_h} \int_{\tau_x\times I_m} \int_{\partial\tau_v} w \frac{\partial u}{\partial \mathbf{n}_v} = \sum_{\tau_x\times I_m} \int_{\tau_x\times I_m} \sum_{e\in\mathscr{E}_v} \int_e [\![w]\!]\mathbf{n}_v \cdot (\nabla_v u)^0.$$

Proof. The regularity of u implies that $\nabla_v u \cdot \mathbf{n}_v$ is continuous across the inter-element boundaries. Hence, we have

$$\sum_{K\in\mathscr{C}_h} \int_{\tau_x\times I_m} \int_{\partial\tau_v} w \frac{\partial u}{\partial \mathbf{n}_v} = \sum_{\tau_x\times I_m} \int_{\tau_x\times I_m} \sum_{e\in\mathscr{E}_v} \int_e [w^+\mathbf{n}_v^+ \cdot \nabla_v u^+ + w^-\mathbf{n}_v^- \cdot \nabla_v u^-]$$

$$= \sum_{\tau_x\times I_m} \int_{\tau_x\times I_m} \sum_{e\in\mathscr{E}_v} \int_e \left[w^+\mathbf{n}_v^+ \cdot \frac{\nabla_v u^+ + \nabla_v u^-}{2} + w^-\mathbf{n}_v^- \cdot \frac{\nabla_v u^+ + \nabla_v u^-}{2} \right]$$

$$= \sum_{\tau_x\times I_m} \int_{\tau_x\times I_m} \sum_{e\in\mathscr{E}_v} \int_e [\![w]\!]\mathbf{n}_v \cdot (\nabla_v u)^0,$$

and the proof is complete. ∎

Lemma 4.5. *Under the assumptions of Theorem 4.3, we have with $\tilde{C} < 1$, that*

$$|B_{DG}(G(f); f, \xi) - B_{DG}(G(f^h); \tilde{f}^h, \xi)| \leq \tilde{C}|||\xi|||^2 + Ch^{2k+1} + C\left[\int_{\partial\Omega_+\times I} \eta^2 |G^h \cdot \mathbf{n}|\, dv\, ds + h^{-1}\|\eta\|_{Q_T}^2 + \sum_{m=0}^{M} |\eta|_m^2 \right] + C(\|\xi\|_{Q_T} + \|\eta\|_{Q_T})\|\xi\|_{Q_T} + Ch(\|\xi\|_{Q_T} + \|\eta\|_{Q_T})^2.$$

Proof. Once again by the definition of the interpolation error η we may write

$$B_{DG}(G(f); f, \xi) - B_{DG}(G(f^h); \tilde{f}^h, \xi) = B_{DG}(G(f^h); \eta, \xi) + B_{DG}(G(f); f, \xi) - B_{DG}(G(f^h); f, \xi)$$
$$:= T_1 + T_2 - T_3.$$

To estimate the term $T_2 - T_3$, we proceed as in the proof of Lemma 3.4 [cf. (3.18)]. For the term T_1 we have

$$T_1 = \langle \eta_+, \xi \rangle_0 + \sum_{K \in \mathscr{C}_h} \Bigg[(\eta_t + G(f^h)\nabla\eta, \xi + h(\xi_t + G(f^h)\nabla\xi))_K$$

$$+ \int_{\partial K_-(G)'} [\eta]\xi_+ |\mathbf{n}_t + G^h \cdot \mathbf{n}|\, d\nu - h\sigma(\Delta_v \eta, \xi_t + G(f^h)\nabla\xi)_K + \sigma(\nabla_v \eta, \nabla_v \xi)_K$$

$$+ \lambda\sigma \sum_{e \in \mathscr{E}_v} (r_e(\eta), r_e(\xi))_K + \sigma(R(\eta), \nabla_v \xi)_K + \sigma(\nabla_v \eta, R(\xi))_K \Bigg] := \sum_{i=1}^{8} S_i. \quad (4.8)$$

Thus, we need to estimate S_i, $1 \le i \le 8$. For the term S_1 we have

$$|S_1| \le |\eta_+|_0^2 + |\xi_+|_0^2. \quad (4.9)$$

Integration by parts leads to an estimate for $S_2 + S_3$,

$$|S_2 + S_3| \le \left| \sum_{K \in \mathscr{C}_h} \int_{\partial K_-(G)'} \eta_-[\xi] |\mathbf{n}_t + G^h \cdot \mathbf{n}|\, d\nu \right| + \left| \int_{\partial\Omega_+ \times I} \eta_- \xi_- |G^h \cdot \mathbf{n}|\, d\nu\, ds \right|.$$

To bound the first term on the right-hand side above, the crucial part is to estimate a term of the form

$$T = \sum_{K \in \mathscr{C}_h} \int_{\partial K_-(G)''} \eta_-[\xi] |G^h \cdot \mathbf{n}|\, d\nu.$$

To this approach using Cauchy-Schwartz inequality, we have for $\delta > 0$ that

$$|T| \le \frac{C}{\delta} \sum_{K \in \mathscr{C}_h} \int_{\partial K_-(G)''} |\eta_-|^2 |G^h \cdot \mathbf{n}|\, d\nu + C\delta \sum_{K \in \mathscr{C}_h} \int_{\partial K_-(G)''} [\xi]^2 |G^h \cdot \mathbf{n}|\, d\nu, \quad (4.10)$$

where the last sum can be hidden in $|||\xi|||^2$, and the first sum is estimated below

$$\sum_{K \in \mathscr{C}_h} \int_{\partial K_-(G)''} |\eta_-|^2 |G^h \cdot \mathbf{n}|\, d\nu \le \|\eta\|_\infty^2 \sum_{K \in \mathscr{C}_h} \left[\int_{\partial K_-(G)''} |G^h \cdot \mathbf{n}|^2\, d\nu + \int_{\partial K_-(G)''} d\nu \right]$$

$$\le C\|\eta\|_\gamma^2 \sum_{K \in \mathscr{C}_h} [Ch^{-1} \|G(f^h)\|_K^2 + Ch^{2d}], \quad (4.11)$$

where $d = 1, 2, 3$. Further, the interpolation error η satisfies

$$\|\eta\|_\infty \le Ch^{k+1} \|f\|_{k+1,\infty}. \quad (4.12)$$

Hence, by (4.10)–(4.12) and by assumptions of the lemma, we obtain

$$|T| \leq Ch^{2k+1} + \frac{1}{C_1} |||\xi|||^2, \tag{4.13}$$

where C_1 is a sufficiently large constant. So that using once again the Cauchy-Schwartz inequality, we obtain for $S_2 + S_3$ the estimate

$$|S_2 + S_3| \leq Ch^{2k+1} + \frac{1}{C_1} |||\xi|||^2 + C \int_{\partial\Omega_+ \times I} \eta^2 |G^h \cdot \mathbf{n}| \, dv \, ds + C \sum_{m=1}^{M} |\eta|_m^2.$$

The terms S_4 and S_5 are estimated as in Lemma 3.4. Moreover, from the definition of operators R and r_e and from the fact that η is a continuous function we can easily deduce that $S_6 = 0$ and $S_7 = 0$. Thus, it remains to estimate the term S_8. To this end we use (4.3), (4.4), the inverse inequality and assumption on σ to obtain

$$\begin{aligned} |S_8| &\leq \sum_{K \in \mathscr{C}_h} \sigma \|\nabla_v \eta\|_K \|R(\xi)\|_K \leq \sum_{K \in \mathscr{C}_h} \left(C\sigma \|\nabla_v \eta\|_K^2 + \frac{\sigma}{C_1} \|R(\xi)\|_K^2 \right) \\ &\leq Ch^{-1} \|\eta\|_{Q_T}^2 + C_2 \sigma \sum_{e \in \mathscr{E}_v} \|r_e(\xi)\|_{Q_T}^2, \end{aligned} \tag{4.14}$$

where, as above, C_1 is taken to be large enough. $T_2 - T_3$ is estimated in a similar way as in Section 3. Finally combining the estimates for the terms T_1 and $T_2 - T_3$, we obtain the desired result. ■

Now we are ready to prove our error estimate.

Proof of Theorem 4.3. From the definition of B_{DG} and Lemma 4.4, we deduce that the exact solution f satisfies the variational formulation

$$B_{DG}(G(f); f, g) - K(f, g) = L(g) \qquad \forall g \in W_h.$$

So using Lemma 4.1 and some algebraic labor, we get

$$\begin{aligned} \alpha |||\xi|||^2 &\leq B_{DG}(G(f^h); f^h - \tilde{f}^h, \xi) = L(\xi) + K(f^h, \xi) - B_{DG}(G(f^h); \tilde{f}^h, \xi) \\ &= B_{DG}(G(f); f, \xi) - B_{DG}(G(f^h); \tilde{f}^h, \xi) + K(f^h, \xi) - K(f, \xi) \qquad := \Delta B + \Delta K. \end{aligned}$$

Here the term ΔK is similar to the one given in the SD-method and, therefore, is estimated in an analogous way as in the proof of Lemma 3.5. Furthermore, a bound for the term ΔB is given by Lemma 4.5. Note that the coefficients are chosen in a way that the contribution to $|||\xi|||^2$ on the right-hand side is dominated by $\alpha|||\xi|||^2$. Now moving all these contributions to the left-hand side, we complete the proof by a similar argument used in the proof of Theorem 3.3. ■

Conclusion and Final Comments

Two finite element schemes, the streamline diffusion and discontinuous Galerkin, are analyzed for the Vlasov-Fokker-Planck system, which is interpreted as a convection dominated convection diffusion problem of degenerate type. The stability properties of convection dominated problems cause the standard Galerkin finite element method to be nonoptimal compared to interpolation. To improve stability is often obtained at the price of decreased accuracy. For example, increasing artificially the diffusion term (e.g., by simply adding $\sigma\Delta_x f$ to the Fokker-Planck term and then setting $\sigma \sim \delta \sim h$) will increase the stability of the Galerkin method, but may also decrease accuracy and prevent sharp resolution of layers mentioned in the introduction. Thus, the objective is to improve stability without a significant reduce of the accuracy.

To circumvent accuracy reductions, while enhancing the stability of the standard Galerkin, one may introduce two modifications on the test functions connected to the residual so that: the modification (1) introduces weighted least square terms increasing stability through least squares control of the residual and modification (2) introduces artificial viscosity based on the residual, which adds to the stability by introducing an elliptic term with the size of the diffusion coefficient or *viscosity,* depending on the residual with the effect that viscosity is added when the residual is large. Both modifications improve stability without a strong effect on the accuracy.

In general, SD strategy is based on these kind of modifications. However, for this study we do not need to involve the residual based modifications and to obtain higher order accuracy, we simply assume sufficiently regular data and let δ in the streamline diffusion test function $g + \delta\omega \cdot \mathcal{D}g$ to be of order h.

In the SD-method the test and trial function spaces are different and the test functions are continuous in space and velocity domains and may be discontinuous in time. In the discontinuous Galerkin test and trial function spaces are the same (this has some advantages in the analysis) and there are allowed discontinuities across the interelement boundaries in space, velocity, and time. Therefore, jump discontinuities are introduced presenting a different, and extra, diffusion term which is a further source of enhancing of the stability.

For both, SD and DG, schemes we derive optimal convergence rates of order $\mathcal{O}(h^{k+1/2})$, provided that the exact solution f is in the Sobolev space $H^{k+1}(Q_T)$, where h is the global mesh size and k is the order of approximation polynomial. The corresponding error estimate for the standard Galerkin is only of order $\mathcal{O}(h^k)$. This study is unifying the convergence theory for the SD and DG finite element methods for some fluid- and gas-dynamic problems. See, e.g., [23, 26–28].

Some simple and special cases of our model problem are implemented in [30] and [31]. These models, although simple, are good examples supporting the theory. Further, numerical considerations are currently under development.

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