SIAM J. NUMER. ANAL.
Vol. 26, No. 1, pp. 66-87, February 1989

005

L_p AND EIGENVALUE ERROR ESTIMATES FOR THE DISCRETE ORDINATES METHOD FOR TWO-DIMENSIONAL NEUTRON TRANSPORT*

MOHAMMAD ASADZADEH†

Abstract. The convergence of the discrete ordinates method is studied for angular discretization of the neutron transport equation for a two-dimensional model problem with the constant total cross section and isotropic scattering. Considering a symmetric set of quadrature points on the unit circle, error estimates are derived for the scalar flux in L_p norms for $1 \leqq p \leqq \infty$. A postprocessing procedure giving improved L_∞ estimates is also analyzed. Finally error estimates are given for simple isolated eigenvalues of the solution operator.

Key words. neutron transport equation, discrete ordinates method, scalar flux, L_p estimates, eigenvalue estimates, postprocessing, quadrature rule

AMS(MOS) subject classifications. primary 65N15, 65N30

Introduction. In this paper we study the convergence of the Nyström discrete ordinates method for angular discretization of the neutron transport equation for a two-dimensional model problem. We extend the L_2 error analysis of [9] to L_p norms, $1 \leqq p \leqq \infty$. We also analyze a postprocessing procedure and obtain improved rates of convergence in L_∞ for the postprocessed solution. Further we derive error estimates for isolated eigenvalues with algebraic multiplicity one.

These studies contain the following important aspects.

(1) Studying L_1 estimates is of practical interest because the eigenvalue estimates are based on L_1 results. To derive L_1 estimates the function spaces involved are interpolation spaces but not Sobolev spaces. Working with the interpolation spaces on the topic of the neutron transport equation is new.

(2) L_∞, being the strongest norm, is the one we are most motivated to consider; however, we should expect lower rates of convergence than in the L_1 case. Previous L_∞ results have been without rates of convergence. In this paper we derive L_∞ rates of convergence and employ a postprocessing procedure to improve these rates and obtain L_∞ error estimates with the same rates as in L_1.

(3) The classical techniques in [6] and [17] are not useful in deriving eigenvalue error estimates for this problem. This depends on the behavior of the operators involved in the problem. A new functional analysis approach has been made to show the equality of the dimensions of some eigenspaces.

The steady state one-velocity process of transport of neutrons in a substance surrounded by vacuum can be formulated as follows. Given the source f and the coefficients α and σ, find the angular flux u satisfying

$$\mu\cdot\nabla u(x,\mu)+\alpha(x)u(x,\mu)=\int_{S^2}\sigma(x,\mu,\eta)u(x,\eta)\,d\eta+f(x,\mu), \quad (x,\mu)\in\Omega\times S^2, \tag{0.1}$$

$$u(x,\mu)=0 \quad \text{for } x\in\Gamma_\mu^-=\{x\in\Gamma\colon \mu\cdot n(x)<0\},$$

where Ω is a domain in R^3, $\Gamma=\partial\Omega$, S^2 is the unit sphere in R^3, $n(x)$ is the outward

* Received by the editors March 18, 1987; accepted for publication (in revised form) December 4, 1987.

† Chalmers University of Technology, Göteborg, Sweden, and University of Göteborg, Department of Mathematics, S-412 96 Göteborg, Sweden.

unit normal to Γ at $x \in \Gamma$, and

$$\mu \cdot \nabla = \sum_{i=1}^{3} \mu_i \frac{\partial}{\partial x_i}.$$

In the discrete ordinates method the integral in (0.1) is replaced, using a quadrature approximation, by a discrete sum involving a finite set of directions. L_∞ convergence of the discrete ordinates method for a two-dimensional problem is studied, e.g., in [11], and for a three-dimensional problem in [19] with no convergence rates. Combined spatial and angular discretizations in the slab case are studied in [14], where error estimates are derived for both the scalar flux in L_p norms, $1 \leqq p \leqq \infty$, and for critical eigenvalues. In [9] an L_2 analysis of a fully discrete scheme for a two-dimensional model problem is carried out, where the discrete ordinates method for the angular variable is combined with the discontinuous Galerkin finite element method for the spatial variable. In [2] the results of [9] are extended to a case where the angular variable varies on the unit disc in R^2 while the spatial variable remains two-dimensional as in [9]. This corresponds to the three-dimensional problem (0.1) with Ω being an infinite cylindrical domain where all functions involved are assumed to be constant along the axis of the cylinder.

An outline of this paper is as follows. In § 1 we present a two-dimensional model problem obtained by taking, in (0.1), $x \in \Omega \subset R^2$, $\mu \in S = \{\mu \in R^2 \colon |\mu| = 1\}$, $\alpha \equiv 1$, and $\sigma \equiv \lambda$, and we reformulate this problem as a Fredholm integral equation of the second kind with a compact integral operator T. In § 2 we formulate a semidiscrete analogue of the model problem by applying a quadrature rule and we note that the semidiscrete analogue can also be formulated as an integral equation involving a certain operator T_N, where N indicates the number of quadrature points. We prove that our integral operator T is self-adjoint in $L_2(\Omega)$ and assuming a symmetric distribution of the quadrature points this is also true for the approximate operator T_N. In § 3 we derive L_p error estimates, $1 \leqq p < \infty$, for the semidiscrete problem. We also prove that if $\lambda^{-1} \notin \sigma(T)$ and N is sufficiently large then $\lambda^{-1} \notin \sigma(T_N)$, where $\sigma(T)$ denotes the spectrum of T. Section 4 is devoted to a postprocessing procedure giving improved rates of convergence in L_∞ for the scalar flux. In § 5 we prove error estimates for the corresponding approximation of simple isolated eigenvalues of our integral operator T. In the concluding § 6, we give some numerical results testing the analysis of §§ 3 and 4.

1. A model problem. We will consider the following two-dimensional model problem. Given a function f and a parameter λ, find $u(x, \mu)$ such that

$$\begin{aligned} &\mu \cdot \nabla u(x, \mu) + u(x, \mu) = \lambda \int_S u(x, \eta)\, d\eta + f(x), \quad \text{for } (x, \mu) \in \Omega \times S, \\ &u(x, \mu) = 0 \quad \text{on } \Gamma_\mu^- = \{x \in \Gamma \colon \mu \cdot n(x) < 0\}, \end{aligned} \tag{1.1}$$

where Ω is a bounded convex polygonal domain in R^2 with boundary Γ, S is the unit circle and $n(x)$ is the outward unit normal to Γ at $x \in \Gamma$.

When we introduce the scalar flux

$$U(x) = \int_S u(x, \eta)\, d\eta, \tag{1.2}$$

(1.1) takes the following form:

$$\begin{aligned} &\mu \cdot \nabla u(x, \mu) + u(x, \mu) = (\lambda U + f)(x), \qquad (x, \mu) \in \Omega \times S, \\ &u(x, \mu) = 0 \quad \text{on } \Gamma_\mu^-. \end{aligned} \tag{1.3}$$

Introducing the solution operator T_μ for the equation $\mu\cdot\nabla u+u=g$, in Ω, $u=0$ on Γ_μ^-, given by

$$T_\mu g(x)=\int_0^{d(x,\mu)} e^{-s}g(x-s\mu)\,ds, \tag{1.4}$$

with

$$d(x,\mu)=\inf\{s>0\colon (x-s\mu)\notin\Omega\}, \tag{1.5}$$

we can write (1.3) as

$$u(x,\mu)=T_\mu(\lambda U+f)(x), \qquad (x,\mu)\in\Omega\times S. \tag{1.6}$$

Integrating (1.6) over S, we get the following equation for the scalar flux U:

$$(I-\lambda T)U=Tf, \tag{1.7}$$

where

$$T=\int_S T_\mu\,d\mu. \tag{1.8}$$

When we use (1.4), it is easy to see that

$$Tg(x)=\int_\Omega \frac{e^{-|x-y|}}{|x-y|}g(y)\,dy. \tag{1.9}$$

Thus, T is an integral operator with weakly singular kernel, and hence, T is a compact operator on $L_p(\Omega)$, $1\leqq p\leqq\infty$. Consequently, (1.7) is a Fredholm integral equation of the second kind (see, e.g., [18]).

Let us introduce some notation to be used below. For $1\leqq p\leqq\infty$, we denote by $\sigma_p(K)$ the spectrum of an operator $K:L_p(\Omega)\to L_p(\Omega)$, defined by $\sigma_p(K)=\{z\in\mathbb{C}\colon (K-zI)$ is not invertible as an operator on $L_p(\Omega)\}$. $\|\cdot\|_p$ and $\|\cdot\|_{W_p^m}$, m a positive integer, denote the usual L_p norms and the Sobolev norms of order m, respectively. $\|\cdot\|_p$ will also denote the operator norm $\|\cdot\|_{(L_p,L_p)}$ on $L_p(\Omega)$. C will denote positive constants, not necessarily the same at each occurrence, independent of the parameters N and ε. Note that since $\sigma_p(T)$ is independent of p, below we will use the notation $\sigma(T)$ instead of $\sigma_p(T)$.

We will assume that the parameter λ^{-1} does not belong to $\sigma(T)$. Then for $1\leqq p\leqq\infty$, $(I-\lambda T)\colon L_p(\Omega)\to L_p(\Omega)$ is invertible, Range $(I-\lambda T)=L_p(\Omega)$, and there is a constant C such that $\|(I-\lambda T)^{-1}\|_p\leqq C$, $1\leqq p\leqq\infty$. In particular

$$\|(I-\lambda T)v\|_p\geqq C\|v\|_p \quad \forall v\in L_p(\Omega). \tag{1.10}$$

Hence the integral equation (1.7) has a unique solution $U=(I-\lambda T)^{-1}Tf$.

2. The quadrature rule. Let $Q\equiv Q_N=\{\mu_1,\cdots,\mu_N\}$ be a set of points on the unit circle S with the property that if $\mu\in Q$ then $-\mu\in Q$. Consider the quadrature rule

$$\int_S u(x,\mu)\,d\mu\sim\sum_{\mu\in Q}u(x,\mu)\omega_\mu, \tag{2.1}$$

where $\omega_\mu=2\pi/N$. Other standard quadrature rules for the neutron transport equation are discussed by Lewis and Miller in [10]. For the semidiscrete approximation of the scalar flux, we set

$$U_N(x)=\sum_{\mu\in Q}u_N(x,\mu)\omega_\mu, \tag{2.2}$$

with $u_N(x, \mu)$ satisfying

$$u_N(x, \mu) = T_\mu(\lambda U_N + f)(x), \qquad x \in \Omega, \quad \mu \in Q. \tag{2.3}$$

Multiplying (2.3) by ω_μ and summing over Q we obtain the integral equation

$$(I - \lambda T_N)U_N = T_N f, \tag{2.4}$$

where

$$T_N = \sum_{\mu \in Q} T_\mu \omega_\mu. \tag{2.5}$$

For the error in the quadrature (2.1) we have the following estimate (see, e.g., [7]). For $k = 1, 2$, there exists a constant C such that

$$\left| \int_S u(x, \mu)\, d\mu - \sum_{\mu \in Q} u(x, \mu)\omega_\mu \right| \leqq \frac{C}{N^k} \int_0^{2\pi} \left| \frac{\partial^k u}{\partial \alpha^k}(x, \alpha) \right| d\alpha, \tag{2.6a}$$

where $\mu = (\cos \alpha, \sin \alpha)$. Also

$$\Sigma''_\varepsilon \omega_\mu \omega_\nu \to 0 \quad \text{as } \max\left(\frac{1}{N}, \varepsilon\right) \to 0, \tag{2.6b}$$

where for $\varepsilon > 0$ the sum $\sum_{(\mu,\nu)\in Q^2}$ is split as follows:

$$\sum_{(\mu,\nu)\in Q^2} = \Sigma'_\varepsilon + \Sigma''_\varepsilon \equiv \sum_{(\mu,\nu)\in I'_\varepsilon} + \sum_{(\mu,\nu)\in I''_\varepsilon}$$

with

$$I'_\varepsilon = \{(\mu, \nu) \in Q^2 : \min(\sin \gamma(\mu, \nu), \sin \gamma(\mu, d_n), \sin \gamma(\nu, d_n)) \geqq \varepsilon,\ n = 1, 2, \cdots, P_0\}$$

and

$$I''_\varepsilon = Q^2 \backslash I'_\varepsilon,$$

where $\gamma(\mu, \nu)$ is the smallest angle between μ and ν, d_n, $n = 1, \cdots, p_0$, are the directions of the sides of Ω, and P_0 is the number of sides of Ω.

Next we recall a result of Anselone [1]. We then show that the continuous integral operator T is self-adjoint in $L_2(\Omega)$ and that T_N is self-adjoint in $L_2(\Omega)$, since $-\mu \in Q$ if $\mu \in Q$.

PROPOSITION 2.1. *Let* $1 \leqq p \leqq \infty$ *and let* $T: L_p(\Omega) \to L_p(\Omega)$ *be a bounded linear operator such that for some positive constant* C, (1.10) *is valid, i.e.,*

$$\|(I - \lambda T)v\|_p \geqq C\|v\|_p \quad \forall v \in L_p(\Omega),$$

and let $\{T_N\}_{N=1}^\infty$ *be a uniformly bounded sequence of linear operators on* $L_p(\Omega)$ *such that for some positive integer* m,

$$\varepsilon_N = \|(T - T_N)T_N^m\|_p \to 0 \quad \text{as } N \to \infty. \tag{2.7}$$

Then there is a positive constant C_1 *such that for* N *large enough*

$$\|(I - \lambda T_N)v\|_p \geqq C_1\|v\|_p \quad \forall v \in L_p(\Omega). \tag{2.8}$$

In the sequel T^* *denotes the adjoint operator of* $T: L_p(\Omega) \to L_p(\Omega)$.

LEMMA 2.1. *The integral operators* T *and* T_N *are self-adjoint on* $L_2(\Omega)$.

Proof. Recalling the representation (1.9) of T,

$$Tg(x) = \int_\Omega \frac{e^{-|x-y|}}{|x-y|} g(y)\, dy,$$

we have for $f, g \in L_2(\Omega)$,

$$(Tf, g) = \int_\Omega [Tf(x)]g(x)\,dx = \int_\Omega \left(\int_\Omega \frac{e^{-|x-y|}}{|x-y|} f(y)\,dy \right) g(x)\,dx$$

and

$$(f, Tg) = \int_\Omega f(x)[Tg(x)]\,dx = \int_\Omega \left(\int_\Omega \frac{e^{-|x-y|}}{|x-y|} g(y)\,dy \right) f(x)\,dx,$$

which proves that $T = T^*$. When we use (2.5), the adjoint of T_N is

$$T_N^* = \sum_{\mu \in Q} T_\mu^* \omega_\mu.$$

Moreover,

$$(T_\mu f, g) = \int_\Omega \left(\int_0^{d(x,\mu)} e^{-s} f(x - s\mu)\,ds \right) g(x)\,dx.$$

Making the substitution $y = x - s\mu$, we note that as s varies, y varies on the line segment $[\bar{x}, x]$, $\bar{x} = x - d(x, \mu)\mu$. Thus for a given $y \in \Omega$ we have $s = |x - y|$ with $x \in \Omega \cap L_\mu(y)$, where $L_\mu(y)$ is the half-line parallel to μ starting at y. Hence by the definition (1.5) of d, and since $0 \leqq s \leqq d(y, -\mu)$,

$$(T_\mu f, g) = \int_\Omega \int_0^{d(y,-\mu)} e^{-s} f(y) g(y - s(-\mu))\,ds\,dy = (f, T_{-\mu} g),$$

so that

$$T_\mu^* = T_{-\mu}. \tag{2.9}$$

Multiplying by ω_μ and summing over $\mu \in Q$, we obtain $T_N^* = T_N$, since $\mu \in Q$ implies $-\mu \in Q$. □

Note that by (1.4) we have the following stability estimate for the solution operator T_μ:

$$\|\mu \cdot \nabla T_\mu g\|_p + \|T_\mu g\|_p \leqq C\|g\|_p, \qquad 1 \leqq p \leqq \infty. \tag{2.10}$$

3. L_p error estimates. In this section we extend the L_2 error estimates for the discrete ordinates method of [9] to L_p norms, $1 \leqq p < \infty$. Our main result is Theorem 3.1. We also prove, using Proposition 2.1, that if $\lambda^{-1} \notin \sigma(T)$ then for $1 \leqq p \leqq \infty$ the operator $(I - \lambda T_N): L_p(\Omega) \to L_p(\Omega)$ is invertible if N is large enough and thus (2.4) has a unique solution $U_N \in L_p(\Omega)$.

Observe that the maximum regularity of the scalar flux U, what we can expect in general, is $U \in W_p^1(\Omega)$ for $1 \leqq p < \infty$ and $U \in W_1^{2-\delta}(\Omega)$ for $\delta > 0$ (see [13]). Theorem 3.1 is stated accordingly. Here $W_1^{2-\delta}(\Omega)$ is defined by the K method of interpolation (see Bergh and Löfström [5]).

THEOREM 3.1. *Suppose that $\lambda^{-1} \notin \sigma(T)$ and let $1 \leqq p < \infty$. Let U be the solution of* (1.7). *Then there exists an integer N_λ such that* (2.4) *has a unique solution $U_N \in L_p(\Omega)$ for $N \geqq N_\lambda$. Further, there is a constant C such that for $N \geqq N_\lambda$ and $f \in W_p^1(\Omega)$,*

$$\|U - U_N\|_{L_p(\Omega)} \leqq CN^{-1}\|\lambda U + f\|_{W_p^1(\Omega)}. \tag{3.1}$$

There exists a constant C and for all $\theta > 0$, there exists $\delta > 0$ such that for $N \geqq N_\lambda$ and $f \in W_1^{2-\delta}(\Omega)$,

$$\|U - U_N\|_{L_1(\Omega)} \leqq CN^{-2+\theta}\|\lambda U + f\|_{W_1^{2-\delta}(\Omega)}. \tag{3.2}$$

The proof of Theorem 3.1 is based on the following two results.

LEMMA 3.1. *For $1 \leqq p < \infty$ there is a constant $C = C(p)$ such that for $g \in W_p^1(\Omega)$,*

$$\|(T - T_N)g\|_{L_p(\Omega)} \leqq CN^{-1}\|g\|_{W_p^1(\Omega)}. \tag{3.3}$$

Further there is a constant C such that for $g \in W_1^2(\Omega)$,

$$\|(T - T_N)g\|_{L_1(\Omega)} \leqq CN^{-2}(\log N)\|g\|_{W_1^2(\Omega)}. \tag{3.4}$$

LEMMA 3.2. *If $\lambda^{-1} \notin \sigma(T)$, then for $1 \leqq p \leqq \infty$ there is an integer N_λ and a constant C such that for $N \geqq N_\lambda$, $\|(I - \lambda T_N)^{-1}\|_p \leqq C$.*

Let us postpone the proofs of these results and first show that Theorem 3.1 follows from them.

Proof of Theorem 3.1. We have, using (1.7) and (2.4),

$$\begin{aligned} U - U_N &= \lambda TU + Tf - \lambda T_N U_N - T_N f \\ &= \lambda(T - T_N)U + \lambda T_N(U - U_N) + (T - T_N)f, \end{aligned}$$

and thus

$$(I - \lambda T_N)(U - U_N) = (T - T_N)(\lambda U + f).$$

Hence using Lemma 3.2 and (3.3), with $g = \lambda U + f$, we can verify (3.1). Interpolating between (3.3) and (3.4), we obtain (3.2). □

Below $\psi_j \equiv \psi_j(\alpha)$ will denote the angle between the direction of the jth side of Ω and $\mu = (\cos\alpha, \sin\alpha)$. In the proof of Lemma 3.1, we will use the following lemma.

LEMMA 3.3. *For $1 \leqq p < \infty$ there exists a constant C such that if $u(x, \mu) = T_\mu g(x)$, then for $g \in W_p^1(\Omega)$,*

$$\int_0^{2\pi} \left\|\frac{\partial u}{\partial \alpha}(\cdot, \alpha)\right\|_{L_p(\Omega)} d\alpha \leqq C\|g\|_{W_p^1(\Omega)}. \tag{3.5}$$

Further there is a constant C such that for $g \in W_1^2(\Omega)$,

$$\left\|\frac{\partial^2 u}{\partial \alpha^2}(\cdot, \alpha)\right\|_{L_1(\Omega)} \leqq C(\min_j |\sin \psi_j(\alpha)|)^{-1}\|g\|_{W_1^2(\Omega)}. \tag{3.6}$$

Proof. By the same argument as in the proof of Lemma 4.4 in [9], we have

$$u(x, \alpha) = \int_0^{d(x,\alpha)} e^{-s} g(x - s\mu)\, ds,$$

where $d(x, \alpha) \equiv d(x, \mu)$, so that

$$\frac{\partial}{\partial \alpha} u(x, \alpha) = e^{-d(x,\alpha)} g(\bar{x}_\alpha) \frac{\partial}{\partial \alpha} d(x, \alpha) + \int_0^{d(x,\mu)} e^{-s} s \frac{\partial}{\partial \mu'} g(x - s\mu)\, ds. \tag{3.7}$$

Here $\bar{x}_\alpha = x - d(x, \alpha)\mu \in \Gamma$ and $\mu' = (\sin\alpha, -\cos\alpha)$ is orthogonal to μ. Further,

$$d(x, \alpha) = \frac{a_j(x)}{\sin \psi_j(\alpha)} \quad \text{for } x \in \Omega_{\alpha,j} = \{x \in \Omega : \bar{x}_\alpha \in S_j\}, \tag{3.8}$$

where the S_j is a side of Ω, $\psi_j(\alpha)$ the angle between S_j and μ, and $a_j(x)$ is the distance from x to the straight line given by S_j. Hence raising the absolute values of both sides of (3.7) to the power p, integrating over $\Omega_{\alpha,j}$, using an orthogonal coordinate system

(ξ_1, ξ_2) with the ξ_1 axis along S_j, and the fact that the boundedness of Ω implies $a_j(x) \leqq C|\sin \psi_j|$ and thus $|(\partial/\partial\alpha)\, d(x, \alpha)| \leqq C|\sin \psi_j|^{-1}$ for $x \in \Omega_{\alpha,j}$, we get

$$\int_{\Omega_{\alpha,j}} \left|\frac{\partial}{\partial\alpha} u(x, \alpha)\right|^p dx \leqq C\left[\int_{\Omega_{\alpha,j}} |g(\bar{x}_\alpha)|^p |\sin \psi_j|^{-p}\, d\xi_1\, d\xi_2 + \int_{\Omega_{\alpha,j}} |\nabla g(x)|^p\, dx\right]$$

$$\leqq C\left[\int_{S_j} |g|^p\, d\xi_1 \int_0^{C|\sin\psi_j|} |\sin \psi_j|^{-p}\, d\xi_2 + \|\nabla g\|_p^p\right]$$

$$\leqq C[\|g\|^p_{L_p(\Gamma)} |\sin \psi_j|^{1-p} + \|\nabla g\|_p^p].$$

Recalling the trace estimate

$$\|g\|_{L_p(\Gamma)} \leqq C\|g\|_{W_p^1(\Omega)}, \qquad 1 \leqq p \leqq \infty,$$

we have summing over j,

$$\left\|\frac{\partial}{\partial\alpha} u(\cdot, \alpha)\right\|_p \leqq C(\min_j |\sin \psi_j(\alpha)|)^{-1+1/p} \|g\|_{W_p^1(\Omega)}. \tag{3.9}$$

Finally we complete the proof of (3.5) by integrating with respect to α. To prove (3.6) we differentiate (3.7) and (3.8) to get

$$\frac{\partial^2 u}{\partial\alpha^2}(x, \alpha) = -e^{-d(x,\alpha)} g(\bar{x}_\alpha)\left(\frac{\partial}{\partial\alpha} d(x, \alpha)\right)^2 + e^{-d(x,\alpha)} g(\bar{x}_\alpha) \frac{\partial^2 d}{\partial\alpha^2}(x, \alpha)$$
$$+ 2\, e^{-d(x,\alpha)} \frac{\partial}{\partial\alpha} g(\bar{x}_\alpha) \frac{\partial d}{\partial\alpha}(x, \alpha) + \int_0^{d(x,\mu)} e^{-s} \frac{\partial^2}{\partial\alpha^2} g(x - s\mu)\, ds \tag{3.10}$$

and

$$\frac{\partial^2}{\partial\alpha^2} d(x, \alpha) = \frac{a_j(x)(1 + \cos^2 \psi_j)}{\sin^3 \psi_j}, \tag{3.11}$$

so that by the same argument as in the proof of (3.5),

$$\int_{\Omega_{\alpha,j}} \left|\frac{\partial^2}{\partial\alpha^2} u(x, \alpha)\right| dx \leqq C\left[\int_{S_j} (|g| + |\nabla g|)\, d\xi_1 \int_0^{C|\sin\psi_j|} |\sin \psi_j|^{-2}\, d\xi_2 + \|g\|_{W_1^2(\Omega)}\right].$$

Using the trace estimate once more and summing over j, we get

$$\left\|\frac{\partial^2}{\partial\alpha^2} u(\cdot, \alpha)\right\|_{L_1(\Omega)} \leqq C(\min_j |\sin \psi_j(\alpha)|)^{-1} \|g\|_{W_1^2(\Omega)}, \tag{3.12}$$

and the proof is complete. □

Proof of Lemma 3.1. Writing $u(x, \alpha) = T_\mu g(x)$ with $\mu = (\cos \alpha, \sin \alpha)$ and using (2.6a) with $k = 1$, we have

$$\|(T - T_N)g\|_{L_p(\Omega)} = \left\|\int_S T_\mu g(\cdot)\, d\mu - \sum_{\mu\in Q} T_\mu g(\cdot)\omega_\mu\right\|_{L_p(\Omega)}$$

$$= \left\|\int_S u(\cdot, \mu)\, d\mu - \sum_{\mu\in Q} u(\cdot, \mu)\omega_\mu\right\|_{L_p(\Omega)}$$

$$\leqq CN^{-1} \left\|\int_0^{2\pi} \frac{\partial u}{\partial\alpha}(\cdot, \alpha)\, d\alpha\right\|_{L_p(\Omega)}$$

$$\leqq CN^{-1} \int_0^{2\pi} \left\|\frac{\partial u}{\partial\alpha}(\cdot, \alpha)\right\|_{L_p(\Omega)} d\alpha,$$

and (3.3) then follows from (3.5). To prove (3.4) we define for $k=1,\cdots,N$,

$$I_k=\left[\frac{2(k-1)\pi}{N},\frac{2k\pi}{N}\right],$$

and let A_j be the union of the I_k containing the direction of S_j and the adjacent I_l closest to the direction of S_j. Let

$$S_0=\bigcup_{j=1}^{P_0} A_j,$$

and

$$Q_0=Q\cap S_0.$$

We have, using (2.6a), (3.6), and (3.9),

$$\begin{aligned}\|(T-T_N)g\|_{L_1(\Omega)}&\leqq\left\|\int_{S\setminus S_0} T_\mu g(\cdot)\,d\mu-\sum_{\mu\in Q\setminus Q_0} T_\mu g(\cdot)\omega_\mu\right\|_{L_1(\Omega)}\\&\quad+\left\|\int_{S_0} T_\mu g(\cdot)\,d\mu-\sum_{\mu\in Q_0} T_\mu g(\cdot)\omega_\mu\right\|_{L_1(\Omega)}\\&\leqq\frac{C}{N^2}\int_{S\setminus S_0}\left\|\frac{\partial^2 u(\cdot,\alpha)}{\partial\alpha^2}\right\|_{L_1(\Omega)} d\alpha+\frac{C}{N}\int_{S_0}\left\|\frac{\partial u(\cdot,\alpha)}{\partial\alpha}\right\|_{L_1(\Omega)} d\alpha\\&\leqq\frac{C}{N^2}\left[\int_{S\setminus S_0}(\min_j|\sin\psi_j(\alpha)|)^{-1}\,d\alpha\right]\|g\|_{W_1^2(\Omega)}+\frac{C}{N}|S_0|\,\|g\|_{W_1^1(\Omega)}\\&\leqq\frac{C}{N^2}(\log N)\,\|g\|_{W_1^2(\Omega)}+\frac{C}{N}|S_0|\,\|g\|_{W_1^1(\Omega)},\end{aligned}$$

where $|S_0|$ is the length of S_0. Since $|S_0|\sim 1/N$ we obtain the desired result. □

Let us now turn to the proof of Lemma 3.2. We want to prove for $1\leqq p\leqq\infty$ that if $\lambda^{-1}\notin\sigma(T)$ and N is sufficiently large depending on λ and p, then $(I-\lambda T_N)$ is invertible as an operator on $L_p(\Omega)$. Lemma 3.4 below together with Proposition 2.1 implies that $(I-\lambda T_N)$ is one to one. To show that it is also onto we will use Proposition 3.1 below (see Rudin [16, Thm. 4.15, p. 97]). Note that $T_N:L_p(\Omega)\to L_p(\Omega)$ is not compact.

LEMMA 3.4. *$\|(T-T_N)T_N^2\|_p\to 0$, as $N\to\infty$, $1\leqq p\leqq\infty$.*

PROPOSITION 3.1. *Suppose X and Y are Banach spaces and F is a bounded linear operator from X into Y; then*

(a) Range $(F)=Y$

if and only if there exists a constant C such that

(b) $\|F^*y^*\|\geqq C\|y^*\|$ *for* $y^*\in Y^*$.

Proof of Lemma 3.2. It remains to prove that Range $(I-\lambda T_N)=L_p(\Omega)$. Hence it suffices to prove (b) in Proposition 3.1 with F replaced by $(I-\lambda T_N)$. Now Proposition 2.1 applied to T^*, where

$$\|(I-\lambda T^*)v^*\|_p\geqq C\|v^*\|_p\quad\forall v^*\in L_p(\Omega),$$

and the same argument as in the proof of Lemma 3.4 below yield

$$\varepsilon_N^*=\|(T^*-T_N^*)T_N^{*2}\|_p\to 0\quad\text{as } N\to\infty.$$

Thus the conclusion of Proposition 2.1 holds for the adjoint operator T_N^*, i.e., there exists a constant C such that (b) in Proposition 3.1 is valid with F^* and y^* replaced by $(I-\lambda T_N)^*$ and v^*, respectively. Observe that since L_∞ is not a reflexive Banach space, the conclusion $\|(I-\lambda T_N)^{-1}\|_p \leqq C$ is only valid for $1\leqq p<\infty$. However, the constants involved in Proposition 2.1 as well as in this proof are independent of p. Thus letting $p\to\infty$ we also obtain the result for $p=\infty$. □

The proof of Lemma 3.4 is based on (3.3) and the following lemma.

LEMMA 3.5. *There is a constant C such that if* $(\mu, \nu)\in I'_\varepsilon$ *and* $g\in L_p(\Omega)$, $1\leqq p\leqq\infty$, *then*

$$\|T_\mu T_\nu g\|_{W_p^1(\Omega)} \leqq C\varepsilon^{-2+1/p}\|g\|_{L_p(\Omega)}.$$

Remark 3.1. Note that the operator T_μ regularizes in the direction μ and thus for two nonparallel directions μ and ν, we have that $T_\mu T_\nu$ regularizes in all directions (with a constant depending on the smallest angle between μ and ν). In the proof of Lemma 3.4, we will use this regularizing property of T_μ to show that T_N^2 can be split as

$$T_N^2 = A_N + B_N,$$

with $A_N : L_p(\Omega)\to W_p^1(\Omega)$; i.e., A_N is compact, and $\|B_N\|\to 0$, as $N\to\infty$.

Proof of Lemma 3.4. For $1\leqq p<\infty$ we have, using (3.3), Lemma 3.5 and (2.10),

$$\begin{aligned}\|(T-T_N)T_N^2 g\|_{L_p(\Omega)} &= \Big\|(T-T_N)\sum_{(\mu,\nu)\in Q^2}\omega_\mu\omega_\nu T_\mu T_\nu g\Big\|_{L_p(\Omega)}\\ &= \|(T-T_N)(\Sigma'_\varepsilon+\Sigma''_\varepsilon)\omega_\mu\omega_\nu T_\mu T_\nu g\|_{L_p(\Omega)}\\ &\leqq C\Sigma'_\varepsilon\omega_\mu\omega_\nu N^{-1}\|T_\mu T_\nu g\|_{W_p^1(\Omega)} + C\Sigma''_\varepsilon\omega_\mu\omega_\nu\|g\|_{L_p(\Omega)}\\ &\leqq C(N^{-1}\varepsilon^{-2+1/p}+\Sigma''_\varepsilon\omega_\mu\omega_\nu)\|g\|_{L_p(\Omega)}.\end{aligned}$$

Now taking $\varepsilon = N^{-\delta}$ with $\delta<p(p-1)^{-1}$ and using (2.6b), we obtain the desired result. For $p=\infty$, instead of (3.3), we use Lemma 4.1 below and the proof is complete. □

In the proof of Lemma 3.5 we will refer to the following result.

LEMMA 3.6. *There is a constant C such that*

$$\left(\int_\Gamma |T_\mu g|^p |n\cdot\mu|\, d\sigma\right)^{1/p} \leqq C\|g\|_{L_p(\Omega)},\qquad 1\leqq p\leqq\infty.$$

Proof. It suffices to give the proof with Γ replaced by $\Gamma' = \{x\in\Gamma\colon n(x)\cdot\mu\neq 0\}$. Let $p=1$. For $x\in\Gamma'$, we have $d(x,\mu)\neq 0$, and if $0\leqq s\leqq d(x,\mu)$ then $x-s\mu\in[x, x-d(x,\mu)\mu]\subset\Omega$. Thus using (1.4) and Fubini's theorem, we find that

$$\begin{aligned}\int_{\Gamma'}|T_\mu g(x)||n\cdot\mu|\,d\sigma &= \int_{\Gamma'}\left|\int_0^{d(x,\mu)} e^{-s}g(x-s\mu)\,ds\right||n\cdot\mu|\,d\sigma\\ &\leqq \int_0^{d(x,\mu)} e^{-s}\int_{\Gamma'}|g(x-s\mu)||n\cdot\mu|\,d\sigma\,ds\\ &\leqq C\|g\|_{L_1(\Omega)}.\end{aligned}$$

For $p=\infty$ we have

$$\sup_{x\in\Gamma'}|T_\mu g(x)||n\cdot\mu| = \sup_{x\in\Gamma'}\left|\int_0^{d(x,\mu)} e^{-s}g(x-s\mu)\,ds\right||n\cdot\mu| \leqq C\sup_{x\in\Omega}|g(x)|.$$

Interpolating between $p=1$ and $p=\infty$ we obtain the desired result. □

Proof of Lemma 3.5. By an orthogonal coordinate transformation we may assume that $\mu=(1,0)$. Since $(\mu,\nu)\in I'_\varepsilon$ we have by (2.10),

$$\|\nabla(T_\mu T_\nu g)\|_{L_p(\Omega)}\leqq\frac{C}{\varepsilon}\left[\|g\|_{L_p(\Omega)}+\left\|\frac{\partial}{\partial\nu}(T_\mu T_\nu g)\right\|_{L_p(\Omega)}\right].\tag{3.13}$$

Replacing g in (1.4) by $T_\nu g$ gives

$$\frac{\partial}{\partial\nu}(T_\mu T_\nu g(x))=e^{-d(x,\mu)}T_\nu g(\bar{x})\frac{\partial}{\partial\nu}d(x,\mu)+\int_0^{d(x,\mu)}e^{-s}\frac{\partial}{\partial\nu}(T_\nu g(x-s\mu))\,ds,\tag{3.14}$$

where $\bar{x}=x-d(x,\mu)\mu$. It is easy to verify that

$$\frac{\partial d}{\partial\nu}=\nu\cdot\nabla_x d(x,\mu)=\frac{n\cdot\nu}{n\cdot\mu},\tag{3.15}$$

where $n=(n_1,n_2)$ is the outward unit normal to Γ at $\bar{x}$. Raising the absolute values of both sides of (3.14) to the power of p and using the facts that $dx_2=\mu\cdot n\,d\sigma$ on Γ and $|\mu\cdot n|\geqq\varepsilon$ we find that

$$\left\|\frac{\partial}{\partial\nu}(T_\mu T_\nu g)\right\|_{L_p(\Omega)}\leqq C\varepsilon^{-1+1/p}\left(\int_\Gamma|T_\nu g|^p|n\cdot\nu|\,d\sigma\right)^{1/p}+C\|g\|_{L_p(\Omega)},$$

where we have also used (2.10). Thus by Lemma 3.6 and (3.13)

$$\|\nabla(T_\mu T_\nu g)\|_{L_p(\Omega)}\leqq C\varepsilon^{-2+1/p}\|g\|_{L_p(\Omega)}.$$

Further by (2.10), $\|T_\mu T_\nu g\|_{L_p(\Omega)}\leqq C\|g\|_{L_p(\Omega)}$ and the proof is complete. □

4. Iterative improvement. In the previous section we have proved, with maximally available regularity of the scalar flux U, i.e., $U\in W_p^1(\Omega)$, for $1\leqq p<\infty$, and $U\in W_1^{2-\delta}(\Omega)$ with $\delta>0$, that

$$\|U-U_N\|_{L_p(\Omega)}\leqq CN^{-1},\tag{4.1}$$

and

$$\|U-U_N\|_{L_1(\Omega)}\leqq CN^{-2+\theta},\tag{4.2}$$

where C depends on θ and p. Further for $\theta>0$, we have from (4.1) (see also Lemma 4.1 below) that

$$\|U-U_N\|_{L_\infty(\Omega)}\leqq CN^{-1+\theta}.\tag{4.3}$$

In this section we will prove that it is possible by a simple postprocessing to produce an improved solution U_N^* for which

$$\|U-U_N^*\|_{L_\infty(\Omega)}\leqq CN^{-2+\theta},\tag{4.4}$$

that is, for which the rate of convergence of U_N^* in L_∞ is the same as the rate in L_1 for the original solution U_N.

The postprocessed solution U_N^* is defined as follows:

$$U_N^*=T_M(\lambda U_M^{(2)}+f),\tag{4.5}$$

where

$$\begin{aligned}&U_M^{(k+1)}=T_M(\lambda U_M^{(k)}+f),\qquad k=0,1,2,\cdots,\\&U_M^{(0)}=U_N,\end{aligned}\tag{4.6}$$

and $M=N^2$. Thus we compute $U_N^* \equiv U_M^{(3)}$ by applying the operator T_M three times with $M=N^2$, starting from the original solution U_N. The postprocessed solution U_N^* should be compared with the solution U_M of the coupled problem

$$U_M = T_M(\lambda U_M + f). \tag{4.7}$$

By (4.3) and (4.4) we have for both U_N^* and U_M the same rate of convergence $\mathcal{O}(M^{-1+\varepsilon})$ in $L_\infty(\Omega)$, since $M=N^2$. To find U_M requires the solution of a large coupled problem (4.7), while to compute U_N^* we only have to solve the smaller coupled problem (2.4) and then in (4.6) apply the operator T_M a few times. Hence we expect to be able to compute U_N^* with less work than U_M. Note that U_N^* may be viewed as an approximate solution of (4.7) obtained after three fixed-point iterations starting with U_N.

Postprocessing procedures of the form (4.5)–(4.6) have been considered in practical computations and an example is discussed in [11, § VI]. In particular the hope is to decrease in this way the so-called ray effects for media in which absorption dominates scattering.

To estimate $\|U-U_N^*\|_\infty$ we use (2.4) and (4.6) to obtain

$$U = \lambda TU + Tf = \lambda T(\lambda TU + Tf) + Tf,$$

and

$$U_M^{(3)} = \lambda T_M U_M^{(2)} + T_M f = \lambda T_M(\lambda T_M U_M^{(1)} + T_M f) + T_M f.$$

We now split $U-U_N^* \equiv U - U_M^{(3)}$ as follows:

$$\begin{aligned} U-U_N^* &= (T-T_M)f + \lambda T(T-T_M)f + \lambda(T-T_M)T_M f + \lambda^2 T(T-T_M)T_M f \\ &\quad + \lambda^2(T-T_M)T_M^2 f + \lambda^3 T(T-T_M)T_M T_N f + \lambda^3(T-T_M)T_M^2 T_N f \\ &\quad + \lambda^4 T(T-T_M)T_M T_N U_N + \lambda^4(T-T_M)T_M^2 T_N U_N + \lambda^2 T^2(U-U_M^{(1)}) \\ &:= \sum_{j=1}^{7} I_j f + R_1 T_M T_N U_N + R_2 T_M T_N U_N + \lambda^2 T^2(U-U_M^{(1)}), \end{aligned} \tag{4.8}$$

with the obvious meanings of I_j, $j=1,2,\cdots,7$ and R_i, $i=1,2$. Now taking the $L_\infty(\Omega)$-norm of both sides of (4.8), we have

$$\|U-U_N^*\|_{L_\infty(\Omega)} \leqq \sum_{j=1}^{7} \|I_j f\|_{L_\infty(\Omega)} + \sum_{i=1}^{2} \|R_i T_M T_N U_N\|_{L_\infty(\Omega)} + |\lambda|^2 \|T^2(U-U_M^{(1)})\|_{L_\infty(\Omega)}. \tag{4.9}$$

Here the quantities $\|I_j f\|_{L_\infty(\Omega)}$, $j=1,\cdots,7$, will be estimated using Lemma 4.1 below. For $\|R_i T_M T_N U_N\|_{L_\infty(\Omega)}$, $i=1,2$, we will use Lemma 4.2. The last quantity, $\|T^2(U-U_M^{(1)})\|_{L_\infty(\Omega)}$ will be handled using the fact that T is regularizing in the sense that for $\tau>0$, $T^2: L_{1+\tau}(\Omega) \to L_\infty(\Omega)$. In the proof of Lemma 4.2 we will use the following splitting with $\varepsilon \geqq 1/M$:

$$\sum_{(\mu,\nu)\in Q_M\times Q_N} \omega_\mu\omega_\nu = \sum_{(\mu,\nu)\in J_\varepsilon} \omega_\mu\omega_\nu + \sum_{(\mu,\nu)\in J'_\varepsilon} \omega_\mu\omega_\nu, \tag{4.10}$$

where

$$J_\varepsilon = \{(\mu,\nu)\in Q_M\times Q_N : \min(\sin\gamma(\mu,\nu), \sin\gamma(\mu,d_n)) \geqq \varepsilon,\ n=1,\cdots,P_0\},$$

$$J'_\varepsilon = Q_M\times Q_N \backslash J_\varepsilon,$$

$$\omega_\mu = \frac{2\pi}{M} \quad\text{and}\quad \omega_\nu = \frac{2\pi}{N}.$$

We recall that, as in the previous splitting, $\gamma(\mu, \nu)$ is the smallest angle between μ and ν, P_0 is the number of sides of Ω, and d_n are the directions of the sides of Ω.

Now we are prepared to state the main result of this section.

THEOREM 4.1. *Suppose that* $\lambda^{-1} \notin \sigma(T)$ *and let* $\theta > 0$. *Then there exist constants* $\delta > 0$, $\tau > 0$, C_λ *and* N_λ *such that for* $N \geqq N_\lambda$ *and* $M \sim N^2$,

$$\| U - U_N^* \|_{L_\infty(\Omega)} \leqq C_\lambda \left(\frac{1}{N^{2-\theta}} + \frac{1}{N^2} (\log N)^3 \right) (\| g \|_{W^{2-\delta}_{1+\tau}(\Omega)} + \| g \|_{W^1_\infty(\Omega)}),$$

where $g = \lambda U + f$.

The proof of Theorem 4.1 is based on the following three results.

LEMMA 4.1. *There is a constant* $C > 0$ *such that*

$$\| (T - T_M) f \|_{L_\infty(\Omega)} \leqq \frac{C}{M} (\log M) \| f \|_{W^1_\infty(\Omega)}.$$

LEMMA 4.2. *If* $\lambda^{-1} \notin \sigma(T)$, *then there is a constant* C_λ *and an integer* N_λ *such that for* $N \geqq N_\lambda$ *and* $M \sim N^2$,

$$\| (T - T_M) T_M T_N U_N \|_{L_\infty(\Omega)} \leqq C_\lambda \left(\frac{1}{N^2} + \frac{1}{M} (\log N)^3 \right) \| f \|_{L_\infty(\Omega)}.$$

LEMMA 4.3. *For* $\tau > 0$ *there exists a constant* C *such that* $T^2 : L_{1+\tau}(\Omega) \to L_\infty(\Omega)$, *i.e.*,

$$\| T^2 h \|_{L_\infty(\Omega)} \leqq C \| h \|_{L_{1+\tau}(\Omega)}.$$

We postpone the proofs of these results and first show that Theorem 4.1 follows from them.

Proof of Theorem 4.1. We have, using (1.7) and (4.6),

$$U - U_M^{(1)} = \lambda T_M (U - U_N) + (T - T_M)(\lambda U + f). \tag{4.11}$$

Using interpolation and the same technique as in the proof of Theorem 3.1, we can show that for $\theta > 0$ and sufficiently large N there exist constants $\delta > 0$, $\tau > 0$, and C such that

$$\| U - U_N \|_{L_{1+\tau}(\Omega)} \leqq C N^{-2+\theta} \| g \|_{W^{2-\delta}_{1+\tau}(\Omega)}. \tag{4.12}$$

Now by Lemma 4.3, (3.3), (4.11), and (4.12),

$$\| T^2 (U - U_M^{(1)}) \|_{L_\infty(\Omega)} \leqq C \left(\frac{1}{N^{2-\theta}} + \frac{1}{M} \right) (\| g \|_{W^{2-\delta}_{1+\tau}(\Omega)} + \| g \|_{W^1_{1+\tau}(\Omega)}), \tag{4.13}$$

and thus the desired result follows from (4.9), Lemma 4.1, Lemma 4.2, and (4.13). □

Proof of Lemma 4.1. Let S_j be the jth side of Ω and $\psi_j(\alpha)$ be the angle between S_j and $\mu = (\cos \alpha, \sin \alpha)$. Defining for $k = 1, \cdots, M$,

$$J_k = \left[\frac{2(k-1)\pi}{M}, \frac{2k\pi}{M} \right],$$

and A_j as the union of the J_k containing the direction of S_j and the adjacent J_l closest to the direction of S_j, and letting

$$S_0 = \bigcup_{j=1}^{P_0} A_j \quad \text{and} \quad Q_0 = Q_M \cap S_0,$$

we have

$$
\begin{aligned}
|Tf(x)-T_Mf(x)| &= \left|\int_S T_\mu f(x)\,d\mu - \sum_{\mu\in Q_M} T_\mu f(x)\omega_\mu\right| \\
&\leqq \left|\int_{S\setminus S_0} T_\mu f(x)\,d\mu - \sum_{Q_M\setminus Q_0} T_\mu f(x)\omega_\mu\right| \\
&\quad + \left|\int_{S_0} T_\mu f(x)\,d\mu - \sum_{Q_0} T_\mu f(x)\omega_\mu\right| :\equiv A_0 + B_0,
\end{aligned}
\tag{4.14}
$$

with the obvious notation. Now applying (2.6a) to $S\setminus S_0$, we have

$$
|A_0| = \left|\int_{S\setminus S_0} T_\mu f(x)\,d\mu - \sum_{Q_M\setminus Q_0} T_\mu f(x)\omega_\mu\right| \leqq \frac{C}{M}\int_{S\setminus S_0}\left|\frac{\partial}{\partial\alpha}T_\mu f(x)\right| d\alpha
$$

and as in the proof of Lemma 3.3

$$
\left\|\frac{\partial}{\partial\alpha}(T_\mu f)\right\|_{L_\infty(\Omega)} \leqq C(\min_j|\sin\psi_j(\alpha)|)^{-1}\|f\|_{W^1_\infty(\Omega)}.
$$

Thus,

$$
\begin{aligned}
\|A_0\|_{L_\infty(\Omega)} &\leqq \frac{C}{M}\left(\int_{S\setminus S_0}(\min_j|\sin\psi_j(\alpha)|)^{-1}\,d\alpha\right)\|f\|_{W^1_\infty(\Omega)} \\
&\leqq \frac{C}{M}(\log M)|\,\|f\|_{W^1_\infty(\Omega)}.
\end{aligned}
\tag{4.15}
$$

On the other hand, using (2.10),

$$
\begin{aligned}
\|B_0\|_{L_\infty(\Omega)} &= \sup_{x\in\Omega}\left|\int_{S_0} T_\mu f(x)\,d\mu - \sum_{\mu\in Q_0} T_\mu f(x)\omega_\mu\right| \\
&\leqq C\left(P_0\frac{1}{M} + |S_0|\right)\sup_{x\in\Omega}|T_\mu f(x)| \\
&\leqq CP_0\frac{1}{M}\|f\|_{L_\infty(\Omega)} \leqq C\frac{1}{M}\|f\|_{L_\infty(\Omega)},
\end{aligned}
\tag{4.16}
$$

where $|S_0|\sim 1/M$ is the length of S_0. Now (4.14)-(4.16) complete the proof. □

Proof of Lemma 4.2. The splitting (4.10) and repeated application of (2.10) together with Lemma 4.1 yield

$$
\begin{aligned}
\|(T-T_M)T_MT_NU_N\|_\infty &\leqq \|(T-T_M)\sum_{(\mu,\nu)\in J_\varepsilon}\omega_\mu\omega_\nu T_\mu T_\nu U_N\|_\infty \\
&\quad + \|(T-T_M)\sum_{(\mu,\nu)\in J'_\varepsilon}\omega_\mu\omega_\nu T_\mu T_\nu U_N\|_\infty \\
&\leqq \frac{C}{M}(\log M)\sum_{(\mu,\nu)\in J_\varepsilon}\omega_\mu\omega_\nu\|T_\mu T_\nu U_N\|_{W^1_\infty(\Omega)} \\
&\quad + C\varepsilon\|T_\mu T_\nu U_N\|_\infty \\
&\leqq \frac{C}{M}(\log M)\sum_{(\mu,\nu)\in J_\varepsilon}\omega_\mu\omega_\nu\|\nabla_x(T_\mu T_\nu U_N)\|_\infty \\
&\quad + C\left[\frac{1}{M}(\log M)+\varepsilon\right]\|U_N\|_\infty.
\end{aligned}
\tag{4.17}
$$

Observe that since $(\mu, \nu) \in J_\varepsilon$, we have $\sin\gamma(\mu, \nu) \geqq \varepsilon$ and for $n = 1, 2, \cdots, P_0$, $\sin\gamma(\mu, d_n) \geqq \varepsilon$. Now assuming $\mu = (1, 0)$ and using (2.10), we find that

$$\|\nabla_x(T_\mu T_\nu U_N)\|_\infty \leqq C|\sin\gamma(\mu, \nu)|^{-1}\left(\|U_N\|_\infty + \left\|\frac{\partial}{\partial\nu}(T_\mu T_\nu U_N)\right\|_\infty\right).$$

Further recalling (3.15) and using the same argument as in the proof of Lemma 3.5, we have

$$\left\|\frac{\partial}{\partial\nu}(T_\mu T_\nu U_N)\right\|_\infty \leqq C(\min_n|\sin\gamma(\mu, d_n)|)^{-1}\|U_N\|_\infty,$$

so that

$$\|\nabla_x(T_\mu T_\nu U_N\|_\infty \leqq C(\min_n|\sin\gamma(\mu, \nu)||\sin\gamma(\mu, d_n)|)^{-1}\|U_N\|_\infty.$$

Summing now first over ν and then over μ, we obtain

$$\begin{aligned} \sum_{(\mu,\nu)\in J_\varepsilon} \omega_\mu\omega_\nu\|\nabla_x(T_\mu T_\nu U_N)\|_\infty &\leqq C \sum_{(\mu,\nu)\in J_\varepsilon} \min_n \frac{\omega_\mu\omega_\nu}{\sin\gamma(\mu, \nu)\sin\gamma(\mu, d_n)}\|U_N\|_\infty \\ &\leqq C\left|\int_\varepsilon^1 \frac{d\alpha}{\alpha}\right|\left|\int_\varepsilon^1 \frac{d\beta}{\beta}\right|\|U_N\|_\infty \qquad (4.18) \\ &\leqq C|\log\varepsilon|^2\|U_N\|_\infty. \end{aligned}$$

Moreover, since $\lambda^{-1} \notin \sigma(T)$, Lemma 3.2 implies that there exists an integer N_λ such that for $N \geqq N_\lambda$, $(I - \lambda T_N)^{-1}$ exists and is a uniformly bounded operator on $L_p(\Omega)$, $1 \leqq p \leqq \infty$. Thus by (2.4) and (2.10) for $N \geqq N_\lambda$

$$\|U_N\|_\infty \leqq \|(I - \lambda T_N)^{-1}\|_\infty\|T_N f\|_\infty \leqq C_\lambda\|f\|_\infty. \qquad (4.19)$$

Taking finally $\varepsilon \sim 1/N^2$ and combining (4.17)-(4.19) the desired result follows. □

Proof of Lemma 4.3. Recall that if h is zero in the complement of Ω, then

$$Th(x) = \int_\Omega \frac{e^{-|x-y|}}{|x-y|}h(y)\,dy \equiv \int_\Omega k(x-y)h(y)\,dy = (k * h)(x), \qquad h(x) = 0, \quad x \notin \Omega.$$

Thus by Young's inequality

$$\|T^2h\|_\infty \leqq \|k * k * h\|_\infty \leqq \|k\|_p\|k * h\|_q, \frac{1}{p} + \frac{1}{q} = 1, \quad Th(x) = 0, \quad x \notin \Omega$$

and

$$\|k * h\|_q \leqq \|k\|_r\|h\|_s, \qquad 1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{s}.$$

In the above inequalities, k is our kernel function. We set $r = p$ to obtain $2/p + 1/s = 2$ and, with $s = 1 + \tau$, we have $p = 1 + 1/(1 + 2\tau) \in (1, 2)$. With this p, $k \in L_p(\Omega)$ and the proof is complete. □

5. Eigenvalue estimates. In this section we prove error estimates for the approximation of simple isolated eigenvalues of our integral operator T. Eigenvalues of T are of physical interest and in particular the smallest positive eigenvalue is related to criticality.

Since T_N does not converge in the operator norm to T, we cannot directly apply the known standard arguments as in [4] and [6] to derive the desired eigenvalue estimates. What is lacking is a proof of an equality of the dimensions of certain eigenspaces of T_N and T. However, by Lemma 5.1, T_N^3 converges in the operator norm to T^3 and using this result we are able to verify the crucial condition concerning the dimensions of eigenspaces of T_N and T.

Because of the limited regularity of the scalar flux U, by Theorem 3.1 the sharpest estimate for the convergence of eigenvalues is obtained for $p=1$.

THEOREM 5.1. *Let λ be an isolated eigenvalue of T with algebraic multiplicity* 1 *and let $\Gamma \subset \rho(T)$ be a circle centered at λ. Then there exists an integer N_λ such that for $N \geqq N_\lambda$, T_N has exactly one eigenvalue $\lambda_N \in \operatorname{Int} B(\lambda, \Gamma)$ with algebraic multiplicity* 1. *Further assume corresponding eigenfunction $g \in W_1^{2-\delta}(\Omega)$ for some $\delta > 0$. Then for $\theta > 0$ there exists a constant $C_{\lambda,\theta}$ such that*

$$|\lambda - \lambda_N| \leqq C_{\lambda,\theta} N^{2-\theta}, \qquad N \geqq N_\lambda. \tag{5.1}$$

Here $\rho(T)$ is the resolvent set of T and $B(\lambda, \Gamma)$ is the disc centered at λ with $\partial B = \Gamma$.

We first review, for the sake of completeness, a rather standard and known argument giving a general form of Theorem 5.1 for linear operators on Banach spaces (Theorem 5.2).

Let X be a complex Banach space with norm $\|\cdot\|$; $F: X \to X$ a bounded linear operator and $\{F_N\}_{N=1}^{\infty}$ a family of bounded linear operators on X such that for $g \in X$,

$$\|Fg - F_N g\| \to 0 \quad \text{as } N \to \infty. \tag{5.2}$$

We assume that λ is an isolated eigenvalue of F with index ν and finite algebraic multiplicity $m \geqq \nu$. Then there exists a circle Γ in the complex plane centered at λ, which separates λ from $\sigma(F)\backslash\{\lambda\}$. We denote by $P(\lambda, F)$ the spectral projection $(1/2\pi i)\int_\Gamma (z-F)^{-1}\, dz$ associated with the eigenspace

$$X(\lambda, F) \equiv \operatorname{null}(\lambda - F)^\nu$$

and let $E(\lambda, F) = \operatorname{Range}(P(\lambda, F))$ be the corresponding generalized eigenspace. It is easy to verify that

$$\begin{aligned} &E \equiv E(\lambda, F) = X(\lambda, F),\\ &\dim E(\lambda, F) = m,\\ &(\lambda - F)^\nu P(\lambda, F) = 0 \quad \text{and} \quad (\lambda - F)^{\nu-1} P(\lambda, F) \neq 0 \end{aligned}$$

(see, e.g., [6, Chap. 5] and [8, p. 573]). Now let us assume that there exists a constant C and an integer N_0 such that for $N \geqq N_0$

$$\|(z - F_N)^{-1}\| \leqq C \quad \forall z \in \Gamma. \tag{5.3}$$

Here $\|\cdot\|$ is the operator norm defined by

$$\|A\| = \sup\left\{\frac{\|Ag\|}{\|g\|} : g \in X, g \neq 0\right\}.$$

Considering (5.3) we may define the projection operator

$$P(\lambda, F_N) = \frac{1}{2\pi i}\int_\Gamma (z - F_N)^{-1}\, dz,$$

associated with the eigenspace

$$E_N \equiv E(\sigma_N, F_N) = \operatorname{null}(\lambda_1 - F_N)^{\nu_1} \oplus \cdots \oplus \operatorname{null}(\lambda_r - F_N)^{\nu_r},$$

where $\sigma_N = \sigma(F_N) \cap B(\lambda, \Gamma)$, $B(\lambda, \Gamma)$ is the disc centered at λ with $\partial B = \Gamma$ and $\lambda_j \in \sigma_N$ are eigenvalues of F_N with algebraic multiplicities m_j and indices ν_j. Finally we assume that for sufficiently large N,

$$m = \dim E(\lambda, F) = \dim E(\sigma_N, F_N) = \sum_{j=1}^{r} m_j. \tag{5.4}$$

We are now ready to formulate the following general result.

THEOREM 5.2. *Let λ be an isolated eigenvalue of $F: X \to X$ with finite algebraic multiplicity m and assume that* (5.2)–(5.4) *hold. Then there exist exactly m eigenvalues, counted with their multiplicities, $\lambda_N \in \sigma_N$ of F_N and a constant C such that*

$$\max_{\lambda_N \in \sigma_N} |\lambda - \lambda_N| \leqq C \|F - F_N\|_E, \tag{5.5}$$

where $\|\cdot\|_E$ denotes the operator norm restricted to E.

The proof of Theorem 5.2 is based on Propositions 5.1–5.3 below. See also, e.g., [4], [6], and [12].

PROPOSITION 5.1. *If* (5.2) *and* (5.3) *hold then*

$$\|(P_N - P)P\| \to 0 \quad \text{as } N \to \infty,$$

where $P \equiv P(\lambda, F)$ and $P_N \equiv P(\lambda, F_N)$.

Proof. Since $(z-F)^{-1}$ and P commute we have for $u \in X$

$$(P - P_N)Pu = \frac{1}{2\pi i} \int_\Gamma [(z-F)^{-1} - (z-F_N)^{-1}]Pu\, dz$$
$$= \frac{1}{2\pi i} \int_\Gamma (z-F_N)^{-1}(F-F_N)P(z-F)^{-1}u\, dz,$$

and

$$\|(P-P_N)Pu\| \leqq C \sup_{z\in\Gamma} \|(z-F_N)^{-1}\| \, \|(F-F_N)P\| \sup_{z \in \Gamma} \|(z-F)^{-1}u\|.$$

When we use (5.3) there exists a constant C such that for sufficiently large N

$$\|(P-P_N)Pu\| \leqq C\|(F-F_N)P\| \, \|u\|.$$

Since the dimension of the range of P is finite, P is compact. Thus by (5.2) we have

$$\|(F-F_N)P\| \to 0 \quad \text{as } N \to \infty,$$

and the proof is complete. □

We define the operator $B_N: E \to E_N$ as the restriction of P_N to E, $B_N u = P_N u$, for $u \in E$.

PROPOSITION 5.2. *If* (5.2)–(5.4) *hold, then there exist an integer N_0 and a constant C such that for $N \geqq N_0$*

(a) *B_N is an isomorphism from E onto E_N,*

(b) $\|B_N^{-1}\| \leqq C$.

Proof. Let $\varphi_1, \cdots, \varphi_m$ be a basis of the space E. Since P is a projection onto E

$$P\varphi_i = \varphi_i, \qquad i = 1, \cdots, m.$$

Set

$$\varphi_{i,N} = B_N \varphi_i \equiv P_N \varphi_i, \qquad i = 1, \cdots, m.$$

We have, using Proposition 5.1, that for $1 \leqq i \leqq m$,

$$\|\varphi_i - \varphi_{i,N}\| = \|(P-P_N)\varphi_i\| = \|(P-P_N)P\varphi_i\| \to 0 \quad \text{as } N \to \infty.$$

Since $\{\varphi_i\}_{i=1}^m$ is a basis of E, this proves that for sufficiently large N, $\varphi_{i,N}$, $i=1,\cdots,m$, are linearly independent and using (5.4) we conclude that $\{\varphi_{i,N}\}_{i=1}^m$ is a basis of E_N. Hence (a) is proved. The proof of (b) follows easily from the fact that E and E_N are finite-dimensional. □

Now we consider the operators A and $A_N: E \to E$ defined by

$$Au = Fu \quad \text{for } u \in E,$$

and

$$A_N u = B_N^{-1} F_N B_N u \quad \text{for } u \in E.$$

The operators A and A_N are well defined because E and E_N are invariant under F and F_N, respectively. Here λ is the only eigenvalue of A and the eigenvalues of A_N are those of F_N in $B(\lambda, \Gamma)$.

PROPOSITION 5.3. *Assume that* (5.2)–(5.4) *are valid. Then for sufficiently large* N *we have*

$$\|A - A_N\|_E \leqq \|F - F_N\|_E.$$

Proof. Since P_N and F_N commute we have for $u \in E$,

$$(A - A_N)u = Fu - B_N^{-1} F_N P_N u = Fu - B_N^{-1} P_N F_N u.$$

Observe that there is no guarantee that $F_N u \in E$ and consequently $B_N^{-1} P_N$ cannot be replaced by the identity operator in this last relation. However, we have

$$(A - A_N)u = Fu - B_N^{-1} P_N F_N u = B_N^{-1}(B_N Fu - P_N F_N u) = B_N^{-1} P_N (F - F_N)u.$$

Now using Proposition 5.2(b) together with the fact that P_N is uniformly bounded (because of (5.2)) we obtain for $u \in E$ and for sufficiently large N,

$$\|(A - A_N)u\| \leqq C\|(F - F_N)u\|,$$

and this gives the desired result. □

Proof of Theorem 5.2. We have by Proposition 5.3

$$\|A - A_N\| \leqq C \sup_{u \in E} \{\|(F - F_N)u\|, \|u\| = 1\}.$$

Since E is finite-dimensional, A and A_N as operators on E can be represented by matrices with λ and λ_N as eigenvalues. Hence by an error estimate for eigenvalues of matrices given in, e.g., Wilkinson [20],

$$\max_{\lambda_N \in \sigma_N} |\lambda - \lambda_N| = \|A - A_N\| \leqq C\|(F - F_N)u\|, \quad u \in E \text{ and } \|u\| = 1,$$

and the proof is complete. □

Let us return to our special case with the operators F and F_N replaced by T and T_N, respectively, and $m = 1$. As stated in the beginning of this section, to derive the sharpest estimate for the convergence of eigenvalues we now take $X = L_1(\Omega)$. For our problem, condition (5.2) follows from the fact that T_N converges pointwise to T and (5.3) is a result of Lemma 3.2. Below we prove Theorem 5.1 by verifying (5.4) for our operators T and T_N with $m = 1$. Condition (5.4) may be proved directly if $\|F - F_N\| \to 0$, as $N \to \infty$, but this is not necessarily true in our case. However, we will prove that $\|T^3 - T_N^3\| \to 0$, as $N \to \infty$. Using this we prove (5.4) with $m = 1$ first for T^3 and T_N^3 and then for T and T_N.

LEMMA 5.1. *We have*

$$\|T^3 - T_N^3\|_p \to 0 \quad \textit{as } N \to \infty, \quad 1 \leqq p < \infty.$$

Proof. We have

$$\|T^3 - T_N^3\|_p = \|T^2(T-T_N) + T(T-T_N)T_N + (T-T_N)T_N^2\|_p.$$

Since T and T^* are compact we see that

$$\|T^2(T-T_N)\|_p = \|(T^* - T_N^*)T^{*2}\|_{p'} \to 0 \quad \text{as } N \to \infty$$

and

$$\|T(T-T_N)T_N\|_p = \|T_N^*(T^* - T_N^*)T^*\|_{p'} \to 0 \quad \text{as } N \to \infty.$$

Thus Lemma 3.4 gives the desired result. □

Below, for $0 \neq \lambda \in \sigma(T)$, we let $\tilde{\Gamma}$ denote a circle in the complex plane centered at λ^3, separating λ^3 from $\sigma(T^3)\backslash\{\lambda^3\}$ and set $\tilde{\sigma}_N = \sigma(T_N^3) \cap \operatorname{Int} B(\lambda^3, \tilde{\Gamma})$.

LEMMA 5.2. *For sufficiently large N*

$$\dim E(\lambda^3, T^3) = \dim E(\tilde{\sigma}_N, T_N^3).$$

Proof. Let $K = T^3$, $K_N = T_N^3$ and for $0 \leqq t \leqq 1$ define

$$K_{N,t} = (1-t)K + tK_N;$$

note that $K_{N,0} = K$, $K_{N,1} = K_N$, and $\|K_{N,t}\| \leqq C$. We denote by $\sigma_{N,t}$ the part of the spectrum of $K_{N,t}$, $\sigma(K_{N,t})$, contained in the interior of $B(\lambda^3, \tilde{\Gamma})$. The projection operator

$$P(\sigma_{N,t}, K_{N,t}) = \frac{1}{2\pi i}\int_{\tilde{\Gamma}} (z - K_{N,t})^{-1}\, dz$$

is well defined and using Lemma 5.1 we can show that it is a continuous function of t (see [3, Thm. 3a]). Further by a result of Riesz and Nagy [15, p. 268] we have the following: If the difference of two projections is of norm less than 1, then their ranges are of equal dimensions. (In the estimate of the norm of the difference of $P(\sigma_{N,t}, K_{N,t})$ and $P(\sigma_{N,s}, K_{N,s})$, $|s-t|$ and the length of $\tilde{\Gamma}$ are involved in the constant on the right-hand side. This constant, because of the length of $\tilde{\Gamma}$, can be made less than one.) Hence for sufficiently large N,

$$\begin{aligned}\dim E(\lambda^3, T^3) &= \dim KX = \dim K_{N,0}X = \dim K_{N,1}X = \dim K_N X\\ &= \dim E(\tilde{\sigma}_N, T_N^3).\end{aligned}$$

□

Now we are prepared to complete the proof of our eigenvalue estimate.

Proof of Theorem 5.1. By the spectral mapping theorem $\lambda \in \sigma(T)$ if and only if $\lambda^3 \in \sigma(T^3)$. Now we use the equality

$$T^3 - \lambda^3 = (T-\lambda)(T - \lambda\, e^{(2\pi/3)i})(T - \lambda\, e^{(4\pi/3)i}),$$

to obtain the decomposition

$$E(\lambda^3, T^3) = E(\lambda, T) \oplus E(\lambda\, e^{(2\pi/3)i}, T) \oplus E(\lambda\, e^{(4\pi/3)i}, T) \tag{5.6}$$

(see, e.g., [17, Thm. 5.9-D]). In our case T as an operator on $L_1(\Omega)$ has only real eigenvalues. To see this, let $v \in L_1(\Omega)$ be an eigenfunction of T corresponding to the eigenvalue λ, so that

$$Tv = \lambda v. \tag{5.7}$$

Now arguing as in the proof of Lemma 4.3, $T^2 \colon L_1(\Omega) \to L_2(\Omega)$, and by (5.7), $T^2 v = \lambda^2 v$; thus $v \in L_2(\Omega)$, and hence (5.7) implies that v is also an eigenfunction of T as an operator on $L_2(\Omega)$ with the same eigenvalue λ and since T is self-adjoint on $L_2(\Omega)$, we have $\lambda \in R$. Thus the two last eigenspaces in (5.6) are empty and

$$\dim E(\lambda, T) = \dim E(\lambda^3, T^3). \tag{5.8}$$

Since $\dim E(\lambda, T)=1$, Lemma 5.2 together with (5.8) imply that $\tilde{\sigma}_N$ is a real number. This follows since because of the structure of T_N (real positive weights), if $\tilde{\sigma} \in \sigma(T_N^3)$ then $\overline{\tilde{\sigma}_N} \in \sigma(T_N^3)$, so that if $\overline{\tilde{\sigma}_N} \neq \tilde{\sigma}_N$, $\dim E(\tilde{\sigma}_N, T_N^3) \geqq 2$, which contradicts Lemma 5.2 and (5.8). The analogue of (5.6) for $\tilde{\sigma}_N$ and T_N^3 is

$$(5.9) \quad E(\tilde{\sigma}_N, T_N^3) = E(\sqrt[3]{\tilde{\sigma}_N}, T_N) \oplus E(\sqrt[3]{\tilde{\sigma}_N}\, e^{(2\pi/3)i}, T_N) \oplus E(\sqrt[3]{\tilde{\sigma}_N}\, e^{(4\pi/3)i}, T_N).$$

Now if one of the last two eigenspaces in the right-hand side of (5.9) is nonempty, say $E(\sqrt[3]{\tilde{\sigma}_N}\, e^{(4\pi/3)i}, T_N) \neq \varnothing$, then $\sqrt[3]{\tilde{\sigma}_N}\, e^{(4\pi/3)i} \in \sigma(T_N)$ and again because of the structure of T_N, $\overline{\sqrt[3]{\tilde{\sigma}_N}\, e^{(4\pi/3)i}} \in \sigma(T_N)$; hence $E(\sqrt[3]{\tilde{\sigma}_N}\, e^{(2\pi/3)i}, T_N) \neq \varnothing$. Consequently $\dim E(\tilde{\sigma}_N, T_N^3) \geqq 2$, and this is again a contradiction. We conclude that

$$\dim E(\sqrt[3]{\tilde{\sigma}_N}, T_N) = \dim E(\tilde{\sigma}_N, T_N^3).$$

This completes the proof of (5.4) for our case $m=1$. Now Theorem 3.1 gives the desired result since for the normalized eigenfunction $g \in W_1^{2-\delta}(\Omega)$ corresponding to the eigenvalue λ, (3.2) implies that

$$|\lambda - \lambda_N| \leqq \|(T-T_N)g\|_{L_1(\Omega)} \leqq CN^{-2+\theta}. \qquad \square$$

6. Numerical results. In order to determine the rate of convergence for the discrete ordinates method in some concrete cases and also test the efficiency of the postprocessing procedure, we have performed some numerical computations on the following two-dimensional neutron transport equation:

$$(6.1) \quad \begin{aligned} &\mu \cdot \nabla u(x,\mu) + u(x,\mu) = \lambda U(x) + f(x), \qquad x \in \Omega := I^2, \\ &u(x,\mu) = 0, \qquad x \in \Gamma_\mu^- = \{x \in \Gamma = \partial\Omega : \mu \cdot n(x) < 0\}, \end{aligned}$$

where $I^2 = [0,1]\times[0,1]$, $\mu \in S = \{\mu \in \mathbb{R}^2 : |\mu| = 1\}$, and $n(x)$ is the outward unit normal to Γ at $x \in \Gamma$.

This problem is equivalent to the following integral equation for the scalar flux U (see (1.7)):

$$(6.2) \qquad (I - \lambda T)U = Tf.$$

The discrete ordinates method gives the following semidiscrete analogue of (6.2) (see (2.4)):

$$(6.3) \qquad (I - \lambda T_N)U_N = T_N f,$$

where N is the number of discrete directions on S.

We compute U_N using the iteration below:

$$(6.4) \quad \begin{aligned} &U_N^{(m+1)} = T_N(\lambda U_N^{(m)} + f), \qquad m = 0, 1, 2, \cdots, \\ &U_N^{(0)} = 0. \end{aligned}$$

The iterations are continued until $m+1 = L$, where

$$(6.5) \qquad \|U_N^{(L)} - U_N^{(L-1)}\|_{L_2(\Omega)} \leqq 10^{-6},$$

and U_N is defined to be $U_N^{(L)}$.

A postprocessed solution U_N^* is also computed (see (4.5)-(4.6)), i.e.,

$$(6.6) \quad \begin{aligned} &U_N^* = T_M(\lambda U_M^{(2)} + f), \\ &U_M^{(k+1)} = T_M(\lambda U_M^{(k)} + f), \qquad k = 0, 1, \end{aligned}$$

where $U_M^{(0)} = U_N$ is computed as in (6.4)-(6.5) and $M = N^2$.

Remark 6.1. In the computations we also discretize in the space variable using the discontinuous Galerkin finite element method with mesh parameter $h = 1/20$, $1/50$ and uniform triangulations of I^2 (see [2]).

6.1. Data. We perform the iteration (6.4)-(6.5) with $N=6$, 9, 12, 18, 36 and the postprocessing procedure (6.6) with $N=6$ and $M=36$ for the following data. Let $f:=f_D$ be the characteristic function of the domain $D\subset I^2$. We consider the following cases:

(D1) $\lambda=0.2$ and $D=\{(x_1,x_2)\colon (x_1-3/4)^2+(x_2-3/4)^2\leqq(0.2)^2\}$,
(D2) $\lambda=0.2$ and $D=\{(x_1,x_2)\colon (x_1-1/2)^2+(x_2-1/2)^2\leqq(0.2)^2\}$,
(D3) $\lambda=0.3$ and $D=\{(x_1,x_2)\colon (x_1-1/2)^2+(x_2-1/2)^2\leqq(0.3)^2\}$.

Each of the cases (D1)-(D3) is solved twice, once with $h=1/20$ and the second time with $h=1/50$.

6.2. Results. In Tables 1-6 we compare $U:=U_{36}$ with U_N and U_6^* where $N=6$, 9, 12, 18. Below $e_N:=U-U_N$, $e^*:=U-U_6^*$ and L denotes the number of iterations in (6.4) for $N=36$.

TABLE 1
Case D1, *with* $h=1/20$; $L=16$.

Error \ Norm	L_1	L_2	L_∞
U	0.494163	0.611624	1.665514
e_6	0.066632	0.079378	0.245124
e_9	0.028889	0.035967	0.101383
e_{12}	0.014218	0.017859	0.064952
e_{18}	0.006515	0.008202	0.025387
e^*	0.009415	0.011435	0.027280

TABLE 2
Case D2, *with* $h=1/20$; $L=17$.

Error \ Norm	L_1	L_2	L_∞
U	0.601861	0.694524	1.736332
e_6	0.065268	0.079722	0.306480
e_9	0.026942	0.034496	0.114590
e_{12}	0.019370	0.025200	0.084097
e_{18}	0.007661	0.009823	0.035274
e^*	0.009452	0.011680	0.030788

TABLE 3
Case D3, *with* $h=1/20$; $L=35$.

Error \ Norm	L_1	L_2	L_∞
U	2.454003	2.613289	4.522570
e_6	0.095097	0.115746	0.428698
e_9	0.049798	0.062952	0.217018
e_{12}	0.038930	0.050942	0.190969
e_{18}	0.015510	0.019816	0.063142
e^*	0.054172	0.066849	0.153723

TABLE 4
Case D1, *with* $h = 1/50$; $L = 16$.

Error \ Norm	L_1	L_2	L_∞
U	0.534893	0.658920	1.726658
e_6	0.073383	0.086619	0.271352
e_9	0.032836	0.040532	0.111557
e_{12}	0.017447	0.022298	0.095111
e_{18}	0.009490	0.011772	0.039077
e^*	0.010576	0.012690	0.029262

TABLE 5
Case D2, *with* $h = 1/50$; $L = 17$.

Error \ Norm	L_1	L_2	L_∞
U	0.636361	0.735342	1.791415
e_6	0.073927	0.090938	0.329555
e_9	0.032881	0.041574	0.129138
e_{12}	0.025804	0.033993	0.110757
e_{18}	0.010783	0.013486	0.046468
e^*	0.010629	0.013895	0.038291

TABLE 6
Case D3, *with* $h = 1/50$; $L = 35$.

Error \ Norm	L_1	L_2	L_∞
U	2.462085	2.625772	4.526402
e_6	0.109765	0.134892	0.496076
e_9	0.062966	0.079828	0.250217
e_{12}	0.047498	0.061768	0.207581
e_{18}	0.020606	0.025912	0.076828
e^*	0.054509	0.071086	0.167479

6.3. Conclusion. By Theorem 3.1, for the discrete ordinates method, we expect the convergence rates $1/_N\alpha$, where $\alpha \approx 2$ for the L_1 estimate and $\alpha \approx 1$ in the L_∞ case. Tables 1-6 show the convergence rates $1/_N\alpha$ with $\alpha \approx 1.6$-2.2 and $\alpha \approx 1.3$-1.8 for L_1 and L_∞, respectively. The difference between the theory and computations may depend on the choice of U_{36}, which is also discretized in space, as the "exact" scalar flux U. Observe that in (D1)-(D3), $f \in H^{1/2}(\Omega)$. Hence by (6.2) the exact scalar flux U has the required regularity in Theorem 3.1 ($U \in W_1^{2-\delta}(\Omega)$, $\delta > 0$ or $U \in W_p^1(\Omega)$, $1 \leqq p < \infty$).

As for the postprocessing procedure we see that the L_∞ errors for e^* are considerably less than those for e_6, in particular if f has small support. Further we have $\|e^*\|_\infty \sim \|e_6\|_1$ (cf. Theorem 4.1).

Acknowledgments. I am indebted to Professor Claes Johnson for introducing me to the subject of this paper and for encouraging me in this work. Helpful discussions about eigenvalue estimates with Professor Jacques Rappaz are gratefully acknowledged.

REFERENCES

[1] P. M. ANSELONE, *Collectively Compact Operator Approximation Theory and Applications to Integral Equations*, Prentice-Hall, Englewood Cliffs, NJ, 1971.

[2] M. ASADZADEH, *Analysis of a fully discrete scheme for the neutron transport in two-dimensional geometry*, SIAM J. Numer. Anal., 23 (1986), pp. 543–561.

[3] K. E. ATKINSON, *The numerical solution of the eigenvalue problem for compact integral operators*, Trans. Amer. Math. Soc., 129 (1967), pp. 458–465.

[4] ———, *Convergence rates for approximate eigenvalue of compact integral operators*, SIAM J. Numer. Anal., 12 (1975), pp. 213–222.

[5] J. BERGH AND J. LÖFSTRÖM, *Interpolation Spaces*, Springer-Verlag, Berlin, Heidelberg, New York, 1976.

[6] F. CHATELIN, *Spectral Approximation of Linear Operators*, Academic Press, New York, 1983.

[7] R. A. DE VORE AND L. R. SCOTT, *Error bounds for Gaussian quadrature and weighted-L_1 polynomial approximation*, SIAM J. Numer. Anal., 21 (1984), pp. 400–412.

[8] N. DUNFORD AND J. SCHWARTZ, *Linear Operators*, Vol. I, Interscience, New York, 1966.

[9] C. JOHNSON AND J. PITKÄRANTA, *Convergence of a fully discrete scheme for two-dimensional neutron transport*, SIAM J. Numer. Anal., 20 (1983), pp. 951–966.

[10] E. E. LEWIS AND W. F. MILLER, JR., *Computational Methods of Neutron Transport*, John Wiley, New York, 1984.

[11] P. NELSON AND H. D. VICTORY, JR., *Convergence of two-dimensional Nyström discrete-ordinates method in solving the linear transport equation*, Numer. Math., 34 (1979), pp. 200–218.

[12] J. E. OSBORN, *Spectral approximation for compact operators*, Math. Comp., 29 (1975), pp. 712–725.

[13] J. PITKÄRANTA, *Estimates for the derivatives of solutions to weakly singular Fredholm integral equations*, SIAM J. Math. Anal., 11 (1980), pp. 925–968.

[14] J. PITKÄRANTA AND L. R. SCOTT, *Error estimates for the combined spatial and angular approximations of the transport equation for slab geometry*, SIAM J. Numer. Anal., 20 (1983), pp. 922–950.

[15] F. RIESZ AND B. SZ. NAGY, *Leçon d'analyse fonctionelle*, 6 ed., Gauthier-Villars, Budapest, 1972.

[16] W. RUDIN, *Functional Analysis*, McGraw-Hill, New York, 1977.

[17] A. E. TAYLOR, *Introduction to Functional Analysis*, John Wiley, New York, 1958.

[18] G. M. VAINIKKO AND A. PEDAS, *The properties of solutions of weakly singular integral equations*, J. Austral. Math. Soc. Ser. B., 22 (1981), pp. 419–430.

[19] H. D. VICTORY, JR., *Convergence properties of discrete ordinates solutions for neutron transport in three-dimensional media*, SIAM J. Numer. Anal., 17 (1980), pp. 71–83.

[20] J. H. WILKINSON, *The Algebraic Eigenvalue Problem*, Oxford University Press, London, 1965.