Lp AND EIGENVALUE ERROR ESTIMATES FOR THE DISCRETE ORDINATES METHOD FOR TWO-DIMENSIONAL NEUTRON TRANSPORT*

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Abstract. The convergence of the discrete ordinates method is studied for angular discretization of the neutron transport equation for a two-dimensional model problem with the constant total cross section and isotropic scattering. Considering a symmetric set of quadrature points on the unit circle, error estimates are derived for the scalar flux in Lp norms for 1 ≤ p ≤ ∞. A postprocessing procedure giving improved L∞ estimates is also analyzed. Finally error estimates are given for simple isolated eigenvalues of the solution operator.

Key words. neutron transport equation, discrete ordinates method, scalar flux, Lp estimates, eigenvalue estimates, postprocessing, quadrature rule

AMS(MOS) subject classifications. primary 65N15, 65N30

Introduction. In this paper we study the convergence of the Nyström discrete ordinates method for angular discretization of the neutron transport equation for a two-dimensional model problem. We extend the L2 error analysis of [9] to Lp norms, 1 ≤ p ≤ ∞. We also analyze a postprocessing procedure and obtain improved rates of convergence in L∞ for the postprocessed solution. Further we derive error estimates for isolated eigenvalues with algebraic multiplicity one.

These studies contain the following important aspects.

(1) Studying L1 estimates is of practical interest because the eigenvalue estimates are based on L1 results. To derive L1 estimates the function spaces involved are interpolation spaces but not Sobolev spaces. Working with the interpolation spaces on the topic of the neutron transport equation is new.

(2) L∞, being the strongest norm, is the one we are most motivated to consider; however, we should expect lower rates of convergence than in the L1 case. Previous L∞ results have been without rates of convergence. In this paper we derive L∞ rates of convergence and employ a postprocessing procedure to improve these rates and obtain L∞ error estimates with the same rates as in L1.

(3) The classical techniques in [6] and [17] are not useful in deriving eigenvalue error estimates for this problem. This depends on the behavior of the operators involved in the problem. A new functional analysis approach has been made to show the equality of the dimensions of some eigenspaces.

The steady state one-velocity process of transport of neutrons in a substance surrounded by vacuum can be formulated as follows. Given the source f and the coefficients α and σ, find the angular flux u satisfying

\[ \mu \cdot \nabla u(x, \mu) + \alpha(x) u(x, \mu) = \int_{S^2} \sigma(x, \mu, \eta) u(x, \eta) \, d\eta + f(x, \mu), \]

(0.1) \( (x, \mu) \in \Omega \times S^2, \)

u(x, \mu) = 0 \text{ for } x \in \Gamma^– = \{ x \in \Gamma: \mu \cdot n(x) < 0 \},

where \( \Omega \) is a domain in \( \mathbb{R}^3 \), \( \Gamma = \partial \Omega \), \( S^2 \) is the unit sphere in \( \mathbb{R}^3 \), \( n(x) \) is the outward
unit normal to $\Gamma$ at $x \in \Gamma$, and
\[
\mu \cdot \nabla = \sum_{i=1}^{3} \mu_i \frac{\partial}{\partial x_i}.
\]

In the discrete ordinates method the integral in (0.1) is replaced, using a quadrature approximation, by a discrete sum involving a finite set of directions. $L^\infty$ convergence of the discrete ordinates method for a two-dimensional problem is studied, e.g., in [11], and for a three-dimensional problem in [19] with no convergence rates. Combined spatial and angular discretizations in the slab case are studied in [14], where error estimates are derived for both the scalar flux in $L_p$ norms, $1 \leq p \leq \infty$, and for critical eigenvalues. In [9] an $L^2$ analysis of a fully discrete scheme for a two-dimensional model problem is carried out, where the discrete ordinates method for the angular variable is combined with the discontinuous Galerkin finite element method for the spatial variable. In [2] the results of [9] are extended to a case where the angular variable varies on the unit disc in $\mathbb{R}^2$ while the spatial variable remains two-dimensional as in [9]. This corresponds to the three-dimensional problem (0.1) with $\Omega$ being an infinite cylindrical domain where all functions involved are assumed to be constant along the axis of the cylinder.

An outline of this paper is as follows. In § 1 we present a two-dimensional model problem obtained by taking, in (0.1), $x \in \Omega \subset \mathbb{R}^2$, $\mu \in S = \{ \mu \in \mathbb{R}^2 : |\mu| = 1 \}$, $\alpha = 1$, and $\sigma = \lambda$, and we reformulate this problem as a Fredholm integral equation of the second kind with a compact integral operator $T$. In § 2 we formulate a semidiscrete analogue of the model problem by applying a quadrature rule and we note that the semidiscrete analogue can also be formulated as an integral equation involving a certain operator $T_N$, where $N$ indicates the number of quadrature points. We prove that our integral operator $T$ is self-adjoint in $L^2(\Omega)$ and assuming a symmetric distribution of the quadrature points this is also true for the approximate operator $T_N$. In § 3 we derive $L_p$ error estimates, $1 \leq p < \infty$, for the semidiscrete problem. We also prove that if $\lambda^{-1} \not\in \sigma(T)$ and $N$ is sufficiently large then $\lambda^{-1} \not\in \sigma(T_N)$, where $\sigma(T)$ denotes the spectrum of $T$. Section 4 is devoted to a postprocessing procedure giving improved rates of convergence in $L^\infty$ for the scalar flux. In § 5 we prove error estimates for the corresponding approximation of simple isolated eigenvalues of our integral operator $T$. In the concluding § 6, we give some numerical results testing the analysis of §§ 3 and 4.

1. A model problem. We will consider the following two-dimensional model problem. Given a function $f$ and a parameter $\lambda$, find $u(x, \mu)$ such that

\[
\mu \cdot \nabla u(x, \mu) + u(x, \mu) = \lambda \int_S u(x, \eta) \, d\eta + f(x), \quad \text{for } (x, \mu) \in \Omega \times S,
\]

(1.1)

\[
u(x, \mu) = 0 \quad \text{on } \Gamma^- \mu = \{ x \in \Gamma : \mu \cdot n(x) < 0 \},
\]

where $\Omega$ is a bounded convex polygonal domain in $\mathbb{R}^2$ with boundary $\Gamma$, $S$ is the unit circle and $n(x)$ is the outward unit normal to $\Gamma$ at $x \in \Gamma$.

When we introduce the scalar flux

\[
U(x) = \int_S u(x, \eta) \, d\eta,
\]

(1.2)

(1.1) takes the following form:

\[
\mu \cdot \nabla u(x, \mu) + u(x, \mu) = (\lambda U + f)(x), \quad (x, \mu) \in \Omega \times S,
\]

(1.3)

\[
u(x, \mu) = 0 \quad \text{on } \Gamma^- \mu.
\]
Introducing the solution operator $T_\mu$ for the equation $\mu \cdot \nabla u + u = g$, in $\Omega$, $u = 0$ on $\Gamma_\mu^-$, given by

\begin{equation}
T_\mu g(x) = \int_0^{d(x, \mu)} e^{-s} g(x - s\mu) \, ds, \tag{1.4}
\end{equation}

with

\begin{equation}
d(x, \mu) = \inf \{ s > 0 : (x - s\mu) \notin \Omega \}, \tag{1.5}
\end{equation}

we can write (1.3) as

\begin{equation}
u(x, \mu) = T_\mu (\lambda U + f)(x), \quad (x, \mu) \in \Omega \times S. \tag{1.6}
\end{equation}

Integrating (1.6) over $S$, we get the following equation for the scalar flux $U$:

\begin{equation}
(I - \lambda T) U = Tf, \tag{1.7}
\end{equation}

where

\begin{equation}
T = \int_S T_\mu \, d\mu. \tag{1.8}
\end{equation}

When we use (1.4), it is easy to see that

\begin{equation}
T g(x) = \int_\Omega \frac{e^{-|x-y|}}{|x-y|} g(y) \, dy. \tag{1.9}
\end{equation}

Thus, $T$ is an integral operator with weakly singular kernel, and hence, $T$ is a compact operator on $L^p(\Omega)$, $1 \leq p \leq \infty$. Consequently, (1.7) is a Fredholm integral equation of the second kind (see, e.g., [18]).

Let us introduce some notation to be used below. For $1 \leq p \leq \infty$, we denote by $\sigma_p(K)$ the spectrum of an operator $K : L^p(\Omega) \to L^p(\Omega)$, defined by $\sigma_p(K) = \{ z \in \mathbb{C} : (K - zI) \text{ is not invertible as an operator on } L^p(\Omega) \}$. $\| \cdot \|_p$ and $\| \cdot \|_{w^m_p}$, $m$ a positive integer, denote the usual $L^p$ norms and the Sobolev norms of order $m$, respectively. $\| \cdot \|_p$ will also denote the operator norm $\| \cdot \|_{L^p(\Omega)}$ on $L^p(\Omega)$. $C$ will denote positive constants, not necessarily the same at each occurrence, independent of the parameters $N$ and $\varepsilon$. Note that since $\sigma_p(T)$ is independent of $p$, below we will use the notation $\sigma(T)$ instead of $\sigma_p(T)$.

We will assume that the parameter $\lambda^{-1}$ does not belong to $\sigma(T)$. Then for $1 \leq p \leq \infty$, $(I - \lambda T) : L^p(\Omega) \to L^p(\Omega)$ is invertible, Range $(I - \lambda T) = L^p(\Omega)$, and there is a constant $C$ such that $\| (I - \lambda T)^{-1} \|_p \leq C$, $1 \leq p \leq \infty$. In particular

\begin{equation}
\| (I - \lambda T) v \|_p \geq C \| v \|_p \quad \forall v \in L^p(\Omega). \tag{1.10}
\end{equation}

Hence the integral equation (1.7) has a unique solution $U = (I - \lambda T)^{-1} T f$.

2. The quadrature rule. Let $Q \equiv Q_N = \{ \mu_1, \cdots, \mu_N \}$ be a set of points on the unit circle $S$ with the property that if $\mu \in Q$ then $-\mu \in Q$. Consider the quadrature rule

\begin{equation}
\int_S u(x, \mu) \, d\mu \sim \sum_{\mu \in Q} u(x, \mu) \omega_\mu, \tag{2.1}
\end{equation}

where $\omega_\mu = 2\pi / N$. Other standard quadrature rules for the neutron transport equation are discussed by Lewis and Miller in [10]. For the semidiscrete approximation of the scalar flux, we set

\begin{equation}
U_N(x) = \sum_{\mu \in Q} u_N(x, \mu) \omega_\mu, \tag{2.2}
\end{equation}
with \( u_N(x, \mu) \) satisfying
\[
(2.3) \quad u_N(x, \mu) = T_\mu(\lambda U_N + f)(x), \quad x \in \Omega, \quad \mu \in Q.
\]
Multiplying (2.3) by \( \omega_\mu \) and summing over \( Q \) we obtain the integral equation
\[
(2.4) \quad (I - \lambda T_N) U_N = T_N f,
\]
where
\[
(2.5) \quad T_N = \sum_{\mu \in Q} T_\mu \omega_\mu.
\]
For the error in the quadrature (2.1) we have the following estimate (see, e.g., [7]). For \( k = 1, 2 \), there exists a constant \( C \) such that
\[
(2.6a) \quad \left| \int_S u(x, \mu) \, d\mu - \sum_{\mu \in Q} u(x, \mu) \omega_\mu \right| \leq \frac{C}{N^k} \int_0^{2\pi} \left| \frac{\partial^k u}{\partial \alpha^k}(x, \alpha) \right| \, d\alpha,
\]
where \( \mu = (\cos \alpha, \sin \alpha) \). Also
\[
(2.6b) \quad \sum_{\mu \in Q} \omega_\mu \omega_\nu \to 0 \quad \text{as} \quad \left( \frac{1}{N}, \varepsilon \right) \to 0,
\]
where for \( \varepsilon > 0 \) the sum \( \sum_{(\mu, \nu) \in Q^2} \) is split as follows:
\[
\sum_{(\mu, \nu) \in Q^2} = \sum'_{(\mu, \nu) \in I_\varepsilon} + \sum''_{(\mu, \nu) \in I_\varepsilon},
\]
with
\[
I_\varepsilon = \{ (\mu, \nu) \in Q^2 : \min(\sin \gamma(\mu, \nu), \sin \gamma(\mu, d_n), \sin \gamma(\nu, d_n)) \geq \varepsilon, \, n = 1, 2, \ldots, P_0 \}
\]
and
\[
I''_\varepsilon = Q^2 \setminus I_\varepsilon,
\]
where \( \gamma(\mu, \nu) \) is the smallest angle between \( \mu \) and \( \nu, \, d_n, \, n = 1, \ldots, P_0 \), are the directions of the sides of \( \Omega \), and \( P_0 \) is the number of sides of \( \Omega \).

Next we recall a result of Anselone [1]. We then show that the continuous integral operator \( T \) is self-adjoint in \( L_2(\Omega) \) and that \( T_N \) is self-adjoint in \( L_2(\Omega) \), since \( -\mu \in Q \) if \( \mu \in Q \).

**Proposition 2.1.** Let \( 1 \leq p \leq \infty \) and let \( T: L_p(\Omega) \to L_p(\Omega) \) be a bounded linear operator such that for some positive constant \( C \), (1.10) is valid, i.e.,
\[
\| (I - \lambda T) v \|_p \geq C \| v \|_p \quad \forall v \in L_p(\Omega),
\]
and let \( \{ T_N \}_{N=1}^\infty \) be a uniformly bounded sequence of linear operators on \( L_p(\Omega) \) such that for some positive integer \( m \),
\[
(2.7) \quad \varepsilon_N = \| (T - T_N) T_N^m \|_p \to 0 \quad \text{as} \quad N \to \infty.
\]
Then there is a positive constant \( C_1 \) such that for \( N \) large enough
\[
(2.8) \quad \| (I - \lambda T_N) v \|_p \geq C_1 \| v \|_p \quad \forall v \in L_p(\Omega).
\]
In the sequel \( T^* \) denotes the adjoint operator of \( T: L_p(\Omega) \to L_p(\Omega) \).

**Lemma 2.1.** The integral operators \( T \) and \( T_N \) are self-adjoint on \( L_2(\Omega) \).

**Proof.** Recalling the representation (1.9) of \( T \),
\[
T g(x) = \int_\Omega e^{-|x-y|} |x-y| g(y) \, dy,
\]
we have for \( f, g \in L_2(\Omega) \),

\[
(Tf, g) = \int_{\Omega} \left( \int_{\Omega} e^{-|x-y|} f(y) \, dy \right) g(x) \, dx
\]

and

\[
(f, Tg) = \int_{\Omega} f(x) \left( \int_{\Omega} e^{-|x-y|} g(y) \, dy \right) g(x) \, dx,
\]

which proves that \( T = T^* \). When we use (2.5), the adjoint of \( T_N \) is

\[
T_N^* = \sum_{\mu \in Q} T_\mu^* \omega_\mu.
\]

Moreover,

\[
(T_\mu f, g) = \int_{\Omega} \left( \int_0^{d(x, \mu)} e^{-s} f(x - s\mu) \, ds \right) g(x) \, dx.
\]

Making the substitution \( y = x - s\mu \), we note that as \( s \) varies, \( y \) varies on the line segment \([\tilde{x}, x]\), \( \tilde{x} = x - d(x, \mu) \mu \). Thus for a given \( y \in \Omega \) we have \( s = |x - y| \) with \( x \in \Omega \cap L_\mu(y) \), where \( L_\mu(y) \) is the half-line parallel to \( \mu \) starting at \( y \). Hence by the definition (1.5) of \( d \), and since \( 0 \leq s \leq d(y, -\mu) \),

\[
(T_\mu f, g) = \int_{\Omega} \int_0^{d(y, -\mu)} e^{-s} f(y) g(y - s(-\mu)) \, ds \, dy = (f, T_{-\mu} g),
\]

so that

\[
T_\mu^* = T_{-\mu}.
\]

Multiplying by \( \omega_\mu \) and summing over \( \mu \in Q \), we obtain \( T_N^* = T_N \), since \( \mu \in Q \) implies \( -\mu \in Q \). \( \Box \)

Note that by (1.4) we have the following stability estimate for the solution operator \( T_\mu \):

\[
\| \mu \cdot \nabla T_\mu g \|_p + \| T_\mu g \|_p \lesssim C \| g \|_p, \quad 1 \leq p \leq \infty.
\]

### 3. \( L_p \) error estimates.

In this section we extend the \( L_2 \) error estimates for the discrete ordinates method of [9] to \( L_p \) norms, \( 1 \leq p < \infty \). Our main result is Theorem 3.1. We also prove, using Proposition 2.1, that if \( \lambda^{-1} \notin \sigma(T) \) then for \( 1 \leq p \leq \infty \) the operator \( (I - \lambda T_N) : L_p(\Omega) \to L_p(\Omega) \) is invertible if \( N \) is large enough and thus (2.4) has a unique solution \( U_N \in L_p(\Omega) \).

Observe that the maximum regularity of the scalar flux \( U \), what we can expect in general, is \( U \in W^1_\theta(\Omega) \) for \( 1 \leq p < \infty \) and \( U \in W^{1,\delta}_0(\Omega) \) for \( \delta > 0 \) (see [13]). Theorem 3.1 is stated accordingly. Here \( W^{1,\delta}_0(\Omega) \) is defined by the \( K \) method of interpolation (see Bergh and Lofstrom [5]).

**Theorem 3.1.** Suppose that \( \lambda^{-1} \notin \sigma(T) \) and let \( 1 \leq p < \infty \). Let \( U \) be the solution of (1.7). Then there exists an integer \( N_\lambda \) such that (2.4) has a unique solution \( U_N \in L_p(\Omega) \) for \( N \geq N_\lambda \). Further, there is a constant \( C \) such that for \( N \geq N_\lambda \) and \( f \in W^{1,\delta}_0(\Omega) \),

\[
\| U - U_N \|_{L_p(\Omega)} \leq CN^{-1} \| \lambda U + f \|_{W^{1,\delta}_0(\Omega)}.
\]

There exists a constant \( C \) and for all \( \theta > 0 \), there exists \( \delta > 0 \) such that for \( N \geq N_\lambda \) and \( f \in W^{1,\delta}_0(\Omega) \),

\[
\| U - U_N \|_{L_1(\Omega)} \leq CN^{1-\theta} \| \lambda U + f \|_{W^{1,\delta}_0(\Omega)}.
\]

The proof of Theorem 3.1 is based on the following two results.
Lemma 3.1. For $1 \leq p < \infty$ there is a constant $C = C(p)$ such that for $g \in W^1_p(\Omega)$,

\[(3.3) \quad \| (T - T_N) g \|_{L_p(\Omega)} \leq CN^{-1} \| g \|_{w^1_p(\Omega)}.\]

Further there is a constant $C$ such that for $g \in W^2_1(\Omega)$,

\[(3.4) \quad \| (T - T_N) g \|_{L_1(\Omega)} \leq CN^{-2} (\log N) \| g \|_{w^1_1(\Omega)}.\]

Lemma 3.2. If $\lambda^{-1} \notin \sigma(T)$, then for $1 \leq p \leq \infty$ there is an integer $N_\lambda$ and a constant $C$ such that for $N \geq N_\lambda$, $\| (I - \lambda T_N)^{-1} \|_p \leq C$.

Let us postpone the proofs of these results and first show that Theorem 3.1 follows from them.

Proof of Theorem 3.1. We have, using (1.7) and (2.4),

\[
U - U_N = \lambda T U + T f - \lambda T_N U_N - T_N f
\]

\[
= \lambda (T - T_N) U + \lambda T_N (U - U_N) + (T - T_N) f,
\]

and thus

\[
(I - \lambda T_N)(U - U_N) = (T - T_N)(\lambda U + f).
\]

Hence using Lemma 3.2 and (3.3), with $g = \lambda U + f$, we can verify (3.1). Interpolating between (3.3) and (3.4), we obtain (3.2). 

Below $\psi_j \equiv \psi_j(\alpha)$ will denote the angle between the direction of the $j$th side of $\Omega$ and $\mu = (\cos \alpha, \sin \alpha)$. In the proof of Lemma 3.1, we will use the following lemma.

Lemma 3.3. For $1 \leq p < \infty$ there exists a constant $C$ such that if $u(x, \mu) = T_\mu g(x)$, then for $g \in W^1_p(\Omega)$,

\[(3.5) \quad \int_0^{2\pi} \left\| \frac{\partial u}{\partial \alpha} (\cdot, \alpha) \right\|_{L_p(\Omega)} d\alpha \leq C \| g \|_{w^1_p(\Omega)}.
\]

Further there is a constant $C$ such that for $g \in W^2_1(\Omega)$,

\[(3.6) \quad \left\| \frac{\partial^2 u}{\partial \alpha^2} (\cdot, \alpha) \right\|_{L_1(\Omega)} \leq C (\min_j |\sin \psi_j(\alpha)|)^{-1} \| g \|_{w^1_1(\Omega)}.
\]

Proof. By the same argument as in the proof of Lemma 4.4 in [9], we have

\[
u(x, \alpha) = \int_0^{d(x, \alpha)} e^{-\gamma} g(x - s\mu) \, ds,
\]

where $d(x, \alpha) = d(x, \mu)$, so that

\[(3.7) \quad \frac{\partial}{\partial \alpha} u(x, \alpha) = e^{-d(x, \alpha)} g(\tilde{x}_\alpha) \frac{\partial}{\partial \alpha} d(x, \alpha) + \int_0^{d(x, \mu)} e^{-\gamma} - \frac{\partial}{\partial \mu} g(x - s\mu) \, ds.
\]

Here $\tilde{x}_\alpha = x - d(x, \alpha) \mu \in \Gamma$ and $\mu' = (\sin \alpha, -\cos \alpha)$ is orthogonal to $\mu$. Further,

\[(3.8) \quad d(x, \alpha) = \frac{a_j(x)}{\sin \psi_j(\alpha)} \quad \text{for} \ x \in \Omega_{\alpha,j} = \{ x \in \Omega : \tilde{x}_\alpha \in S_j \},
\]

where the $S_j$ is a side of $\Omega$, $\psi_j(\alpha)$ the angle between $S_j$ and $\mu$, and $a_j(x)$ is the distance from $x$ to the straight line given by $S_j$. Hence raising the absolute values of both sides of (3.7) to the power $p_i$ integrating over $\Omega_{\alpha,j}$, using an orthogonal coordinate system.
(ξ_1, ξ_2) with the ξ_1 axis along S_α, and the fact that the boundedness of Ω implies
a_j(x) ≤ C|sin ψ_j| and thus |(∂/∂α) d(x, α)| ≤ C|sin ψ_j|^{-1} for x ∈ Ω_{α,j}, we get

\[ \int_{Ω_{α,j}} |\frac{∂}{∂α} u(x, α)|^p dx \leq C \left[ \int_{Ω_{α,j}} |g(\xi_α)|^p |sin ψ_j|^{-p} dξ_1 dξ_2 + \int_{Ω_{α,j}} |\nabla g(x)|^p dx \right] \]

\[ \leq C \left[ \int_{S_j} |g|^p dξ_1 \int_0^{C|sin ψ_j| |sin ψ_j|^{-p} dξ_2 + \|\nabla g\|^p_p \right] \]

\[ \leq C \|g\|_{L_p(1)} \|sin ψ_j\|^{-1-p} + \|\nabla g\|^p_p \].

Recalling the trace estimate

we have summing over j,

\[ \|g\|_{L_p(1)} \leq C \|g\|_{W^1_p(Ω)}, \quad 1 \leq p \leq ∞, \]

Finally we complete the proof of (3.5) by integrating with respect to α. To prove (3.6)
we differentiate (3.7) and (3.8) to get

\[ \frac{∂^2 u}{∂α^2} (x, α) = -e^{-d(x,α)} g(\xi_α) \left( \frac{∂}{∂α} d(x, α) \right)^2 + e^{-d(x,α)} g(\xi_α) \frac{∂^2 d}{∂α^2} (x, α) \]

\[ + 2 e^{-d(x,α)} \frac{∂}{∂α} g(\xi_α) \frac{∂d}{∂α} (x, α) + \int_0^{d(x,μ)} e^{-s} \frac{∂^2}{∂α^2} g(x-sμ) ds \]

and

\[ \frac{∂^2}{∂α^2} d(x, α) = \frac{a_j(x)(1 + cos^2 ψ_j)}{sin^3 ψ_j}, \]

so that by the same argument as in the proof of (3.5),

\[ \int_{Ω_{α,j}} \left| \frac{∂^2}{∂α^2} u(x, α) \right| dx \leq C \left[ \int_{S_j} \left( |g| + |\nabla g| \right) dξ_1 \int_0^{C|sin ψ_j|} |sin ψ_j|^{-2} dξ_2 + \|g\|_{W^2_1(Ω)} \right] \]

Using the trace estimate once more and summing over j, we get

\[ \left\| \frac{∂^2}{∂α^2} u(\cdot, α) \right\|_{L_p(Ω)} \leq C (min_j |sin ψ_j(α)|)^{-1} \|g\|_{W^2_1(Ω)}, \]

and the proof is complete. \( \square \)

**Proof of Lemma 3.1.** Writing \( u(x, α) = T_μ g(x) \) with \( μ = (cos α, sin α) \) and using
(2.6a) with \( k = 1 \), we have

\[ \| (T - T_N) g \|_{L_p(Ω)} = \left\| \int_{S} T_μ g(\cdot) dμ - \sum_{μ ∈ Q} T_μ g(\cdot) ω_μ \right\|_{L_p(Ω)} \]

\[ = \left\| \int_{S} u(\cdot, μ) dμ - \sum_{μ ∈ Q} u(\cdot, μ) ω_μ \right\|_{L_p(Ω)} \]

\[ \leq CN^{-1} \left\| \int_0^{2π} \frac{∂u}{∂α}(\cdot, α) dα \right\|_{L_p(Ω)} \]

\[ \leq CN^{-1} \left\| \int_0^{2π} \frac{∂u}{∂α}(\cdot, α) dα \right\|_{L_p(Ω)}, \]
and (3.3) then follows from (3.5). To prove (3.4) we define for \( k = 1, \cdots, N, \)
\[
I_k = \left[ \frac{2(k-1)}{N}, \frac{2k}{N} \right],
\]
and let \( A_j \) be the union of the \( I_k \) containing the direction of \( S_j \) and the adjacent \( I_i \) closest to the direction of \( S_j \). Let
\[
S_0 = \bigcup_{j=1}^{n_0} A_j,
\]
and
\[
Q_0 = Q \cap S_0.
\]
We have, using (2.6a), (3.6), and (3.9),
\[
\left\| (T - T_N) g \right\|_{L^p(\Omega)} \leq \int_{S_0} T_{\mu} g(\cdot) d\mu - \sum_{\mu \in Q \setminus Q_0} T_{\mu} g(\cdot) \omega_{\mu}
\]
\[
+ \int_{S_0} T_{\mu} g(\cdot) d\mu - \sum_{\mu \in Q_0} T_{\mu} g(\cdot) \omega_{\mu}
\]
\[
= C \frac{N^2}{N} \int_{S \setminus S_0} \frac{\partial^2 u(\cdot, \alpha)}{\partial \alpha^2} \left|_{L^1(\Omega)} \right| d\alpha + \frac{C}{N} \int_{S_0} \frac{\partial u(\cdot, \alpha)}{\partial \alpha} \left|_{L^1(\Omega)} \right| d\alpha
\]
\[
\leq C \frac{N^2}{N} \left[ \int_{S \setminus S_0} \left( \min |\sin \psi_j(\alpha)| \right)^{-1} d\alpha \right] \left\| g \right\|_{L^1_0(\Omega)} + \frac{C}{N} |S_0| \left\| g \right\|_{L^1_0(\Omega)}
\]
\[
\leq C \frac{N^2}{N} (\log N) \left\| g \right\|_{L^1_0(\Omega)} + \frac{C}{N} |S_0| \left\| g \right\|_{L^1_0(\Omega)},
\]
where \( |S_0| \) is the length of \( S_0 \). Since \( |S_0| \sim 1/N \) we obtain the desired result. \( \square \)

Let us now turn to the proof of Lemma 3.2. We want to prove for \( 1 \leq p \leq \infty \) that if \( \lambda^{-1} \not\in \sigma(T) \) and \( N \) is sufficiently large depending on \( \lambda \) and \( p \), then \( (I - \lambda T_N) \) is invertible as an operator on \( L_p(\Omega) \). Lemma 3.4 below together with Proposition 2.1 implies that \( (I - \lambda T_N) \) is one to one. To show that it is also onto we will use Proposition 3.1 below (see Rudin [16, Thm. 4.15, p. 97]). Note that \( T_N : L_p(\Omega) \rightarrow L_p(\Omega) \) is not compact.

**Lemma 3.4.** \( \|(T - T_N) T_N^*\|_p \rightarrow 0 \), as \( N \rightarrow \infty \), \( 1 \leq p \leq \infty \).

**Proposition 3.1.** Suppose \( X \) and \( Y \) are Banach spaces and \( F \) is a bounded linear operator from \( X \) into \( Y \); then

(a) \( \text{Range } (F) = Y \)

if and only if there exists a constant \( C \) such that

(b) \( \|F y^*\| \geq C \|y^*\| \) for \( y^* \in Y^* \).

**Proof of Lemma 3.2.** It remains to prove that \( \text{Range } (I - \lambda T_N) = L_p(\Omega) \). Hence it suffices to prove (b) in Proposition 3.1 with \( F \) replaced by \( (I - \lambda T_N) \). Now Proposition 2.1 applied to \( T^* \), where

\[
\|(I - \lambda T^*) v^*\|_p \geq C \|\lambda^* v^*\|_p \quad \forall v^* \in L_p(\Omega),
\]
and the same argument as in the proof of Lemma 3.4 below yield

\[
\varepsilon_N^k = \|(T^* - T_N^*) T_N^{\lambda^*} \|_p \rightarrow 0 \quad \text{as } N \rightarrow \infty.
\]
Thus the conclusion of Proposition 2.1 holds for the adjoint operator $T^*_N$, i.e., there exists a constant $C$ such that (b) in Proposition 3.1 is valid with $F^*$ and $y^*$ replaced by $(I - \lambda T_N)^*$ and $v^*$, respectively. Observe that since $L_\infty$ is not a reflexive Banach space, the conclusion $\|(I - \lambda T_N)^{-1}\|_p \leq C$ is only valid for $1 \leq p < \infty$. However, the constants involved in Proposition 2.1 as well as in this proof are independent of $p$. Thus letting $p \to \infty$ we also obtain the result for $p = \infty$. □

The proof of Lemma 3.4 is based on (3.3) and the following lemma.

**Lemma 3.5.** There is a constant $C$ such that if $(\mu, \nu) \in I_\nu'$ and $g \in L_\mu(\Omega)$, $1 \leq p \leq \infty$, then

$$\|T_\mu T_\nu g\|_{L_p(\Omega)} \leq C e^{-2(1/p)} \|g\|_{L_p(\Omega)}.$$

**Remark 3.1.** Note that the operator $T_\mu$ regularizes in the direction $\mu$ and thus for two nonparallel directions $\mu$ and $\nu$, we have that $T_\mu T_\nu$ regularizes in all directions (with a constant depending on the smallest angle between $\mu$ and $\nu$). In the proof of Lemma 3.4, we will use this regularizing property of $T_\mu$ to show that $T^*_N$ can be split as

$$T^2_N = A_N + B_N,$$

with $A_N : L_\mu(\Omega) \to W^*_p(\Omega)$; i.e., $A_N$ is compact, and $\|B_N\| \to 0$, as $N \to \infty$.

**Proof of Lemma 3.4.** For $1 \leq p < \infty$ we have, using (3.3), Lemma 3.5 and (2.10),

$$\|(T - T_N)T^2_N g\|_{L_p(\Omega)} = \left\|(T - T_N) \sum_{(\mu, \nu) \in Q^2} \omega_\mu \omega_\nu T_\mu T_\nu g\right\|_{L_p(\Omega)}$$

$$\leq C \| \Sigma'_\mu \omega_\mu \omega_\nu T_\mu T_\nu g\|_{L_p(\Omega)} + C \| \Sigma''_\mu \omega_\mu \omega_\nu\|_{L_p(\Omega)}.$$

Now taking $e = N^{-\delta}$ with $\delta < p/(p - 1)^{-1}$ and using (2.6b), we obtain the desired result. For $p = \infty$, instead of (3.3), we use Lemma 4.1 below and the proof is complete. □

In the proof of Lemma 3.5 we will refer to the following result.

**Lemma 3.6.** There is a constant $C$ such that

$$\left( \int_{\Gamma'} |T_\mu g|^p |n \cdot \mu| \, d\sigma \right)^{1/p} \leq C \|g\|_{L_p(\Omega)}, \quad 1 \leq p \leq \infty.$$

**Proof.** It suffices to give the proof with $\Gamma'$ replaced by $\Gamma'' = \{x \in \Gamma : n(x) \cdot \mu \neq 0\}$. Let $p = 1$. For $x \in \Gamma''$, we have $d(x, \mu) \neq 0$, and if $0 \leq s \leq d(x, \mu)$ then $x - s\mu \in [x, x - d(x, \mu)\mu] \subset \Omega$. Thus using (1.4) and Fubini's theorem, we find that

$$\int_{\Gamma''} |T_\mu g(x)| |n \cdot \mu| \, d\sigma = \left( \int_0^{d(x, \mu)} \int_{\Gamma''} e^{-s} g(x - s\mu) \, d\sigma \right) |n \cdot \mu| \, ds$$

$$\leq C \|g\|_{L_1(\Omega)}.$$

For $p = \infty$ we have

$$\sup_{x \in \Gamma''} \|T_\mu g(x)\| = \sup_{x \in \Gamma''} \| \int_0^{d(x, \mu)} e^{-s} g(x - s\mu) \, ds \| \leq C \|g\|_{L_\infty(\Omega)}.$$
Proof of Lemma 3.5. By an orthogonal coordinate transformation we may assume that \( \mu = (1, 0) \). Since \((\mu, \nu) \in I_\nu^\nu \) we have by (2.10),

\[
\left\| \nabla (T_\mu T_{\nu} g) \right\|_{L_p(\Omega)} \leq \frac{C}{\varepsilon} \left[ \left\| g \right\|_{L_p(\Omega)} + \left\| \frac{\partial}{\partial \nu} (T_\mu T_{\nu} g) \right\|_{L_p(\Omega)} \right].
\]

Replacing \( g \) in (1.4) by \( T_\nu g \) gives

\[
\frac{\partial}{\partial \nu} (T_\mu T_{\nu} g(x)) = e^{-d(x, \nu)T_\mu T_{\nu} g(\tilde{x})} \frac{\partial}{\partial \nu} d(x, \mu) + \int_0^{d(x, \mu)} e^{-s} \frac{\partial}{\partial \nu} (T_\nu g(x - s\mu)) \, ds,
\]

where \( \tilde{x} = x - d(x, \mu) \). It is easy to verify that

\[
\frac{\partial d}{\partial \nu} = \nu \cdot \nabla_d d(x, \mu) = \frac{n \cdot \nu}{n \cdot \mu},
\]

where \( n = (n_1, n_2) \) is the outward unit normal to \( \Gamma \) at \( \tilde{x} \). Raising the absolute values of both sides of (3.14) to the power of \( p \) and using the facts that \( dx_2 = \mu \cdot n \, d\sigma \) on \( \Gamma \) and \( |\mu \cdot n| \geq \varepsilon \) we find that

\[
\left\| \frac{\partial}{\partial \nu} (T_\mu T_{\nu} g) \right\|_{L_p(\Omega)} \leq C \varepsilon^{-1/p} \left( \int_\Gamma |T_\nu g| |n \cdot \nu| \, d\sigma \right)^{1/p} + C \left\| g \right\|_{L_p(\Omega)},
\]

where we have also used (2.10). Thus by Lemma 3.6 and (3.13)

\[
\left\| \nabla (T_\mu T_{\nu} g) \right\|_{L_p(\Omega)} \leq C \varepsilon^{-2/p} \left\| g \right\|_{L_p(\Omega)}.
\]

Further by (2.10), \( \left\| T_\mu T_{\nu} g \right\|_{L_p(\Omega)} \leq C \left\| g \right\|_{L_p(\Omega)} \) and the proof is complete. \( \square \)

4. Iterative improvement. In the previous section we have proved, with maximally available regularity of the scalar flux \( U \), i.e., \( U \in W^1_p(\Omega) \), for \( 1 \leq p < \infty \), and \( U \in W^{2,\delta}_1(\Omega) \) with \( \delta > 0 \), that

\[
\left\| U - U_N \right\|_{L_p(\Omega)} \leq C N^{-1},
\]

and

\[
\left\| U - U_N \right\|_{L_1(\Omega)} \leq C N^{-2+\theta},
\]

where \( C \) depends on \( \theta \) and \( p \). Further for \( \theta > 0 \), we have from (4.1) (see also Lemma 4.1 below) that

\[
\left\| U - U_N \right\|_{L_\infty(\Omega)} \leq C N^{-1+\theta}.
\]

In this section we will prove that it is possible by a simple postprocessing to produce an improved solution \( U^*_N \) for which

\[
\left\| U - U^*_N \right\|_{L_\infty(\Omega)} \leq C N^{-2+\theta},
\]

that is, for which the rate of convergence of \( U^*_N \) in \( L_\infty \) is the same as the rate in \( L_1 \) for the original solution \( U_N \).

The postprocessed solution \( U^*_N \) is defined as follows:

\[
U^*_N = T_M(\lambda U^{(2)}_M + f),
\]

where

\[
U^{(k+1)}_M = T_M(\lambda U^{(k)}_M + f), \quad k = 0, 1, 2, \ldots,
\]

\[
U^{(0)}_M = U_N,
\]
and $M = N^2$. Thus we compute $U_N^{(3)} = U_N^{(1)}$ by applying the operator $T_M$ three times with $M = N^2$, starting from the original solution $U_N$. The postprocessed solution $U_N^{*}$ should be compared with the solution $U_M$ of the coupled problem

\[(4.7)\quad U_M = T_M(\lambda U_M + f).\]

By (4.3) and (4.4) we have for both $U_N^{(3)}$ and $U_M$ the same rate of convergence $O(M^{-1+\varepsilon})$ in $L_\infty(\Omega)$, since $M = N^2$. To find $U_M$, requires the solution of a large coupled problem (4.7), while to compute $U_N^{(3)}$ we only have to solve the smaller coupled problem (2.4) and then in (4.6) apply the operator $T_M$ a few times. Hence we expect to be able to compute $U_N^{(3)}$ with less work than $U_M$. Note that $U_N^{(3)}$ may be viewed as an approximate solution of (4.7) obtained after three fixed-point iterations starting with $U_N$.

Postprocessing procedures of the form (4.5)–(4.6) have been considered in practical computations and an example is discussed in [11, § VI]. In particular the hope is to decrease in this way the so-called ray effects for media in which absorption dominates scattering.

To estimate $\|U - U_N^{(3)}\|_\infty$ we use (2.4) and (4.6) to obtain

\[U = \lambdaTU + Tf = \lambda T(\lambda TU + Tf) + Tf,\]

and

\[U^{(3)}(M) = \lambda T_MU + T_Mf = \lambda T_M(\lambda T_MU^{(1)} + T_Mf) + T_Mf.\]

We now split $U - U_N^{(3)} = U - U_M^{(3)}$ as follows:

\begin{align*}
U - U_N^{(3)} & = (T - T_M)f + \lambda(T - T_M)f + \lambda(T - T_M)T_Mf + \lambda^2(T - T_M)^2T_Mf \\
& \quad + \lambda^3(T - T_M)^3T_Mf + \lambda^4(T - T_M)^4T_Mf
\end{align*}

\[(4.8)\]

Now taking the $L_\infty(\Omega)$-norm of both sides of (4.8), we have

\begin{align*}
\|U - U_N^{(3)}\|_{L_\infty(\Omega)} & \leq \sum_{j=1}^7 \|I_jf\|_{L_\infty(\Omega)} + \sum_{i=1}^2 \|R_iT_MT_NU_N\|_{L_\infty(\Omega)} + \|T^2(U - U_M^{(1)})\|_{L_\infty(\Omega)}.
\end{align*}

Here the quantities $\|I_jf\|_{L_\infty(\Omega)}$, $j = 1, \ldots, 7$, will be estimated using Lemma 4.1 below. For $\|R_iT_MT_NU_N\|_{L_\infty(\Omega)}$, $i = 1, 2$, we will use Lemma 4.2. The last quantity, $\|T^2(U - U_M^{(1)})\|_{L_\infty(\Omega)}$ will be handled using the fact that $T$ is regularizing in the sense that for $\tau > 0$, $T^2 : L_{1+\tau}(\Omega) \to L_\infty(\Omega)$. In the proof of Lemma 4.2 we will use the following splitting with $\varepsilon \equiv 1/M$:

\begin{align*}
\sum_{(\mu, \nu) \in Q_M \times Q_N} \omega_\mu \omega_\nu = \sum_{(\mu, \nu) \in J_r} \omega_\mu \omega_\nu + \sum_{(\mu, \nu) \in J'_r} \omega_\mu \omega_\nu,
\end{align*}

where

\[
J_r = \{(\mu, \nu) \in Q_M \times Q_N : \min(\sin(\gamma(\mu, \nu)), \sin(\gamma(\mu, \nu), d_n)) \geq \varepsilon, \; n = 1, \ldots, P_0\},
\]

\[
J'_r = Q_M \times Q_N \setminus J_r,
\]

\[
\omega_\mu = \frac{2\pi}{M} \quad \text{and} \quad \omega_\nu = \frac{2\pi}{N}.
\]
We recall that, as in the previous splitting, $\gamma(\mu, \nu)$ is the smallest angle between $\mu$ and $\nu$, $P_0$ is the number of sides of $\Omega$, and $d_n$ are the directions of the sides of $\Omega$.

Now we are prepared to state the main result of this section.

**Theorem 4.1.** Suppose that $\lambda^{-1} \notin \sigma(T)$ and let $\theta > 0$. Then there exist constants $\delta > 0$, $\tau > 0$, $C_\lambda$ and $N_\lambda$ such that for $N \geq N_\lambda$ and $M \sim N^2$,

$$\|U - U_N\|_{L^\alpha(\Omega)} \leq C_{\lambda} \left( \frac{1}{N^{2-\delta}} + \frac{1}{N^2} (\log N)^2 \right) \left( \|g\|_{W^{2,\alpha}_\gamma(\Omega)} + \|g\|_{W^{\alpha}_\gamma(\Omega)} \right),$$

where $g = \lambda U + f$.

The proof of Theorem 4.1 is based on the following three results.

**Lemma 4.1.** There is a constant $C > 0$ such that

$$\|(T - T_M)f\|_{L^\alpha(\Omega)} \leq \frac{C}{M} (\log M) \|f\|_{W^\alpha_\gamma(\Omega)}.$$  

**Lemma 4.2.** If $\lambda^{-1} \notin \sigma(T)$, then there is a constant $C_\lambda$ and an integer $N_\lambda$ such that for $N \geq N_\lambda$ and $M \sim N^2$,

$$\|(T - T_M) T_M T_N U_N\|_{L^\alpha(\Omega)} \leq C_{\lambda} \left( \frac{1}{N^2} + \frac{1}{M} (\log N)^3 \right) \|f\|_{L^\alpha(\Omega)}.$$  

**Lemma 4.3.** For $\tau > 0$ there exists a constant $C$ such that $T^2 : L^{1+\tau}(\Omega) \to L^{\alpha}(\Omega)$, i.e.,

$$\|T^2 h\|_{L^\alpha(\Omega)} \leq C \|h\|_{L^{1+\tau}(\Omega)}.$$  

We postpone the proofs of these results and first show that Theorem 4.1 follows from them.

**Proof of Theorem 4.1.** We have, using (1.7) and (4.6),

$$U - U_N = \lambda T_M (U - U_N) + (T - T_M)(\lambda U + f).$$  

Using interpolation and the same technique as in the proof of Theorem 3.1, we can show that for $\theta > 0$ and sufficiently large $N$ there exist constants $\delta > 0$, $\tau > 0$, and $C$ such that

$$\|U - U_N\|_{L^{1+\tau}(\Omega)} \leq CN^{-2+\theta} \|g\|_{W^{2,\alpha}_\gamma(\Omega)}.$$  

Now by Lemma 4.3, (3.3), (4.11), and (4.12),

$$\|T^2 (U - U_M^{(1)})\|_{L^\alpha(\Omega)} \leq C \left( \frac{1}{N^{2-\delta}} + \frac{1}{M} \right) \left( \|g\|_{W^{2,\alpha}_\gamma(\Omega)} + \|g\|_{W^{\alpha}_\gamma(\Omega)} \right),$$  

and thus the desired result follows from (4.9), Lemma 4.1, Lemma 4.2, and (4.13). □

**Proof of Lemma 4.1.** Let $S_j$ be the $j$th side of $\Omega$ and $\psi_j(\alpha)$ be the angle between $S_j$ and $\mu = (\cos \alpha, \sin \alpha)$. Defining for $k = 1, \ldots, M$,

$$J_k = \left[ \frac{2(k-1)\pi}{M}, \frac{2k\pi}{M} \right],$$

and $A_j$ as the union of the $J_k$ containing the direction of $S_j$ and the adjacent $J_l$ closest to the direction of $S_j$, and letting

$$S_0 = \bigcup_{j=1}^{P_0} A_j \quad \text{and} \quad Q_0 = Q_M \cap S_0,$$
we have
\[
|T_f(x) - T_M f(x)| = \left| \int_S T_\mu f(x) \, d\mu - \sum_{\mu \in Q_0} T_\mu f(x) \omega_\mu \right|
\]
\[
(4.14)
\]
with the obvious notation. Now applying (2.6a) to $S \setminus S_0$, we have
\[
|A_0| = \left| \int_{S \setminus S_0} T_\mu f(x) \, d\mu - \sum_{\mu \in Q_0 \setminus Q_0} T_\mu f(x) \omega_\mu \right| \leq \frac{C}{M} \int_{S \setminus S_0} \left| \frac{\partial}{\partial \alpha} T_\mu f(x) \right| \, d\alpha
\]
and as in the proof of Lemma 3.3
\[
\left\| \frac{\partial}{\partial \alpha} (T_\mu f) \right\|_{L^\infty(\Omega)} \leq C \left( \min_j \left| \sin \psi_j(\alpha) \right| \right)^{-1} \left\| f \right\|_{W^{1,\infty}(\Omega)}.
\]

Thus,
\[
\left\| A_0 \right\|_{L^\infty(\Omega)} \leq \frac{C}{M} \left( \int_{S \setminus S_0} \left( \min_j \left| \sin \psi_j(\alpha) \right| \right)^{-1} \, d\alpha \right) \left\| f \right\|_{W^{1,\infty}(\Omega)}
\]
\[
(4.15)
\]
\[
\leq \frac{C}{M} (\log M) \left\| f \right\|_{W^{1,\infty}(\Omega)}.
\]

On the other hand, using (2.10),
\[
\left\| B_0 \right\|_{L^\infty(\Omega)} = \sup_{x \in \Omega} \left| \int_{S_0} T_\mu f(x) \, d\mu - \sum_{\mu \in Q_0} T_\mu f(x) \omega_\mu \right|
\]
\[
(4.16)
\]
\[
\leq C \left( P_0 \frac{1}{M} + |S_0| \right) \sup_{x \in \Omega} \left| T_\mu f(x) \right|
\]
\[
\leq CP_0 \frac{1}{M} \left\| f \right\|_{L^\infty(\Omega)} \leq C \frac{1}{M} \left\| f \right\|_{L^\infty(\Omega)},
\]
where $|S_0| \sim 1/M$ is the length of $S_0$. Now (4.14)–(4.16) complete the proof. □

**Proof of Lemma 4.2.** The splitting (4.10) and repeated application of (2.10) together with Lemma 4.1 yield
\[
\left\| (T - T_M) T_M T_N U_N \right\|_\infty \leq \left\| (T - T_M) \sum_{(\mu, \nu) \in J_\nu} \omega_\mu \omega_\nu T_\mu T_\nu U_N \right\|_\infty
\]
\[
+ \left\| (T - T_M) \sum_{(\mu, \nu) \in J_\nu} \omega_\mu \omega_\nu T_\mu T_\nu U_N \right\|_\infty
\]
\[
\leq \frac{C}{M} (\log M) \sum_{(\mu, \nu) \in J_\nu} \omega_\mu \omega_\nu \left\| T_\mu T_\nu U_N \right\|_{W^{1,\infty}(\Omega)}
\]
\[
(4.17)
\]
\[
+ C \varepsilon \left\| T_\mu T_\nu U_N \right\|_\infty
\]
\[
\leq \frac{C}{M} (\log M) \sum_{(\mu, \nu) \in J_\nu} \omega_\mu \omega_\nu \left\| \nabla_x (T_\mu T_\nu U_N) \right\|_\infty
\]
\[
+ C \left[ \frac{1}{M} (\log M) + \varepsilon \right] \left\| U_N \right\|_\infty.
\]
Observe that since \((\mu, \nu) \in J_\nu\), we have \(\sin \gamma(\mu, \nu) \geq \epsilon\) and for \(n = 1, 2, \ldots, p_0\), \(\sin \gamma(\mu, d_n) \geq \epsilon\). Now assuming \(\mu = (1, 0)\) and using (2.10), we find that
\[
\|\nabla_x (T_\mu T_x U_N)\|_\infty \leq C |\sin \gamma(\mu, \nu)|^{-1} \left( \|U_N\|_\infty + \left\| \frac{\partial}{\partial \nu} (T_\mu T_x U_N) \right\|_\infty \right).
\]
Further recalling (3.15) and using the same argument as in the proof of Lemma 3.5, we have
\[
\left\| \frac{\partial}{\partial \nu} (T_\mu T_x U_N) \right\|_\infty \leq C (\min_n |\sin \gamma(\mu, d_n)|)^{-1} \|U_N\|_\infty,
\]
so that
\[
\|\nabla_x (T_\mu T_x U_N)\|_\infty \leq C (\min_n |\sin \gamma(\mu, d_n)|)^{-1} \|U_N\|_\infty.
\]
Summing now first over \(\nu\) and then over \(\mu\), we obtain
\[
\sum_{(\mu, \nu) \in J_\nu} \omega_\mu \omega_\nu \|\nabla_x (T_\mu T_x U_N)\|_\infty \leq C \sum_{(\mu, \nu) \in J_\nu} \min_n \frac{\omega_\mu \omega_\nu}{\sin \gamma(\mu, \nu) \sin \gamma(\mu, d_n)} \|U_N\|_\infty
\]
\begin{equation}
\leq C \left\| \int \frac{1}{\alpha} \int \frac{1}{\beta} \left\| U_N \right\|_\infty
\end{equation}
\begin{equation}
\leq C |\log \epsilon|^2 \|U_N\|_\infty.
\end{equation}
Moreover, since \(\lambda^{-1} \notin \sigma(T)\), Lemma 3.2 implies that there exists an integer \(N_\lambda\) such that for \(N \geq N_\lambda\), \((I - \lambda T_N)^{-1}\) exists and is a uniformly bounded operator on \(L_\mu(\Omega)\), \(1 \leq p \leq \infty\). Thus by (2.4) and (2.10) for \(N \geq N_\lambda\)
\begin{equation}
\|U_N\|_\infty \leq \|(I - \lambda T_N)^{-1}\|_\infty \|T_N f\|_\infty \leq C_\lambda \|f\|_\infty.
\end{equation}
Taking finally \(\epsilon \sim 1/N^2\) and combining (4.17)-(4.19) the desired result follows. \(\square\)

**Proof of Lemma 4.3.** Recall that if \(h\) is zero in the complement of \(\Omega\), then
\[
T(x) = \int_\Omega e^{-\sqrt{|x-y|}/|x-y|} h(y) \, dy = \int_\Omega k(x-y)h(y) \, dy = (k * h)(x), \quad h(x) = 0, \quad x \notin \Omega.
\]
Thus by Young’s inequality
\[
\|T^2 h\|_\infty \leq \|k * k * h\|_\infty \leq \|k\|_p \|k * h\|_q, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad Th(x) = 0, \quad x \notin \Omega
\]
and
\[
\|k * h\|_q \leq \|k\|_r \|h\|_s, \quad 1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}.
\]
In the above inequalities, \(k\) is our kernel function. We set \(r = p\) to obtain \(2/p + 1/s = 2\) and, with \(s = 1 + r\), we have \(p = 1 + 1/(1 + 2\tau) \in (1, 2)\). With this \(p\), \(k \in L_p(\Omega)\) and the proof is complete. \(\square\)

**5. Eigenvalue estimates.** In this section we prove error estimates for the approximation of simple isolated eigenvalues of our integral operator \(T\). Eigenvalues of \(T\) are of physical interest and in particular the smallest positive eigenvalue is related to criticality.
Since \( T_N \) does not converge in the operator norm to \( T \), we cannot directly apply the known standard arguments as in [4] and [6] to derive the desired eigenvalue estimates. What is lacking is a proof of an equality of the dimensions of certain eigenspaces of \( T_N \) and \( T \). However, by Lemma 5.1, \( T_N^3 \) converges in the operator norm to \( T^3 \) and using this result we are able to verify the crucial condition concerning the dimensions of eigenspaces of \( T_N \) and \( T \).

Because of the limited regularity of the scalar flux \( U \), by Theorem 3.1 the sharpest estimate for the convergence of eigenvalues is obtained for \( p = 1 \).

**Theorem 5.1.** Let \( \lambda \) be an isolated eigenvalue of \( T \) with algebraic multiplicity 1 and let \( \Gamma \subset \rho(T) \) be a circle centered at \( \lambda \). Then there exists an integer \( N_\lambda \) such that for \( N \geq N_\lambda \), \( T_N \) has exactly one eigenvalue \( \lambda_N \in \text{Int} B(\lambda, \Gamma) \) with algebraic multiplicity 1. Further assume corresponding eigenfunction \( g \in W^{2, \delta}_1(\Omega) \) for some \( \delta > 0 \). Then for \( \theta > 0 \) there exists a constant \( C_{\lambda, \theta} \) such that

\[
|\lambda - \lambda_N| \leq C_{\lambda, \theta} N^{\frac{\theta}{\delta}}, \quad N \geq N_\lambda.
\]

Here \( \rho(T) \) is the resolvent set of \( T \) and \( B(\lambda, \Gamma) \) is the disc centered at \( \lambda \) with \( \partial B = \Gamma \).

We first review, for the sake of completeness, a rather standard and known argument giving a general form of Theorem 5.1 for linear operators on Banach spaces (Theorem 5.2).

Let \( X \) be a complex Banach space with norm \( \| \cdot \| : F : X \to X \) a bounded linear operator and \( \{ F_N \}_{N=1}^\infty \) a family of bounded linear operators on \( X \) such that for \( g \in X \),

\[
\| Fg - F_N g \| \to 0 \quad \text{as} \quad N \to \infty.
\]

We assume that \( \lambda \) is an isolated eigenvalue of \( F \) with index \( \nu \) and finite algebraic multiplicity \( m \geq \nu \). Then there exists a circle \( \Gamma \) in the complex plane centered at \( \lambda \), which separates \( \lambda \) from \( \sigma(F) \setminus \{ \lambda \} \). We denote by \( P(\lambda, F) \) the spectral projection \((1/2\pi i) \int_{\Gamma} (z - F)^{-1} \, dz \) associated with the eigenspace

\[
X(\lambda, F) = \text{null} (\lambda - F)^\nu
\]

and let \( E(\lambda, F) = \text{Range} (P(\lambda, F)) \) be the corresponding generalized eigenspace. It is easy to verify that

\[
E = E(\lambda, F) = X(\lambda, F),
\]

\[
\dim E(\lambda, F) = m,
\]

\[
(\lambda - F)^\nu P(\lambda, F) = 0 \quad \text{and} \quad (\lambda - F)^{\nu - 1} P(\lambda, F) \neq 0
\]

(see, e.g., [6, Chap. 5] and [8, p. 573]). Now let us assume that there exists a constant \( C \) and an integer \( N_0 \) such that for \( N \geq N_0 \)

\[
\|(z - F_N)^{-1}\| \leq C \quad \forall z \in \Gamma.
\]

Here \( \| \cdot \| \) is the operator norm defined by

\[
\|A\| = \sup \left\{ \frac{\|Ag\|}{\|g\|} : g \in X, \ g \neq 0 \right\}.
\]

Considering (5.3) we may define the projection operator

\[
P(\lambda, F_N) = \frac{1}{2\pi i} \int_{\Gamma} (z - F_N)^{-1} \, dz,
\]

associated with the eigenspace

\[
E_N = E(\sigma_N, F_N) = \text{null} (\lambda_1 - F_N)^\nu \oplus \cdots \oplus \text{null} (\lambda_r - F_N)^\nu,
\]
where $\sigma_N = \sigma(F_N) \cap B(\lambda, \Gamma)$, $B(\lambda, \Gamma)$ is the disc centered at $\lambda$ with $\delta B = \Gamma$ and $\lambda_j \in \sigma_N$ are eigenvalues of $F_N$ with algebraic multiplicities $m_j$ and indices $\nu_j$. Finally we assume that for sufficiently large $N$,

$$m = \dim E(\lambda, F) = \dim E(\sigma_N, F_N) = \sum_{j=1}^r m_j.$$  

We are now ready to formulate the following general result.

**Theorem 5.2.** Let $\lambda$ be an isolated eigenvalue of $F : X \to X$ with finite algebraic multiplicity $m$ and assume that (5.2)–(5.4) hold. Then there exist exactly $m$ eigenvalues, counted with their multiplicities, $\lambda_N \in \sigma_N$ of $F_N$ and a constant $C$ such that

$$\max_{\lambda_N \in \sigma_N} |\lambda - \lambda_N| \leq C \|F - F_N\|_E,$$

where $\|\cdot\|_E$ denotes the operator norm restricted to $E$.

The proof of Theorem 5.2 is based on Propositions 5.1–5.3 below. See also, e.g., [4], [6], and [12].

**Proposition 5.1.** If (5.2) and (5.3) hold then

$$\|P_N - P\| \to 0 \quad \text{as } N \to \infty,$$

where $P = P(\lambda, F)$ and $P_N = P(\lambda, F_N)$.

**Proof.** Since $(z - F)^{-1}$ and $P$ commute we have for $u \in X$

$$(P - P_N)Pu = \frac{1}{2\pi i} \int_I [(z - F)^{-1} - (z - F_N)^{-1}]Pu \, dz$$

$$= \frac{1}{2\pi i} \int_I (z - F_N)^{-1}(F - F_N)P(z - F)^{-1}u \, dz,$$

and

$$\|(P - P_N)Pu\| \leq C \sup_{z \in I} \|(z - F_N)^{-1}\| \|F - F_N\| \sup_{z \in I} \|(z - F)^{-1}u\|.$$  

When we use (5.3) there exists a constant $C$ such that for sufficiently large $N$

$$\|(P - P_N)Pu\| \leq C \|F - F_N\| \|u\|.$$  

Since the dimension of the range of $P$ is finite, $P$ is compact. Thus by (5.2) we have

$$\|(F - F_N)P\| \to 0 \quad \text{as } N \to \infty,$$

and the proof is complete. □

We define the operator $B_N : E \to E_N$ as the restriction of $P_N$ to $E$, $B_Nu = P_Nu$, for $u \in E$.

**Proposition 5.2.** If (5.2)–(5.4) hold, then there exist an integer $N_0$ and a constant $C$ such that for $N \geq N_0$

(a) $B_N$ is an isomorphism from $E$ onto $E_N$,

(b) $\|B_N^{-1}\| \leq C$.

**Proof.** Let $\varphi_1, \cdots, \varphi_m$ be a basis of the space $E$. Since $P$ is a projection onto $E$

$$P\varphi_i = \varphi_i, \quad i = 1, \cdots, m.$$ 

Set

$$\varphi_{i,N} = B_N\varphi_i = P_N\varphi_i, \quad i = 1, \cdots, m.$$ 

We have, using Proposition 5.1, that for $1 \leq i \leq m$

$$\|\varphi_i - \varphi_{i,N}\| = \|(P - P_N)\varphi_i\| = \|(P - P_N)P\varphi_i\| \to 0 \quad \text{as } N \to \infty.$$
Since \( \{\varphi_i\}_{i=1}^m \) is a basis of \( E \), this proves that for sufficiently large \( N \), \( \varphi_{i,N} \), \( i = 1, \ldots, m \), are linearly independent and using (5.4) we conclude that \( \{\varphi_{i,N}\}_{i=1}^m \) is a basis of \( E_N \). Hence (a) is proved. The proof of (b) follows easily from the fact that \( E \) and \( E_N \) are finite-dimensional.

Now we consider the operators \( A \) and \( A_N : E \to E \) defined by
\[
Au = Fu \quad \text{for } u \in E,
\]
and
\[
A_Nu = B_N^{-1}F_NB_Nu \quad \text{for } u \in E.
\]
The operators \( A \) and \( A_N \) are well defined because \( E \) and \( E_N \) are invariant under \( F \) and \( F_N \), respectively. Here \( \lambda \) is the only eigenvalue of \( A \) and the eigenvalues of \( A_N \) are those of \( F_N \) in \( B(\lambda, \Gamma) \).

**Proposition 5.3.** Assume that (5.2)–(5.4) are valid. Then for sufficiently large \( N \) we have
\[
\|A - A_N\|_E \leq \|F - F_N\|_E.
\]

**Proof.** Since \( P_N \) and \( F_N \) commute we have for \( u \in E \),
\[
(A - A_N)u = Fu - B_N^{-1}F_NP_Nu = Fu - B_N^{-1}P_NF_Nu.
\]
Observe that there is no guarantee that \( F_Nu \in E \) and consequently \( B_N^{-1}P_N \) cannot be replaced by the identity operator in this last relation. However, we have
\[
(A - A_N)u = Fu - B_N^{-1}P_NF_Nu = B_N^{-1}(B_NFu - P_NF_Nu) = B_N^{-1}P_N(F - F_N)u.
\]
Now using Proposition 5.2(b) together with the fact that \( P_N \) is uniformly bounded (because of (5.2)) we obtain for \( u \in E \) and for sufficiently large \( N \),
\[
\|(A - A_N)u\| \leq C\|(F - F_N)u\|,
\]
and this gives the desired result.

**Proof of Theorem 5.2.** We have by Proposition 5.3
\[
\|A - A_N\| \leq C \sup_{u \in E} \{\|(F - F_N)u\|, \|u\| = 1\}.
\]
Since \( E \) is finite-dimensional, \( A \) and \( A_N \) as operators on \( E \) can be represented by matrices with \( \lambda \) and \( \lambda_N \) as eigenvalues. Hence by an error estimate for eigenvalues of matrices given in, e.g., Wilkinson [20],
\[
\max_{\lambda \text{ or } \lambda_N \text{ in } u_N} |\lambda - \lambda_N| = \|A - A_N\| \leq C\|(F - F_N)u\|, \quad u \in E \text{ and } \|u\| = 1,
\]
and the proof is complete.

Let us return to our special case with the operators \( F \) and \( F_N \) replaced by \( T \) and \( T_N \), respectively, and \( m = 1 \). As stated in the beginning of this section, to derive the sharpest estimate for the convergence of eigenvalues we now take \( X = L^1(\Omega) \). For our problem, condition (5.2) follows from the fact that \( T_N \) converges pointwise to \( T \) and (5.3) is a result of Lemma 3.2. Below we prove Theorem 5.1 by verifying (5.4) for our operators \( T \) and \( T_N \) with \( m = 1 \). Condition (5.4) may be proved directly if \( \|F - F_N\| \to 0 \), as \( N \to \infty \), but this is not necessarily true in our case. However, we will prove that \( \|T^3 - T_N^3\| \to 0 \), as \( N \to \infty \). Using this we prove (5.4) with \( m = 1 \) first for \( T^3 \) and \( T_N^3 \) and then for \( T \) and \( T_N \).

**Lemma 5.1.** We have
\[
\|T^3 - T_N^3\|_p \to 0 \quad \text{as } N \to \infty, \quad 1 \leq p < \infty.
\]
Proof. We have
\[ \|T^3 - T^3_N\|_p = \|T^2(T - T_N) + T(T - T_N)T_N + (T - T_N)T^2_N\|_p. \]
Since \(T\) and \(T^*\) are compact we see that
\[ \|T^2(T - T_N)\|_p \to 0 \quad \text{as} \quad N \to \infty \]
and
\[ \|T(T - T_N)T_N\|_p \to 0 \quad \text{as} \quad N \to \infty. \]
Thus Lemma 3.4 gives the desired result.

Below, for \(0 < \lambda \in \sigma(T)\), we let \(\hat{\Gamma}\) denote a circle in the complex plane centered at \(\lambda^3\), separating \(\lambda^3\) from \(\sigma(T^3)\setminus\{\lambda^3\}\) and set \(\hat{\sigma}_N = \sigma(T^3) \cap \text{Int } B(\lambda^3, \hat{\Gamma}).\)

**Lemma 5.2.** For sufficiently large \(N\)
\[ \dim E(\lambda^3, T^3) = \dim E(\hat{\sigma}_N, T^3_N). \]

**Proof.** Let \(K = T^3, \quad K_N = T^3_N\) and for \(0 < t \leq 1\) define
\[ K_{N,t} = (1-t)K + tK_N; \]
note that \(K_{N,0} = K, \quad K_{N,1} = K_N, \quad \text{and} \quad \|K_{N,t}\| \leq C.\)
We denote by \(\sigma_{N,t}\) the part of the spectrum of \(K_{N,t}, \sigma(K_{N,t}),\) contained in the interior of \(B(\lambda^3, \hat{\Gamma}).\) The projection operator
\[ P(\sigma_{N,t}, K_{N,t}) = \frac{1}{2\pi i} \int_{\hat{\Gamma}} (z - K_{N,t})^{-1} \, dz \]
is well defined and using Lemma 5.1 we can show that it is a continuous function of \(t\) (see [3, Thm. 3a]). Further by a result of Riesz and Nagy [15, p. 268] we have the following: If the difference of two projections is of norm less than 1, then their ranges are of equal dimensions. (In the estimate of the norm of the difference of \(P(\sigma_{N,t}, K_{N,t})\) and \(P(\sigma_{N,t}, K_{N,t}), |s-t|\) and the length of \(\hat{\Gamma}\) are involved in the constant on the right-hand side. This constant, because of the length of \(\hat{\Gamma},\) can be made less than one.) Hence for sufficiently large \(N,\)
\[ \dim E(\lambda^3, T^3) = \dim KX = \dim K_{N,0}X = \dim K_{N,1}X = \dim K_NX = \dim E(\hat{\sigma}_N, T^3_N). \]

Now we are prepared to complete the proof of our eigenvalue estimate.

**Proof of Theorem 5.1.** By the spectral mapping theorem \(\lambda \in \sigma(T)\) if and only if \(\lambda^3 \in \sigma(T^3).\) Now we use the equality
\[ T^3 - \lambda^3 = (T - \lambda)(T - \lambda e^{(2\pi/3)i})(T - \lambda e^{(4\pi/3)i}), \]
to obtain the decomposition
\[ (5.6) \quad E(\lambda^3, T^3) = E(\lambda, T) \oplus E(\lambda e^{(2\pi/3)i}, T) \oplus E(\lambda e^{(4\pi/3)i}, T) \]
(see, e.g., [17, Thm. 5.9-D]). In our case \(T\) as an operator on \(L_1(\Omega)\) has only real eigenvalues. To see this, let \(v \in L_1(\Omega)\) be an eigenfunction of \(T\) corresponding to the eigenvalue \(\lambda,\) so that
\[ (5.7) \quad Tv = \lambda v. \]
Now arguing as in the proof of Lemma 4.3, \(T^2 : L_1(\Omega) \to L_2(\Omega),\) and by (5.7), \(T^2v = \lambda^2 v;\) thus \(v \in L_2(\Omega),\) and hence (5.7) implies that \(v\) is also an eigenfunction of \(T\) as an operator on \(L_2(\Omega)\) with the same eigenvalue \(\lambda\) and since \(T\) is self-adjoint on \(L_2(\Omega),\) we have \(\lambda \in \mathbb{R}.)\) Thus the two last eigenspaces in (5.6) are empty and
\[ (5.8) \quad \dim E(\lambda, T) = \dim E(\lambda^3, T^3). \]
Since \( \text{dim} E(\lambda, T) = 1 \), Lemma 5.2 together with (5.8) imply that \( \tilde{\sigma}_N \) is a real number. This follows since because of the structure of \( T_N \) (real positive weights), if \( \tilde{\sigma} \in \sigma(T_N^3) \) then \( \tilde{\sigma}_N \in \sigma(T_N^3) \), so that if \( \tilde{\sigma}_N \neq \tilde{\sigma}_N \), \( \text{dim} E(\tilde{\sigma}_N, T_N^3) \leq 2 \), which contradicts Lemma 5.2 and (5.8). The analogue of (5.6) for \( \tilde{\sigma}_N \) and \( T_N^3 \) is

\[
(5.9) \quad E(\tilde{\sigma}_N, T_N^3) = E(\sqrt{\tilde{\sigma}_N}, T_N^3) \oplus E(\sqrt{\tilde{\sigma}_N} e^{(2\pi/3)i}, T_N^3) \oplus E(\sqrt{\tilde{\sigma}_N} e^{(4\pi/3)i}, T_N^3).
\]

Now if one of the last two eigenspaces in the right-hand side of (5.9) is nonempty, say \( E(\sqrt{\tilde{\sigma}_N} e^{(4\pi/3)i}, T_N^3) \neq \emptyset \), then \( \sqrt{\tilde{\sigma}_N} e^{(4\pi/3)i} \in \sigma(T_N^3) \) and again because of the structure of \( T_N, \sqrt{\tilde{\sigma}_N} e^{(4\pi/3)i} \in \sigma(T_N) \); hence \( E(\sqrt{\tilde{\sigma}_N} e^{(2\pi/3)i}, T_N^3) \neq \emptyset \). Consequently \( \text{dim} E(\tilde{\sigma}_N, T_N^3) \leq 2 \), and this is again a contradiction. We conclude that

\[
\text{dim} E(\sqrt{\tilde{\sigma}_N}, T_N^3) = \text{dim} E(\tilde{\sigma}_N, T_N^3).
\]

This completes the proof of (5.4) for our case \( m = 1 \). Now Theorem 3.1 gives the desired result since for the normalized eigenfunction \( g \in W_2^2(\Omega) \) corresponding to the eigenvalue \( \lambda \), (3.2) implies that

\[
|\lambda - \lambda_N| \leq \|T - T_N\|_{L^1(\Omega)} \equiv CN^{-2+\theta}.
\]

6. Numerical results. In order to determine the rate of convergence for the discrete ordinates method in some concrete cases and also test the efficiency of the postprocessing procedure, we have performed some numerical computations on the following two-dimensional neutron transport equation:

\[
\mu \cdot \nabla u(x, \mu) + u(x, \mu) = \lambda U(x) + f(x), \quad x \in \Omega := I^2,
\]

\[
u(x, \mu) = 0, \quad x \in \Gamma_M = \{x \in \Gamma = \partial \Omega; \mu \cdot n(x) < 0\},
\]

where \( I^2 = [0, 1] \times [0, 1] \), \( \mu \in S = \{\mu \in \mathbb{R}^2; |\mu| = 1\} \), and \( n(x) \) is the outward unit normal to \( \Gamma \) at \( x \in \Gamma \).

This problem is equivalent to the following integral equation for the scalar flux \( U \) (see (1.7)):

\[
(I - \lambda T) U = T f.
\]

The discrete ordinates method gives the following semidiscrete analogue of (6.2) (see (2.4)):

\[
(I - \lambda T_N) U_N = T_N f,
\]

where \( N \) is the number of discrete directions on \( S \).

We compute \( U_N \) using the iteration below:

\[
U^{(m+1)}_N = T_N(\lambda U^{(m)}_N + f), \quad m = 0, 1, 2, \ldots,
\]

\[
U^{(0)}_N = 0.
\]

The iterations are continued until \( m + 1 = L \), where

\[
\|U^{(L)}_N - U^{(L-1)}_N\|_{L^2(\Omega)} \leq 10^{-6},
\]

and \( U_N \) is defined to be \( U^{(L)}_N \).

A postprocessed solution \( U^*_N \) is also computed (see (4.5)-(4.6)), i.e.,

\[
U^*_N = T_M(\lambda U^{(2)}_M + f),
\]

\[
U^{(k+1)}_M = T_M(\lambda U^{(k)}_M + f), \quad k = 0, 1,
\]

where \( U^{(0)}_M = U_N \) is computed as in (6.4)-(6.5) and \( M = N^2 \).

Remark 6.1. In the computations we also discretize in the space variable using the discontinuous Galerkin finite element method with mesh parameter \( h = 1/20, 1/50 \) and uniform triangulations of \( I^2 \) (see [2]).
6.1. Data. We perform the iteration (6.4)-(6.5) with $N = 6, 9, 12, 18, 36$ and the postprocessing procedure (6.6) with $N = 6$ and $M = 36$ for the following data. Let $f^* := f_D$ be the characteristic function of the domain $D \subset I^2$. We consider the following cases:

(D1) $\lambda = 0.2$ and $D = \{(x_1, x_2): (x_1 - 3/4)^2 + (x_2 - 3/4)^2 \leq (0.2)^2\}$,
(D2) $\lambda = 0.2$ and $D = \{(x_1, x_2): (x_1 - 1/2)^2 + (x_2 - 1/2)^2 \leq (0.2)^2\}$,
(D3) $\lambda = 0.3$ and $D = \{(x_1, x_2): (x_1 - 1/2)^2 + (x_2 - 1/2)^2 \leq (0.3)^2\}$.

Each of the cases (D1)-(D3) is solved twice, once with $h = 1/20$ and the second time with $h = 1/50$.

6.2. Results. In Tables 1-6 we compare $U := U_{36}$ with $U_N$ and $U_6^*$ where $N = 6, 9, 12, 18$. Below $e_N := U - U_N$, $e^* := U - U_6^*$ and $L$ denotes the number of iterations in (6.4) for $N = 36$.

### Table 1
**Case D1, with $h = 1/20$; $L = 16$.**

<table>
<thead>
<tr>
<th>Error</th>
<th>Norm</th>
<th>$L_1$</th>
<th>$L_2$</th>
<th>$L_{\infty}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U$</td>
<td>0.494163</td>
<td>0.611624</td>
<td>1.665514</td>
<td></td>
</tr>
<tr>
<td>$e_6$</td>
<td>0.066632</td>
<td>0.079378</td>
<td>0.245124</td>
<td></td>
</tr>
<tr>
<td>$e_9$</td>
<td>0.028889</td>
<td>0.035967</td>
<td>0.101383</td>
<td></td>
</tr>
<tr>
<td>$e_{12}$</td>
<td>0.014218</td>
<td>0.017859</td>
<td>0.064952</td>
<td></td>
</tr>
<tr>
<td>$e_{18}$</td>
<td>0.006515</td>
<td>0.008202</td>
<td>0.025387</td>
<td></td>
</tr>
<tr>
<td>$e^*$</td>
<td>0.009415</td>
<td>0.011435</td>
<td>0.027280</td>
<td></td>
</tr>
</tbody>
</table>

### Table 2
**Case D2, with $h = 1/20$; $L = 17$.**

<table>
<thead>
<tr>
<th>Error</th>
<th>Norm</th>
<th>$L_1$</th>
<th>$L_2$</th>
<th>$L_{\infty}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U$</td>
<td>0.601861</td>
<td>0.694524</td>
<td>1.736332</td>
<td></td>
</tr>
<tr>
<td>$e_6$</td>
<td>0.065268</td>
<td>0.079722</td>
<td>0.306480</td>
<td></td>
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<tr>
<td>$e_9$</td>
<td>0.026942</td>
<td>0.034496</td>
<td>0.114590</td>
<td></td>
</tr>
<tr>
<td>$e_{12}$</td>
<td>0.019370</td>
<td>0.025200</td>
<td>0.084097</td>
<td></td>
</tr>
<tr>
<td>$e_{18}$</td>
<td>0.007661</td>
<td>0.009823</td>
<td>0.035274</td>
<td></td>
</tr>
<tr>
<td>$e^*$</td>
<td>0.009452</td>
<td>0.011680</td>
<td>0.030788</td>
<td></td>
</tr>
</tbody>
</table>

### Table 3
**Case D3, with $h = 1/20$; $L = 35$.**

<table>
<thead>
<tr>
<th>Error</th>
<th>Norm</th>
<th>$L_1$</th>
<th>$L_2$</th>
<th>$L_{\infty}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U$</td>
<td>2.454003</td>
<td>2.613289</td>
<td>4.522570</td>
<td></td>
</tr>
<tr>
<td>$e_6$</td>
<td>0.095097</td>
<td>0.115746</td>
<td>0.428698</td>
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<tr>
<td>$e_9$</td>
<td>0.049798</td>
<td>0.062952</td>
<td>0.217018</td>
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<tr>
<td>$e_{12}$</td>
<td>0.038930</td>
<td>0.050942</td>
<td>0.190969</td>
<td></td>
</tr>
<tr>
<td>$e_{18}$</td>
<td>0.015510</td>
<td>0.019816</td>
<td>0.063142</td>
<td></td>
</tr>
<tr>
<td>$e^*$</td>
<td>0.054172</td>
<td>0.066849</td>
<td>0.153723</td>
<td></td>
</tr>
</tbody>
</table>
6.3. Conclusion. By Theorem 3.1, for the discrete ordinates method, we expect the convergence rates $1/N^\alpha$, where $\alpha \approx 2$ for the $L_1$ estimate and $\alpha = 1$ in the $L_\infty$ case. Tables 1–6 show the convergence rates $1/N^\alpha$ with $\alpha \approx 1.6–2.2$ and $\alpha \approx 1.3–1.8$ for $L_1$ and $L_\infty$, respectively. The difference between the theory and computations may depend on the choice of $U_{36}$, which is also discretized in space, as the “exact” scalar flux $U$. Observe that in (D1)–(D3), $f \in H^{1/2}(\Omega)$. Hence by (6.2) the exact scalar flux $U$ has the required regularity in Theorem 3.1 $U \in W^{2-\delta, \psi}(\Omega)$, $\delta > 0$ or $U \in W_p^1(\Omega)$, $1 \leq p < \infty$.

As for the postprocessing procedure we see that the $L_\infty$ errors for $e^*$ are considerably less than those for $e_6$, in particular if $f$ has small support. Further we have $\|e^*\|_\infty \approx \|e_6\|_1$ (cf. Theorem 4.1).

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