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ANALYSIS OF A FULLY DISCRETE SCHEME FOR NEUTRON TRANSPORT IN TWO-DIMENSIONAL GEOMETRY*

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Abstract. We derive error estimates for a fully discrete scheme for the numerical solution of the neutron transport equation in two-dimensional Cartesian geometry obtained by using a special quadrature rule for the angular variable and the discontinuous Galerkin finite element method with piecewise linear trial function for the space variable.

Key words. fully discrete scheme, neutron transport, scalar flux, quadrature rule, finite element method, error estimate

AMS(MOS) subject classifications. Primary 65N15, 65N30

Introduction. The purpose of this paper is to establish error estimates for a fully discrete method for the numerical solution of the neutron transport equation in two-dimensional Cartesian geometry. We use a special quadrature rule for the angular variable, where integration is replaced by a numerical quadrature rule involving a weighted sum over functional values at selected directions, and we apply the discontinuous Galerkin finite element method for the spatial variable.

The stationary one-velocity processes of neutron transport in a substance surrounded by vacuum can be represented by the following integro-differential equation: Given the distributed source density f and the coefficients α and σ, find $u=u(x,\mu)$ such that for $\mu\in S^2$:

$$
\begin{aligned}
&\mu\cdot\nabla u(x,\mu)+\alpha(x)u(x,\mu)=\int_{S^2}\sigma(x,\mu,\mu')u(x,\mu')\,d\mu'+f(x,\mu) &&\text{for } x\in\Omega,\\
&u(x,\mu)=0 &&\text{for } x\in\Gamma_\mu^-,
\end{aligned}
\tag{0.1}
$$

where σ is the transfer kernel describing the distribution of particles arising from scattering, fission and capturing events and α is the total cross-section. Further, Ω is a domain in $\mathbb{R}^3$ with boundary Γ, $S^2=\{\mu\in\mathbb{R}^3: |\mu|=1\}$, $\mu=(\mu_1,\mu_2,\mu_3)$ and

$$\mu\cdot\nabla=\sum_{i=1}^{3}\mu_i\frac{\partial}{\partial x_i},$$

$$\Gamma_\mu^-=\{x\in\Gamma: \mu\cdot n(x)<0\},$$

where $n(x)$ is the outward unit normal to Γ at $x\in\Gamma$, and finally the unknown function $u=u(x,\mu)$ is the density at x of particles moving in the direction μ.

The neutron transport problem was studied analytically by e.g. Davison [3] and Vladimirov [14]. Numerical treatments of the problem have been restricted to separate error estimates for either angular or spatial discretizations. Angular discretizations for a two-dimensional problem were studied by e.g. Nelson and Victory [9] using the Nyström discrete ordinates method. Spatial discretizations in a two-dimensional case for one single direction were studied by e.g. Lesaint and Raviart [6] using the discontinuous Galerkin finite element method.

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An analysis of a fully discrete scheme with combined spatial and angular discretizations for slab geometry (one-dimensional spatial and angular variables) was recently done by Pitkäranta and Scott [11]. In a two-dimensional case, with the angular variable varying on the unit circle i.e., a case where $x \in \Omega \subset \mathbb{R}^2$ and $\mu \in S = \{\mu \in \mathbb{R}^2: |\mu| = 1\}$, a fully discrete scheme for a model problem was analyzed by Johnson and Pitkäranta [5].

In the present text we extend the analysis of [5] to the case of a homogeneous infinite cylindrical domain in $\mathbb{R}^3$. Because of the translational invariance, the relevant spatial domain is again two-dimensional, namely the cross-section of the cylinder; however, the relevant angular domain is now the unit disc in $\mathbb{R}^2$, i.e., the projection of the surface of the unit sphere in $\mathbb{R}^3$ onto the plane of the cross-section of the cylinder. Thus, in the model problem to be considered we will have $x \in \Omega \subset \mathbb{R}^2$ and $\mu \in D = \{\mu \in \mathbb{R}^2: |\mu| \leqq 1\}$.

The plan of the paper is as follows: In § 1 we present the model problem and show that this problem can also be formulated as a Fredholm integral equation of the second kind for the scalar flux. In § 2 we introduce notations, assumptions and some previous results which will be used frequently. In § 3 we study a quadrature approximation of weighted integrals for a class of functions relevant to our purpose and derive some quadrature error estimates. Section 4 is devoted to a semidiscrete problem with angular discretization. The results of this section are essential tools in the proofs of error estimates. In § 5 we study the fully discrete scheme and state our main result: Theorem 5.1. Our concluding § 6 contains error estimates obtained by applying results of the previous sections.

1. A model problem. We shall consider the following model problem: Given a distributed source density f and a real constant λ, find the flux $u(x, \mu)$ such that

$$\tag{1.1} \begin{aligned} &\mu \cdot \nabla u(x,\mu) + u(x,\mu) = \lambda \int_D u(x,\mu')(1-|\mu'|^2)^{-1/2}\, d\mu' + f(x), \quad (x,\mu) \in \Omega \times D, \\ &u(x,\mu) = 0 \qquad \text{on } \Gamma_\mu^-, \end{aligned}$$

where Ω is a bounded convex domain in $\mathbb{R}^2$ with polygonal boundary Γ and

$$D = \{\mu \in \mathbb{R}^2: |\mu| \leqq 1\}, \quad \mu = (\mu_1, \mu_2), \quad \Gamma_\mu^- = \{x \in \Gamma: \mu \cdot n(x) < 0\},$$

where $n(x)$ is the outward unit normal to Γ at $x \in \Gamma$. This problem is obtained from problem (0.1) assuming that Ω in (0.1) is a cylinder (cf. Remark 1.1 and the proof of Lemma 1.1 below). Observe that our model problem corresponds to a problem with isotropic source and a homogeneous medium with isotropic scattering.

Let us reformulate problem (1.1) using the following notation. For $\mu \in D$, $\mu \neq 0$, let T_μ be the solution operator for the problem: Given $g \in L_2(\Omega)$ find u such that

$$\tag{1.2} \begin{aligned} &\mu \cdot \nabla u + u = g \quad \text{in } \Omega, \\ &u = 0 \qquad \text{on } \Gamma_\mu^-, \end{aligned}$$

i.e., $u = T_\mu g$ if u satisfies (1.2). By a simple calculation we find that

$$\tag{1.3} T_\mu g(x) = \int_0^{d(x,\mu)/|\mu|} e^{-s} g(x - s\mu)\, ds,$$

with $d = d(x, \mu)$ denoting the distance of x from the exterior of Ω in the direction $-\mu$,

$$d(x,\mu) = \inf\{s > 0: (x - s\mu/|\mu|) \notin \Omega\}.$$

If g is constant on each vertical line, then the representation (1.3) is valid also for cylindrical domains of the form $C_{\mathbb{R}}=\Omega\times\mathbb{R}$.

Introducing now the *scalar flux* U defined by

$$U(x)=\int_D u(x,\mu)(1-|\mu|^2)^{-1/2}\,d\mu,$$

and letting $g=\lambda U+f$, we have by (1.3)

$$u(x,\mu)=T_\mu(\lambda U+f)(x),\qquad (x,\mu)\in\Omega\times D. \tag{1.4}$$

Multiplying (1.4) by the weight $(1-|\mu|^2)^{-1/2}$ and integrating over D we obtain the following integral equation for the scalar flux U:

$$(I-\lambda T)U=Tf, \tag{1.5}$$

where

$$T=\int_D (1-|\mu|^2)^{-1/2}T_\mu\,d\mu. \tag{1.6}$$

Remark 1.1. The presence of the weight $(1-|\mu|^2)^{-1/2}$ is a consequence of the geometry. Consider the volume element in the spherical coordinate system

$$dV=s^2\sin\theta\,ds\,d\theta\,d\varphi,\qquad 0\leqq s<\infty,\ 0\leqq\theta\leqq\pi \text{ and } 0\leqq\varphi\leqq 2\pi.$$

Let $\mu\in D$ be the projection of $\tilde\mu\in S^2$, then $|\mu|=\sin\theta$. This implies that $\sin\theta\,d\theta=(1-|\mu|^2)^{-1/2}|\mu|\,d|\mu|$. And since $d\mu=d\mu_1 d\mu_2=|\mu|\,d|\mu|\,d\varphi$ we have

$$dV=(1-|\mu|^2)^{-1/2}s^2\,ds\,d\mu.$$

LEMMA 1.1. *The integral operator $T=\int_D(1-|\mu|^2)^{-1/2}T_\mu\,d\mu$ is compact on $L_2(\Omega)$.*

Proof. Using (1.3) and Remark 1.1, we get

$$\begin{aligned} Tg(x)&=\int_D(1-|\mu|^2)^{-1/2}T_\mu g(x)\,d\mu=\int_D\int_0^{d(x,\mu)/|\mu|}(1-|\mu|^2)^{-1/2}\,e^{-s}g(x-s\mu)\,ds\,d\mu\\ &=\int_D\int_0^{d(x,\mu)/|\mu|}\frac{e^{-s}}{s^2}\,g(x-s\mu)\,dV.\end{aligned} \tag{1.7}$$

Let now $\tilde g$ be an extension of g to $C_R=\Omega\times\mathbb{R}$,

$$\tilde g(x,z)=g(x),\quad x\in\Omega,\quad z\in\mathbb{R},$$

and let $\tilde S$ be the upper half of the unit sphere

$$\tilde S=S^2\cap\{(x,z)\in\mathbb{R}^2\times\mathbb{R}:\ z\geqq 0\}.$$

We associate with each $\tilde\mu\in\tilde S$ its orthogonal projection $\mu\in D$: $\tilde\mu\in\tilde S\leftrightarrow D\ni\mu$ and define the distance function $\tilde d=\tilde d(\tilde x,\tilde\mu)$ for $(\tilde x,\tilde\mu)\equiv(x,z,\tilde\mu)\in\Omega\times\mathbb{R}^+\times\tilde S$,

$$\tilde d(\tilde x,\tilde\mu)=\inf\{s>0:\ \tilde x-s\tilde\mu\notin C_{\mathbb{R}}\}.$$

Identifying $x\in\Omega$ and $(x,0)\in C_{\mathbb{R}}$, by an easy argument we get

$$\tilde d(\tilde x,\tilde\mu)=\tilde d(x,\tilde\mu)=d(x,\mu)/|\mu|.$$

Let $\tilde T_{\tilde\mu}$ and $\tilde T$ be the following integral operators:

$$\tilde T_{\tilde\mu}\tilde g(\tilde x)=\int_0^{\tilde d(\tilde x,\tilde\mu)}e^{-s}\tilde g(\tilde x-s\tilde\mu)\,ds,\qquad \tilde T=\int_{\tilde S}\tilde T_{\tilde\mu}\,d\tilde\mu.$$

By the z-independence of $\tilde{g}$ and a geometrical interpretation we have

$$\begin{aligned}
Tg(x) = \tilde{T}\tilde{g}(\tilde{x}) &= \int_{\tilde{\mu}\in\tilde{S}} \int_0^{\tilde{d}(\tilde{x},\tilde{\mu})} \frac{e^{-s}}{s^2} \tilde{g}(\tilde{x}-s\tilde{\mu}) s^2 \, ds \, d\tilde{\mu} \\
&= \int_{C_{\mathbb{R}^+}} \frac{e^{-|\tilde{x}-\tilde{y}|}}{|\tilde{x}-\tilde{y}|} \tilde{g}(\tilde{y}) \, d\tilde{y} = \int_\Omega \left[\int_0^\infty \frac{e^{-|(x-y,z)|}}{|(x-y,z)|^2} dz\right] g(y) \, dy \\
&= \frac{1}{2}\int_\Omega \left[\int_{-\infty}^\infty \frac{e^{-|(x-y,z)|}}{|(x-y,z)|^2} dz\right] g(y) \, dy,
\end{aligned}$$

where $C_{\mathbb{R}^+} = \Omega \times \mathbb{R}^+$ and $\tilde{y} = \tilde{x} - s\tilde{\mu}$. Integrating by parts we get

$$\int_{-\infty}^\infty \frac{e^{-|(x-y,z)|}}{|(x-y,z)|^2} dz = \frac{1}{|x-y|} F(|x-y|)$$

with

$$F(|x-y|) = \int_{-\infty}^\infty \frac{z\, e^{-(|x-y|^2+z^2)^{1/2}}}{(|x-y|^2+z^2)^{1/2}} \arctan \frac{z}{|x-y|} \, dz.$$

Making the substitution of variable $t = \arctan z/|x-y|$ and integrating by parts we get, (compare with [10, (1.4)]).

$$F(|x-y|) = \int_{-\pi/2}^{\pi/2} t\, e^{-|x-y|/\cos t} \frac{|x-y| \sin t}{\cos^2 t} dt = 2\int_0^{\pi/2} e^{-|x-y|/\cos t} \, dt. \tag{1.8}$$

Thus

$$Tg(x) = \frac{1}{2}\int_\Omega \frac{F(|x-y|)}{|x-y|} g(y) \, dy, \tag{1.9}$$

with F as in (1.8). Thus, T is an integral operator with weakly singular kernel and as in [8] one can show that T maps $L_2(\Omega)$ into $H^1(\Omega)$. Hence $T: L_2(\Omega) \to L_2(\Omega)$ is compact and the proof is complete. □

COROLLARY 1.1. *The equation* (1.5) *is a Fredholm equation of the second kind.*

Let us assume that the system described by (1.5) is subcritical, that is $0 \leqq \lambda < 1/\Lambda$, where Λ is the largest eigenvalue of T. Then $(I - \lambda T)$ is invertible and $(I - \lambda T)^{-1}: L_2(\Omega) \to L_2(\Omega)$ is a continuous linear mapping. This implies that:

(i) For a given $f \in L_2(\Omega)$, the problem $(I - \lambda T)U = Tf$ has a unique solution.

(ii) There exists a constant $C > 0$ such that

$$\|(I - \lambda T)v\| \geqq C\|v\| \quad \forall v \in L_2(\Omega), \tag{1.10}$$

where $\|\cdot\|$ denotes the $L_2(\Omega)$ norm.

From now on the letter C will denote various constants not necessarily the same at each occurrence.

2. Notation and preliminaries. In this section we define some function spaces, introduce discrete analogues of (1.5) and finally we state some results which we shall refer to below.

For $s \geqq 0$ let $H^s(\Omega)$ be the Sobolev space with the norm

$$\|v\|_s = \left(\sum_{|\alpha|\leqq s} \|D^\alpha v\|^2\right)^{1/2}, \qquad \|v\|_0 = \|v\|_{L_2(\Omega)} = \|v\|$$

and with corresponding seminorm

$$|v|_s = \left(\sum_{|\alpha|=s} \|D^\alpha v\|^2 \right)^{1/2}.$$

Further, for $s, r \geqq 0$ we define the space $H^{r,s}(\mathbb{R}^2)$ as in [7] with norm

$$\|v\|_{r,s} = \left[\int_{\mathbb{R}} \|v(\cdot, x_2)\|^2_{H^r(\mathbb{R})} \, dx_2 + \int_{\mathbb{R}} \|v(x_1, \cdot)\|^2_{H^s(\mathbb{R})} \, dx_1 \right]^{1/2} \tag{2.1}$$

and corresponding seminorm

$$|v|_{r,s} = \left[\int_{\mathbb{R}} |v(\cdot, x_2)|^2_{H^r(\mathbb{R})} \, dx_2 + \int_{\mathbb{R}} |v(x_1, \cdot)|^2_{H^s(\mathbb{R})} \, dx_1 \right]^{1/2}. \tag{2.2}$$

We shall consider a quadrature rule for weighted integrals of the form

$$\int_D u(\mu)(1-|\mu|^2)^{-1/2} \, d\mu \sim \sum_{\mu \in \Delta} u(\mu)\omega_\mu, \tag{2.3}$$

where $\Delta = \{\mu^1, \cdots, \mu^n\}$ is a discrete set of quadrature points $\mu^i \in D$ with corresponding positive weights ω_μ, $\mu \in \Delta$. The precise choice of quadrature points and weights will be given in § 3 below.

We shall denote by $\{\mathscr{C}_h\}$ a family of quasiuniform triangulations $\mathscr{C}_h = \{K\}$ of Ω indexed by the parameter h, the maximum diameter of triangles $K \in \mathscr{C}_h$. We introduce the finite element space

$$V_h = \{v \in L_2(\Omega) \colon v|_K \text{ is linear}, K \in \mathscr{C}_h\},$$

and a discrete solution operator $T^h_\mu \colon L_2(\Omega) \to V_h$ approximating T_μ which is defined by the following.

Discontinuous Galerkin finite element method for (1.2). Given $\mu \in D$ and $g \in L_2(\Omega)$ find $u^h(\cdot, \mu) = T^h_\mu g$ such that

$$\sum_{K \in \mathscr{C}_h} \left[(\mu \cdot \nabla u^h + u^h, v)_K + \int_{\partial K^-} [u^h] v_+ |\mu \cdot n| \, d\sigma \right] = \int_\Omega g v \, dx, \quad \forall v \in L_2(\Omega), \tag{2.4}$$

where

$$(u, v)_K = \int_K uv \, dx,$$
$$\partial K^- = \{x \in \partial K \colon \mu \cdot n(x) < 0\},$$
$$[v] = v_+ - v_-,$$
$$v_\pm(x) = \lim_{s \to 0\pm} v(x + s\mu), \qquad x \in \partial K,$$

$n = n(x)$ is the outward unit normal to ∂K at $x \in \partial K$ and $u^h_- = 0$ on Γ^-_μ.

We can now state the following discrete analogues of (1.5).

The semidiscrete problem. Find $u_n(x, \mu)$ such that

$$u_n(x, \mu) = T_\mu(\lambda U_n + f)(x), \qquad (x, \mu) \in \Omega \times \Delta, \tag{2.5}$$

where U_n is the quadrature approximation of the scalar flux U,

$$U_n(x) = \sum_{\mu \in \Delta} u_n(x, \mu)\omega_\mu.$$

Multiplying (2.5) by the weight ω_μ and summing over $\mu \in \Delta$ we obtain the following equation: Find $U_n \in L_2(\Omega)$ such that

$$(I - \lambda T_n) U_n = T_n f, \tag{2.6}$$

where $T_n = \sum_{\mu \in \Delta} T_\mu \omega_\mu$.

The fully discrete problem. Find $u_n^h(\cdot, \mu) \in V_h$ such that

$$u_n^h(\cdot, \mu) = T_\mu^h(\lambda U_n^h + f), \qquad \mu \in \Delta, \tag{2.7}$$

where U_n^h is a totally discrete approximation of the scalar flux U,

$$U_n^h = \sum_{\mu \in \Delta} u_n^h(\cdot, \mu)\omega_\mu.$$

Again, multiplying (2.7) by ω_μ and summing over $\mu \in \Delta$, we obtain the following fully discrete analogue of (1.5): Find $U_n^h \in V_h$ such that

$$(I - \lambda T_n^h) U_n^h = T_n^h f, \tag{2.8}$$

where

$$T_n^h = \sum_{\mu \in \Delta} T_\mu^h \omega_\mu.$$

Our main concern will be the fully discrete problem (2.8) and our main results are estimates of the error in scalar flux $U - U_n^h$. To prove these estimates the main step will be to prove that under certain assumptions on the quadrature rule (2.3) and on the relation between n and h, $(I - \lambda T_n^h)^{-1}: L_2(\Omega) \to L_2(\Omega)$ exists and is uniformly bounded.

We shall also use the following propositions, the first of which is due to Anselone [2]. For the second and third propositions we refer to Johnson and Pitkäranta [5] and [4], respectively.

PROPOSITION 2.1. *Let $T: L_2(\Omega) \to L_2(\Omega)$ be a bounded linear operator such that for some positive constant C*

$$\|(I - \lambda T)v\| \geqq C\|v\| \quad \forall v \in L_2(\Omega)$$

and let $\{T_n\}_{n=1}^\infty$ be a uniformly bounded sequence of linear operators on $L_2(\Omega)$ such that for some positive integer m.

$$\varepsilon_n = \|(T - T_n)T_n^m\| \to 0 \quad \text{as } n \to \infty. \tag{2.9}$$

Then there exists a positive constant C_1 such that for n large enough

$$\|(I - \lambda T_n)v\| \geqq C_1\|v\| \quad \forall v \in L_2(\Omega).$$

PROPOSITION 2.2. *There is a constant C such that for $g \in L_2(\Omega)$*

$$\|\mu \cdot \nabla T_\mu g\| + \|T_\mu g\| + \left[\int_\Gamma (T_\mu g)^2 |n \cdot \mu| \, ds\right]^{1/2} \leqq C\|g\|.$$

PROPOSITION 2.3. *Given $g \in L_2(\Omega)$ there is a unique $u^h \equiv T_\mu^h g \in V_h$ satisfying (2.4). Moreover, there is a constant C independent of g, μ, h and Ω such that*

$$|||(T_\mu - T_\mu^h)g|||_\mu \leqq Ch^{1/2}|T_\mu g|_1, \tag{2.10a}$$

$$|||(T_\mu - T_\mu^h)g|||_\mu \leqq Ch^{3/2}|T_\mu g|_2, \tag{2.10b}$$

$$|||T_\mu^h g|||_\mu \leqq C\|g\|, \tag{2.11}$$

where

$$\text{(2.12)}\qquad |||v|||_\mu=\left[\|v\|^2+h\sum_K\|\mu\cdot\nabla v\|_K^2+\sum_K\int_{\partial K}|[v]|^2|\mu\cdot n|\,ds\right]^{1/2},\qquad \|v\|_K=(v,v)_K^{1/2}.$$

3. The quadrature rule. In this section we define the quadrature rule to be used and derive estimates for the quadrature error.

Using polar coordinates $\mu=(r\cos\varphi, r\sin\varphi)$ for the directional variable $\mu\in D$ in the quadrature rule (2.3), we have

$$\text{(3.1)}\qquad \int_0^{2\pi}\int_0^1 u(x,r,\varphi)(1-r^2)^{-1/2}r\,dr\,d\varphi\sim\sum_{\mu\in\Delta}u(x,\mu)\omega_\mu.$$

Let now $\Delta=\{\mu_{kj}\in D\colon \mu_{kj}=r_k(\cos\varphi_j,\sin\varphi_j),\ k=1,\cdots,N \text{ and } j=1,\cdots,M\}$ be the set of quadrature points, where $\varphi_j=2\pi j/M$, $j=1,\cdots,M$ and r_k are the zeros of the orthogonal polynomials associated with the distribution $d\alpha(r)=(1-r^2)^{-1/2}r\,dr$ on the interval $[0,1]$. It is known (see Szegö [13, pp. 121-122]) that

$$\text{(3.2)}\qquad r_k=\sin\theta_k,$$

where θ_k is a certain point in the interval I_k,

$$\text{(3.3)}\qquad \theta_k\in I_k=\left[\frac{(2k-1)\pi}{4N+2},\frac{2k\pi}{4N+2}\right],\qquad k=1,\cdots,N.$$

With each $\mu_{kj}\in\Delta$ we associate the positive weight $\omega_{kj}=A_kW_j$ where $W_j=2\pi/M$ and

$$\text{(3.4)}\qquad A_k=\alpha(s_k)-\alpha(s_{k-1})=\sqrt{1-s_{k-1}^2}-\sqrt{1-s_k^2},\qquad k=1,\cdots,N,$$

where s_k is a certain point in the interval (r_k, r_{k+1}) and $s_0=0$, $s_N=1$ (cf. Szegö [13, pp. 47–50]). This quadrature rule is different from the S_n-rules which are used in practice; however the concrete calculations here are easy to follow. For compatibility reasons we shall assume that $N\sim M$, see [1]. The number of quadrature points in Δ is then

$$\text{(3.5)}\qquad n=MN.$$

Finally, for integer $\alpha\geqq 1$ and for $L_\alpha\geqq 0$ we define the function space $c^{(\alpha)}(L_\alpha;0,1)=\{u\in c^{\alpha-1}([0,1])\colon u^{(\alpha)}$ is piecewise continuous and $|u^{(\alpha)}(r)|\leqq L_\alpha,\ r\in[0,1]\}$. If $u\in c^{(\alpha)}(L_\alpha;0,1)$, then by Taylor's formula we have

$$\text{(3.6)}\qquad u(r)=Q_{\alpha-1}(r)+R_\alpha(r),\ r\in[0,1],$$

where

$$Q_{\alpha-1}(r)=\sum_{l=0}^{\alpha-1}\frac{r^l}{l!}u^{(l)}(0)\quad\text{and}\quad R_\alpha(r)=\frac{1}{(\alpha-1)!}\int_0^r(r-t)^{\alpha-1}u^{(\alpha)}(t)\,dt.$$

Introducing the Peano Kernel function defined by

$$K_\alpha(x)=\begin{cases}\dfrac{1}{(\alpha-1)!}x^{\alpha-1} & \text{for } x\geqq 0,\\ 0 & \text{for } x<0,\end{cases}$$

we have

$$\text{(3.7)}\qquad R_\alpha(r)=\int_0^1 u^{(\alpha)}(t)K_\alpha(r-t)\,dt.$$

We shall use the following result of Stroud and Secrest [12]:

LEMMA 3.1. *Let $u \in c^{(\alpha)}(L_\alpha; 0, 1)$ and w be a weight function and suppose that the quadrature approximation*

$$\int_0^1 u(r)w(r)\,dr \sim \sum_{k=1}^N A_k u(r_k)$$

is exact for all polynomials of degree $\leqq \alpha - 1$. Then

$$\left|\int_0^1 u(r)w(r)\,dr - \sum_{k=1}^N A_k u(r_k)\right| \leqq \int_0^1 |E_\alpha(t)|\,|u^{(\alpha)}(t)|\,dt, \tag{3.8}$$

where

$$E_\alpha(t) = \int_0^1 w(r)K_\alpha(r-t)\,dr - \sum_{k=1}^N A_k K_\alpha(r_k - t).$$

Proof. Since the quadrature approximation is exact for all polynomials of degree $\leqq \alpha - 1$, we have using (3.6), (3.7) and Fubini's Theorem

$$\begin{aligned}
&\int_0^1 u(r)w(r)dr - \sum_{k=1}^N A_k u(r_k)\\
&\quad = \int_0^1 w(r)\int_0^1 u^{(\alpha)}(t)K_\alpha(r-t)\,dt\,dr - \sum_{k=1}^N A_k \int_0^1 u^{(\alpha)}(t)K_\alpha(r_k - t)\,dt\\
&\quad = \int_0^1 u^{(\alpha)}(t)\left[\int_0^1 w(r)K_\alpha(r-t)\,dr - \sum_{k=1}^N A_k K_\alpha(r_k - t)\right]dt\\
&\quad = \int_0^1 E_\alpha(t)u^{(\alpha)}(t)\,dt
\end{aligned}$$

and the proof is complete. □

We shall assume that u is sufficiently smooth. Then by Lemma 3.1 with $w(r) = (1-r^2)^{-1/2}r$ we have

$$\left|\int_0^1 u(x, r, \varphi)(1-r^2)^{-1/2} r\,dr - \sum_{k=1}^N A_k u(x, r_k, \varphi)\right| \leqq \int_0^1 |E_1(t)|\left|\frac{\partial u(x, t, \varphi)}{\partial t}\right| dt, \tag{3.9}$$

where

$$E_1(t) = \int_t^1 (1-r^2)^{-1/2} r\,dr - \sum_{r_k > t} A_k = \sqrt{1-t^2} - \sum_{r_k>t} A_k.$$

Further, it is easy to show that if u is sufficiently smooth, then we have for our choice of uniform quadrature rule,

$$\left|\int_0^{2\pi} u(x, r, \varphi) - \sum_{j=1}^M w_j u(x, r, \varphi_j)\right| \leqq \frac{C}{M}\int_0^{2\pi}\left|\frac{\partial u(x, r, \varphi)}{\partial \varphi}\right| d\varphi. \tag{3.10}$$

LEMMA 3.2. *One has $|E_1(t)| \leqq 3\pi/4N$ for all $t \in (0, 1)$.*

Proof. We assume, without loss of generality, that $r_{l-1} < t < r_l$. Using (3.4) we get

$$E_1(t) = \sqrt{1-t^2} - \sum_{r_k>t} A_k = \sqrt{1-t^2} - \alpha(s_N) + \alpha(s_{l-1}) = \sqrt{1-t^2} - \sqrt{1-s_{l-1}^2}.$$

Since $r_{l-1} < s_{l-1} < r_l$ and $r_{l-1} < t < r_l$ we have

$$|E_1(t)| \leqq \sqrt{1-r_{l-1}^2} - \sqrt{1-r_l^2}$$

and recalling (3.2) and (3.3) we end up with

$$|E_1(t)| \leqq \cos\theta_{l-1} - \cos\theta_l = \int_{\theta_{l-1}}^{\theta_l} \sin\theta\, d\theta \leqq (\theta_l - \theta_{l-1}) \leqq \frac{3\pi}{4N}. \qquad \square$$

4. The semidiscrete problem. Our aim in this section is to establish the stability of the semidiscrete problem (2.6) using Proposition 2.1. For this purpose it suffices to prove (2.9) with $m=2$. Since T_n is not compact, the conclusion of Proposition 2.1:

$$\|(I-\lambda T_n)v\| \geqq C_1\|v\| \quad \forall v \in L_2(\Omega) \tag{4.1}$$

does not (directly) imply the existence of a solution to (2.6). However to prove existence we can argue as in the remark following [5, Lemma 4.1].

We shall use the following splitting of $\sum_{(\mu,\nu)\in\Delta^2}\omega_\mu\omega_\nu$: For $\varepsilon>0$, let

$$\sum_{(\mu,\nu)\in\Delta^2} \omega_\mu\omega_\nu = \sum_\varepsilon{}' \omega_\mu\omega_\nu + \sum_\varepsilon{}'' \omega_\mu\omega_\nu = \sum_{(\mu,\nu)\in I'_\varepsilon} \omega_\mu\omega_\nu + \sum_{(\mu,\nu)\in I''_\varepsilon} \omega_\mu\omega_\nu,$$

$$I'_\varepsilon = \{(\mu,\nu)\in\Delta^2: \min(|\sin(\mu,\nu)|,$$
$$|\sin(\mu,d_k)|, |\sin(\nu,d_k)|, |\mu|, |\nu|) \geqq \varepsilon,\ k=1,\cdots,P\}$$

$$I''_\varepsilon = \{(\mu,\nu)\in\Delta^2: (\mu,\nu)\notin I'_\varepsilon\}.$$

Here $\{d_k\}_{k=1}^P$ are the directions of the sides of Ω. Further we shall assume that $\varepsilon=\varepsilon(n)\leqq \sin\pi/(2N+1)$, in which case the sum

$$\sigma(\varepsilon,n) = \sum_\varepsilon{}''\omega_\mu\omega_\nu \to 0 \quad \text{as } n\to\infty. \tag{4.2}$$

To see this note that I''_ε has in this case at most $M^2 \sim n$ elements, moreover by (3.9) and

$$r_l \leqq s_l \leqq r_{l+1}, \qquad l=1,\cdots,N$$

we have

$$A_l \leqq (1-r_{l-1}^2)^{1/2} - (1-r_{l+1}^2)^{1/2} = \cos\theta_{l-1} - \cos\theta_{l+1}$$
$$= \int_{\theta_{l-1}}^{\theta_{l+1}} \sin\theta\, d\theta \leqq \frac{5\pi}{4N} r_{l+1}, \qquad l=1,\cdots,N.$$

Hence

$$A_l \leqq \frac{5\pi}{4N}, \qquad l=1,\cdots,N, \tag{4.3}$$

and

$$\sum_\varepsilon{}'' \omega_\mu\omega_\nu = \sum_{(\mu_{ij},\nu_{kl})\in I''_\varepsilon} A_i W_j A_k W_l \leqq \frac{C}{M^2N^2} \sum_{(\mu,\nu)\in I''_\varepsilon} 1 = \frac{C}{N^2} = \frac{C}{n}.$$

We now state the main result of this section:

LEMMA 4.1. $\|(T-T_n)T_n^2\| \to 0$ *as* $n\to\infty$.

To prove Lemma 4.1, we need the following two lemmas.

LEMMA 4.2. *There exists a constant C such that for* $(\mu,\nu)\in I'_\varepsilon$ *and* $g\in L_2(\Omega)$,

$$\|T_\mu T_\nu g\|_1 \leqq C\varepsilon^{-7/2}\|g\|.$$

LEMMA 4.3. *There exists a constant C such that for $g \in H^1(\Omega)$,*

$$\|(T-T_n)g\| \leqq C\left(\frac{1}{N}+\frac{1}{M}\right)\|g\|_1.$$

Once the Lemmas 4.2 and 4.3 have been proved, we can prove Lemma 4.1.

Proof of Lemma 4.1. Applying Lemmas 4.2, 4.3 and Proposition 2.2 we have

$$\begin{aligned}
\|(T-T_n)T_n^2 g\| &= \|(T-T_n)\sum_{(\mu,\nu)\in\Delta^2} \omega_\mu\omega_\nu T_\mu T_\nu g\| \\
&\leqq \left\|(T-T_n)\sum_\varepsilon{}' \omega_\mu\omega_\nu T_\mu T_\nu g\right\| + \left\|\sum_\varepsilon{}'' \omega_\mu\omega_\nu (T-T_n) T_\mu T_\nu g\right\| \\
&\leqq C\sum_\varepsilon{}' \omega_\mu\omega_\nu\left(\frac{1}{N}+\frac{1}{M}\right)\|T_\mu T_\nu g\|_1 + C\|g\|\sum_\varepsilon{}'' \omega_\mu\omega_\nu \\
&\leqq C\left[\varepsilon^{-7/2}\left(\frac{1}{N}+\frac{1}{M}\right)+\sigma(\varepsilon, n)\right]\|g\|.
\end{aligned}$$

Choosing now e.g. $\varepsilon = N^{-1/7}$ and using (4.2), we obtain the assertion of Lemma 4.1. □

Proof of Lemma 4.2. By an orthogonal coordinate transformation we may assume that $\mu = (|\mu|, 0)$. If $(\mu, \nu) \in I'_\varepsilon$, then by Proposition 2.2,

$$\begin{aligned}
\|\nabla(T_\mu T_\nu g)\| &\leqq \frac{C}{\varepsilon^2}\left[\left\|\frac{\partial}{\partial\mu}(T_\mu T_\nu g)\right\| + \left\|\frac{\partial}{\partial\nu}(T_\mu T_\nu g)\right\|\right] \\
&\leqq \frac{C}{\varepsilon^2}\left[\|g\| + \left\|\frac{\partial}{\partial\nu}(T_\mu T_\nu g)\right\|\right],
\end{aligned} \tag{4.4}$$

where $\partial/\partial\mu = \mu\cdot\nabla$. Recalling (1.3) we have

$$T_\mu T_\nu g(x) = \int_0^{d(x,\mu)/|\mu|} e^{-s} T_\nu g(x - s\mu)\, ds.$$

Thus

$$\begin{aligned}
\frac{\partial}{\partial\nu}(T_\mu T_\nu g(x)) &= e^{-d/|\mu|} T_\nu g(\bar{x})\frac{\partial}{\partial\nu}\left(\frac{d}{|\mu|}\right) \\
&\quad + \int_0^{d(x,\mu)/|\mu|} e^{-s}\frac{\partial}{\partial\nu}(T_\nu g(x-s\mu))\, ds,
\end{aligned} \tag{4.5}$$

where $\bar{x} = x - (d(x, \mu)/|\mu|)\mu$. By an easy calculation we have

$$\frac{\partial}{\partial\nu}\left(\frac{d}{|\mu|}\right) = \frac{1}{|\mu|}\frac{\partial d}{\partial\nu} = \frac{1}{|\mu|}|\mu|\frac{\nu\cdot n}{\mu\cdot n} = \frac{\nu\cdot n}{\mu\cdot n},$$

where $n = (n_1, n_2)$ is the outward unit normal to Γ at $\bar{x}$. Squaring (4.5) and integrating over Ω, using the facts that $dx_2 = \mu\cdot n/|\mu|\, ds$ on Γ, $|\mu\cdot n| \geqq \varepsilon^2$ and Proposition 2.2 we obtain:

$$\begin{aligned}
\left\|\frac{\partial}{\partial\nu}(T_\mu T_\nu g(x))\right\|^2 &\leqq C\int_\Gamma |T_\nu g|^2\left|\frac{\nu\cdot n}{\mu\cdot n}\right|^2\frac{\mu\cdot n}{|\mu|}\, ds + C\|g\|^2 \\
&\leqq \frac{C}{\varepsilon^3}\int_\Gamma |T_\nu g|^2|\nu\cdot n|\, ds + C\|g\|^2 \leqq \frac{C}{\varepsilon^3}\|g\|^2.
\end{aligned}$$

Thus by (4.4)

$$\|\nabla(T_\mu T_\nu g)\| \leqq \frac{C}{\varepsilon^2}\left(\|g\| + \frac{C}{\varepsilon^{3/2}}\|g\|\right) \leqq C\varepsilon^{-7/2}\|g\|.$$

Since by Proposition 2.2 $\|T_\mu T_\nu g\| \leqq C\|g\|$. This proves the desired result. □

For the proof of Lemma 4.3 we need the following result.

LEMMA 4.4. *There is a constant C such that if $u(x, r, \varphi) = u(x, \mu) = T_\mu g(x)$, where $g \in H^1(\Omega)$ and $\mu = (r\cos\varphi, r\sin\varphi)$, then*

$$\sum_{k=1}^{N} A_k \int_0^{2\pi} \left\|\frac{\partial u(\cdot, r_k, \varphi)}{\partial\varphi}\right\| d\varphi \leqq C\|g\|_1 \tag{4.6}$$

and

$$\int_0^{2\pi}\int_0^1 \left\|\frac{\partial u(\cdot, r, \varphi)}{\partial r}\right\| dr\, d\varphi \leqq C\|g\|_1. \tag{4.7}$$

Proof. Recall that

$$u(x, r, \varphi) = \int_0^{d(x,r,\varphi)/r} e^{-s} g(x - s\mu)\, ds,$$

where $d(x, r, \varphi) \equiv d(x, \mu)$. Thus

$$\begin{aligned}\frac{\partial u}{\partial\varphi} &= \frac{1}{r}e^{-d/r} g\left(x - d\frac{\mu}{r}\right)\frac{\partial d}{\partial\varphi} + \int_0^{d/r} e^{-s}\frac{\partial}{\partial\varphi}(g(x - s\mu))\, ds\\ &= \frac{1}{r}e^{-d/r} g(\bar{x}_\varphi)\frac{\partial d}{\partial\varphi} + \int_0^{d/r} e^{-s}\frac{\partial g}{\partial\mu'}(x - s\mu)\, ds,\end{aligned} \tag{4.8}$$

where $\bar{x}_\varphi = x - d(\mu/r) \in \Gamma^-_\mu$, and $\mu' = (r\sin\varphi, -r\cos\varphi)$ is orthogonal to μ. Estimating $\partial d/\partial\varphi$ in each subdomain $\Omega_{\varphi,j} = \{x \in \Omega\colon \bar{x}_\varphi \in S_j\}$, we obtain

$$\left|\frac{\partial d(x, r, \varphi)}{\partial\varphi}\right| \leqq \frac{C}{|\sin\psi_j(\varphi)|} \quad \text{for } x \in \Omega_{\varphi,j}.$$

Here the S_j's are the sides of Ω and ψ_j is the angle between μ and S_j. By an orthogonal coordinate transformation we may assume that the x_1-axis is parallel to S_j. Then

$$d(x, r, \varphi) = \frac{x_2}{\sin\psi_i(\varphi)}, \qquad x = (x_1, x_2) \in \Omega. \tag{4.9}$$

Squaring (4.8) and integrating over $\Omega_{\varphi,j}$ we get

$$\begin{aligned}\int_{\Omega_{\varphi,j}} \left|\frac{\partial u(x, r, \varphi)}{\partial\varphi}\right|^2 dx \leqq C\Bigg(&\int_{\Omega_{\varphi,j}} \frac{1}{r^2} e^{-2d(x,\mu)/r} g^2(\bar{x}_\varphi)\frac{dx}{\sin^2\psi_j}\\ &+ \int_{\Omega_{\varphi,j}}\int_0^{d(x,\mu)/r} s^2 e^{-2s}\left(\frac{\partial g}{\partial\mu'}\right)^2 ds\, dx\Bigg) = C(I_j + J_j),\end{aligned}$$

with the obvious definitions for I_j and J_j. These terms are estimated as follows:

$$\begin{aligned}I_j &= \int_{\Omega_{\varphi,j}} \frac{1}{r^2}\exp(-2d(x, r, \varphi)/r) g^2(\bar{x}_\varphi)\frac{dx}{\sin^2\psi_j(\varphi)}\\ &\leqq \int_{S_j} g^2(\bar{x}_\varphi)\, dx_1 \int_0^{C|\sin\psi_j|} \frac{-1}{2r\sin\psi_j}\left(\frac{-2\exp((-2x_2)/(r\sin\psi_j))}{r\sin\psi_j}\right) dx_2\\ &\leqq \frac{C}{r|\sin\psi_j|}[1 - e^{-2/r}]\|g\|^2_\Gamma,\end{aligned}$$

and using the trace estimate: $\|g\|_\Gamma \leqq C\|g\|_{1,\Omega}$ we find that

$$I_j \leqq \frac{C}{r|\sin \psi_j|}\|g\|_1^2.$$

For estimating J_j, let us assume that $\mu' \cdot \nabla g = 0$ on the complement of the $\Omega_{\varphi,j}$. Then applying Fubini's theorem we get

$$J_j = \int_{\Omega_{\varphi,j}} \int_0^{d(x,\mu)/r} s^2 e^{-2s}(\mu' \cdot \nabla g)^2 \, ds \, dx \leqq r^2 \int_{\mathbb{R}^2} \int_0^\infty s^2 e^{-2s} |\nabla g(x - s\mu)|^2 \, ds \, dx$$

$$\leqq \left(r^2 \int_0^\infty s^2 e^{-2s} \, ds \right) \|\nabla g\|^2 \leqq Cr^2 \|\nabla g\|^2 \leqq cr^2 \|g\|_1.$$

Summing over j we have

$$\left\| \frac{\partial u(\cdot, r, \varphi)}{\partial \varphi} \right\| \leqq Cr^{-1/2} \min_j |\sin \psi_j(\varphi)|^{-1/2} \|g\|_1,$$

hence using (4.3) we obtain

$$\sum_{i=1}^N A_k \int_0^{2\pi} \left\| \frac{\partial u(\cdot, r_k, \varphi)}{\partial \varphi} \right\| d\varphi$$

$$\leqq C\|g\|_1 \left(\frac{5\pi}{4N} \sum_{k=1}^N \frac{r_{k+1}}{r_k^{1/2}} \right) \int_0^{2\pi} \min_j |\sin \psi_j(\varphi)|^{-1/2} \, d\varphi \leqq C\|g\|_1.$$

To verify (4.7) we observe that

$$\frac{\partial u}{\partial r} = e^{-d/r} g(\bar{x}_\varphi) \left(\frac{1}{r} \frac{\partial d}{\partial r} - \frac{1}{r^2} d \right) + \int_0^{d/r} e^{-s} \frac{\partial}{\partial r} (g(x - s\mu)) \, ds.$$

Moreover $\partial g/\partial r = -(s/r)(\mu \cdot \nabla g)$, and since by (4.9) $\partial d/\partial r = 0$, thus

$$\frac{\partial u}{\partial r} = \frac{-d\, e^{-d/r}}{r^2} g(\bar{x}_\varphi) + \int_0^{d/r} \frac{-s\, e^{-s}}{r} (\mu \cdot \nabla g) \, ds. \tag{4.10}$$

Squaring (4.10) and integrating over $\Omega_{\varphi,j}$ we get

$$\int_{\Omega_{\varphi,j}} \left| \frac{\partial u(x, r, \varphi)}{\partial r} \right|^2 dx \leqq C \Bigg[\int_{\Omega_{\varphi,j}} \frac{d^2 e^{-2d/r}}{r^4} g^2(\bar{x}_\varphi) \, dx$$

$$+ \int_{\Omega_{\varphi,j}} \int_0^{d/r} s^2 e^{-2s} |\nabla g|^2 \, ds \, dx \Bigg] = C(I_j' + J_j').$$

We estimate as before I_j' and J_j' separately:

$$I_j' = \int_{\Omega_{\varphi,j}} \frac{d^2 e^{-2d/r}}{r^4} g^2(\bar{x}_\varphi) \, dx$$

$$\leqq C \int_0^{C|\sin \psi_j|} \frac{x_2^2 \exp((-2x_2)/(r \sin \psi_j))}{r^4 \sin^2 \psi_j} \, dx_2 \int_\Gamma g^2(\bar{x}_\varphi) \, dx_1$$

$$\leqq C\|g\|_\Gamma^2 \frac{-1}{2r^3 \sin \psi_j} \int_0^{C|\sin \psi_j|} x_2^2 \frac{-2 \exp((-2x_2)/(r \sin \psi_j))}{r \sin \psi_j} \, dx_2.$$

Integrating by parts we have

$$I_j' \leqq C\|g\|_\Gamma^2 \frac{|\sin\psi_j|}{r^3}(-e^{-2/r} - r\,e^{-2/r} - r^2\,e^{-2/r} + r^2) \leqq C\frac{|\sin\psi_j|}{r}\|g\|_\Gamma^2.$$

By the same treatment as in the estimate of J_j we find that

$$J_j' \leqq C\|\nabla g\|^2 \leqq C\|g\|_1^2.$$

Again using the trace estimate and summing over j we have

$$\left\|\frac{\partial u(\cdot, r, \varphi)}{\partial r}\right\| \leqq \frac{C}{r^{1/2}}\|g\|_1,$$

thus

$$\int_0^{2\pi}\int_0^1 \left\|\frac{\partial u(\cdot, r, \varphi)}{\partial r}\right\| dr\,d\varphi \leqq C\left(\int_0^1 \frac{dr}{r^{1/2}}\right)\|g\|_1 \leqq C\|g\|_1. \qquad \square$$

Proof of Lemma 4.3. Applying Lemmas 3.1, 3.2 and (3.10) we get

$$\begin{aligned}
|e_{NM}(x)| = |(T - T_n)g(x)| &= \left|\int_D u(x,\mu)(1-|\mu|^2)^{-1/2}d\mu - \sum_{\mu\in\Delta} u(x,\mu)\omega_\mu\right| \\
&= \left|\int_0^{2\pi}\int_0^1 u(x,r,\varphi)(1-r^2)^{-1/2} r\,dr\,d\varphi - \sum_{j=1}^M w_j \sum_{k=1}^N A_k u(x, r_k, \varphi_j)\right| \\
&\leqq \int_0^{2\pi}\left|\int_0^1 u(x,r,\varphi)(1-r^2)^{-1/2} r\,dr - \sum_{k=1}^N A_k u(x,r_k,\varphi)\right| d\varphi \\
&\quad + \sum_{k=1}^N A_k\left[\left|\int_0^{2\pi} u(x,r_k,\varphi)\,d\varphi - \sum_{j=1}^M w_j u(x,r_k,\varphi_j)\right|\right] \\
&\leqq \frac{C}{N}\int_0^{2\pi}\int_0^1\left|\frac{\partial u(x,r,\varphi)}{\partial r}\right| dr\,d\varphi + \frac{C}{M}\sum_{k=1}^N A_k \int_0^{2\pi}\left|\frac{\partial u(x,r_k,\varphi)}{\partial\varphi}\right| d\varphi,
\end{aligned}$$

and the desired result follows from Lemma 4.4. $\square$

5. The fully discrete problem. We shall prove that the fully discrete problem (2.8) has a unique solution in V_h. To do this we shall first show that if h is properly related to n, then

$$\|(T_n - T_n^h)T_n^h\| \to 0 \quad \text{as } n \to \infty. \tag{5.1}$$

By the argument used in the proof of Proposition 2.1 (see proof of [5, Lemma 3.1]) and by (4.1) it then follows that if n is sufficiently large, then there exists a positive constant C such that

$$\|(I - \lambda T_n^h)v\| \geqq C\|v\| \quad \forall v \in L_2(\Omega).$$

Hence, $(I - \lambda T_n^h)$: $L_2(\Omega) \to L_2(\Omega)$ is injective and since T_n^h has finite dimensional range and thus is compact, it follows that $(I - \lambda T_n^h)$ is one-to-one, onto and has a bounded inverse. In particular the fully discrete problem

$$(I - \lambda T_n^h)U_n^h = T_n^h f$$

has a unique solution U_n^h in $L_2(\Omega)$ and since T_n^h: $L_2(\Omega) \to V_h$, we have $U_n^h \in V_h$, i.e., the fully discrete problem (2.8) has a unique solution. To prove (5.1) the main work will be to prove an estimate of the form:

$$\|(T_\mu - T_\mu^h)T_\nu^h g\| \leqq C(\gamma)h^\alpha \|g\|, \tag{5.2}$$

where $\gamma = \gamma(\mu\ \nu) \equiv \min(|\sin d(\mu, \nu)|, |\mu|, |\nu|)$, with $d(\mu, \nu)$ the smallest angle between μ and ν, $C(\gamma) \to \infty$ as $\gamma \to 0$ and $\alpha > 0$. Using (5.2) and the L_2-stability estimate

$$\|(T_\mu - T_\mu^h)T_\nu^h g\| \leqq C\|g\|$$

resulting from Propositions 2.2 and 2.3, in the decomposition

$$\begin{aligned}(T_n - T_n^h)T_n^h g &= \sum_{(\mu,\nu)\in\Delta^2} \omega_\mu\omega_\nu(T_\mu - T_\mu^h)T_\nu^h g \\ &= \left(\sum_{\gamma(\mu,\nu)\geqq\varepsilon} + \sum_{\gamma(\mu,\nu)<\varepsilon}\right)\omega_\mu\omega_\nu(T_\mu - T_\mu^h)T_\nu^h g,\end{aligned}$$

we get

$$\begin{aligned}\|(T_n - T_n^h)T_n^h g\| &\leqq \left[h^\alpha \sum_{\gamma(\mu,\nu)\geqq\varepsilon} C(\gamma(\mu, \nu))\omega_\mu\omega_\nu + C\sum_{\gamma(\mu,\nu)<\varepsilon} \omega_\mu\omega_\nu\right]\|g\| \\ &\equiv [h^\alpha A(\varepsilon, n) + B(\varepsilon, n)]\|g\|.\end{aligned}$$

We shall then choose ε and h related to n in such a way that $h^\alpha A(\varepsilon, n) \to 0$ and $B(\varepsilon, n) \to 0$ as $n \to \infty$, and (5.1) will follow.

To prove (5.2) for $\gamma \geqq \varepsilon$, let us first note that by an orthogonal coordinate transformation we may assume that $\mu = (\mu_1, 0)$. If we reformulate the discrete problem (2.4) using new coordinates $\bar{x}$ given by

$$\bar{x} = Dx, \qquad D = \begin{pmatrix} 1 & -\nu_1/\nu_2 \\ 0 & 1 \end{pmatrix}$$

we obtain with $\bar{\mu} = D\mu = D(\mu_1, 0) = (\mu_1, 0)$ and $\bar{\nu} = D\nu = D(\nu_1, \nu_2) = (0, \nu_2)$ the discrete solution operators $T_{\bar{\mu}}^h$ and $T_{\bar{\nu}}^h$ which are given by application of the discontinuous Galerkin method on a triangulation $\bar{\mathscr{C}}_h = \{\bar{K}\}$ of the domain $\bar{\Omega} = \{Dx: x \in \Omega\}$, where $\bar{K} = \{Dx: x \in K\}$ with $K \in \mathscr{C}_h$. The triangulation $\bar{\mathscr{C}}_h$ has the following properties:

(5.3a) maximal side length $\leqq Ch/\gamma$,

(5.3b) minimal side length $\geqq Ch$,

(5.3c) minimal angle $\geqq C\gamma$,

(5.3d) since $\mathscr{C}_h$ is quasiuniform, the number of triangles of $\bar{\mathscr{C}}_h$ meeting at each node is bounded by a constant independent of h and γ, where $\gamma = \min(|\sin d(\mu, \nu)|, |\mu|, |\nu|)$.

LEMMA 5.1. *Suppose $\bar{\mu} = (\mu_1, 0)$, $\bar{\nu} = (0, \nu_2)$ and the discrete solution operators $T_{\bar{\mu}}^h$ and $T_{\bar{\nu}}^h$ are based on using a triangulation $\bar{\mathscr{C}}_h$ of $\bar{\Omega}$ satisfying (5.3). Then for any $\rho > 0$, there is a constant C independent of h and γ such that if $\theta = T_{\bar{\mu}}f$, $\theta_h = T_{\bar{\mu}}^h f$, where $f = T_{\bar{\nu}}^h g$ with $g \in L_2(\Omega)$, then*

$$\|\theta - \theta_h\| \leqq C \min(1, \gamma^{-3/2}h^{1/8-\rho})\|g\|. \tag{5.4}$$

The proof is the same as that of [5, Lemma 5.1], except that the inequality $\|f\|_{0,s} \leqq C_s h^{-1/4}\|g\|$ in [5], here is replaced by $\|f\|_{0,s} \leqq C_s\gamma^{-1}h^{-1/4}\|g\|$. The appearance of $\gamma^{-3/2}$ in (5.4) is due to the presence of γ^{-1} in the latter inequality, which is the matter of Lemma 5.2.

LEMMA 5.2. *For any $0<s<\frac{1}{2}$ there is a constant C such that, if φ is a piecewise linear function on a triangulation $\bar{\mathscr{C}}_h$ of $\bar{\Omega}$ satisfying* (5.3) *and extended by zero outside $\bar{\Omega}$, then*

$$\|\varphi\|_{0,s} \leqq C\gamma^{-1}h^{-1/4}A, \tag{5.5}$$

where

$$A=\left[\|\varphi\|^2+h\sum_{\bar{K}}\|\nu\cdot\nabla\varphi\|_{\bar{K}}^2+\sum_{\bar{K}}\int_{\partial\bar{K}}[\varphi]^2|n\cdot\bar{\nu}|\,ds\right]^{1/2}$$

and $\nu=(0,\nu_2)$. Observe that if $\varphi=T_{\bar{\nu}}^h g$, then by Proposition 2.3, $A=|||T_{\bar{\nu}}^h\varphi|||_{\bar{\nu}}\leqq C\|g\|$.

Proof. The proof is the same as that of [5, Lemma 5.2]. The appearance of γ^{-1} in (5.5) depends on the presence of $|\bar{\nu}|$ and $1/\nu_2$ in the relations: $dx_1=n_2\,ds=(n\cdot\bar{\nu}/|\bar{\nu}|)\,ds$ and $\varphi'_\xi=(1/\nu_2)(\bar{\nu}\cdot\nabla\varphi)$. Using the same notations as in [5] and repeating the last step of the proof of [5, Lemma 5.2], we obtain

$$\begin{aligned}
\int_{\mathbb{R}} A(\bar{x}_1)^2\,d\bar{x}_1 &\equiv \int_{\mathbb{R}}\left(\|\psi\|^2+h\sum_{i=1}^{M-1}\|\psi'|_{I_i}^2+\sum_{i=1}^{M}[\psi_i]^2\right)d\bar{x}_1\\
&=\int_{\mathbb{R}}\int_{\mathbb{R}}|\psi(\bar{x}_1,\xi)|^2\,d\xi\,d\bar{x}_1+h\sum_{i=1}^{M-1}\int_{\mathbb{R}}\int_{I_i}|\varphi'_\xi(\bar{x}_1,\xi)|^2\,d\xi\,d\bar{x}_1\\
&\quad+\sum_{i=1}^{M}\int_{\mathbb{R}}[\varphi(\bar{x}_1,\xi^+)-\varphi(\bar{x}_1,\xi^-)]^2\,d\bar{x}_1\\
&\leqq \int_{\Omega}|\varphi(\bar{x}_1,\bar{x}_2)|^2\,d\bar{x}_1\,d\bar{x}_2+h\sum_{\bar{K}}\int_{\bar{K}}\left|\frac{1}{\nu_2}(\bar{\nu}\cdot\nabla\varphi)\right|^2d\bar{x}_1\,d\bar{x}_2\\
&\quad+\sum_{i=1}^{M}\int_{\partial\bar{K}}[\varphi]^2\frac{n\cdot\bar{\nu}}{|\bar{\nu}|}\,ds\\
&\leqq \|\varphi\|^2+\frac{h}{|\nu_2|^2}\sum_{\bar{K}}\|\bar{\nu}\cdot\nabla\varphi\|_{\bar{K}}^2+\frac{1}{|\bar{\nu}|}\sum_{\bar{K}}\int_{\partial\bar{K}}[\varphi]^2|\mu\cdot\nu|\,ds\\
&\leqq\frac{1}{|\bar{\nu}|^2}A^2\leqq\frac{1}{\gamma^2}A^2,
\end{aligned}$$

where we have used $\varphi'_\xi=(1/\nu_2)(\bar{\nu}\cdot\nabla\varphi)$ and $\varphi=0$ on $\bar{\Omega}^c$. Using this result we get (see [5])

$$\int_{\mathbb{R}}\|\varphi(\bar{x}_1,\cdot)\|_{H^s(\mathbb{R})}^2\,d\bar{x}_1\leqq C_s h^{-1/2}\int_{\mathbb{R}}A(\bar{x}_1)^2\,d\bar{x}_1\leqq C_s h^{-1/2}\gamma^{-2}A^2$$

and since by (2.1)

$$\|\varphi\|_{0,s}^2=C\|\varphi\|^2+\int_{\mathbb{R}}\|\varphi(\bar{x}_1,\cdot)\|_{H^s(\mathbb{R})}^2\,d\bar{x}_1,$$

we conclude that

$$\|\varphi\|_{0,s}\leqq C_s\gamma^{-1}h^{-1/4}A. \qquad \square$$

Changing coordinates from $\bar{x}$ back to the original coordinates x, we obtain from Lemma 5.1 the following result: For any $\rho>0$ there is a constant C such that for $g\in L_2(\Omega)$

$$\|(T_\mu-T_\mu^h)T_\nu^h g\|\leqq C\min(1,(\gamma(\mu,\nu))^{-3/2}h^{1/8-\rho})\|g\|.$$

From this estimate it follows that for $\rho > 0$

$$\|(T_n - T_n^h)T_n^h g\| \leqq (h^{1/8-\rho}A(\varepsilon, n) + B(\varepsilon, n))\|g\| \tag{5.6}$$

where

$$A(\varepsilon, n) = C \sum_{\gamma(\mu,\nu)\geqq\varepsilon} (\gamma(\mu, \nu))^{-3/2}\omega_\mu\omega_\nu, \tag{5.7a}$$

$$B(\varepsilon, n) = C \sum_{\gamma(\mu,\nu)<\varepsilon} \omega_\mu\omega_\nu. \tag{5.7b}$$

By (4.2), (5.6) and the argument following (5.1) we obtain the basic result of this paper as follows.

THEOREM 5.1. *Suppose that $h = h(n)$ and $\varepsilon = \varepsilon(n)$ have been chosen so that for some $\rho > 0$*

$$h^{1/8-\rho}A(\varepsilon, n) \to 0 \text{ as } n \to \infty \quad \text{and} \quad B(\varepsilon, n) \to 0 \text{ as } n \to \infty,$$

where $A(\varepsilon, n)$ and $B(\varepsilon, n)$ are given by (5.7). Then for sufficiently large n, $(I - \lambda T_n^h)^{-1}$: $L_2(\Omega) \to L_2(\Omega)$ exists and is uniformly bounded.

6. An error estimate. In this section we prove an error estimate for the scalar flux U. We shall use the following splitting:

$$\sum_{\mu\in\Delta} = \sum_{\mu\in J_\varepsilon} + \sum_{\mu\in J'_\varepsilon}, \tag{6.1}$$

where

$$J_\varepsilon = \{\mu \in \Delta: \min(|\sin(\mu, d_k)|) < \varepsilon = \left(\frac{\pi}{M}\right), k = 1, \cdots, P\},$$

$$J'_\varepsilon = \{\mu \in \Delta: \mu \notin J_\varepsilon\}.$$

THEOREM 6.1. *Let U and U_n^h satisfy* (1.5) *and* (2.8) *respectively. Then there is a constant C such that for U and $f \in H^1(\Omega)$,*

$$\|U - U_n^h\| \leqq C\left(\frac{1}{M} + \frac{1}{N} + h^{1/2}\right)(\lambda\|U\|_1 + \|f\|_1).$$

The proof of Theorem 6.1 is based on the following lemma.

LEMMA 6.1. *There is a constant C such that for $g \in H^1(\Omega)$*

$$\sum_{\mu\in J'_\varepsilon} \omega_\mu|T_\mu g|_1 \leqq C\|g\|_1.$$

Let us postpone the proof of Lemma 6.1 and first show that Theorem 6.1 follows from this lemma and the splitting (6.1).

Proof of Theorem 6.1. We have by (1.5) and (2.8)

$$(I - \lambda T_n^h)(U - U_n^h) = (T - T_n)(\lambda U + f) + (T_n - T_n^h)(\lambda U + f) \equiv e_n + e_n^h$$

with the obvious notation. By Lemma 4.3 with $g = \lambda U + f$ we have

$$\|e_n\| = \|(T - T_n)(\lambda U + f)\| \leqq C\left(\frac{1}{M} + \frac{1}{N}\right)(\lambda\|U\|_1 + \|f\|_1). \tag{6.2}$$

To estimate e_n^h we use the splitting (6.1), (2.10a) and the L_2-stability estimate

$$\|(T_\mu - T_\mu^h)g\| \leqq C\|g\|,$$

resulting from Propositions 2.2 and 2.3, to obtain

$$\begin{aligned}\|e_n^h\| &\leqq \sum_{\mu\in\Delta}\omega_\mu\|(T_\mu-T_\mu^h)g\| \leqq \left(\sum_{\mu\in J_\varepsilon}\omega_\mu+\sum_{\mu\in J'_\varepsilon}\omega_\mu\right)\|(T_\mu-T_\mu^h)g\| \\ &\leqq C\|g\|\sum_{\mu\in J_\varepsilon}\omega_\mu + Ch^{1/2}\sum_{\mu\in J'_\varepsilon}\omega_\mu|T_\mu g|_1.\end{aligned}$$

But, for $\varepsilon=\pi/M$, $\sum_{\mu\in J_\varepsilon}$ contains at most Np elements, where p is the number of sides of Ω. Since $\omega_{kj}=W_jA_k=(2\pi/M)A_k$, by (4.3) and Lemma 6.1 we have

$$\|e_n^h\|\leqq\frac{C}{M}\cdot\frac{1}{N}Np\|g\|+Ch^{1/2}\|g\|_1\leqq C\left(\frac{1}{M}+h^{1/2}\right)\|g\|_1$$

and the desired result follows from (6.2) and Theorem 5.1. □

Proof of Lemma 6.1. By an orthogonal coordinate transformation, we may assume that $\mu=(\mu_1,0)$, $|\mu|=r$. Using Proposition 2.2 we have

$$\left\|\frac{\partial(T_\mu g)}{\partial x_1}\right\|\leqq\frac{C}{|\mu|}\|g\|. \tag{6.3}$$

Let $\nu=(0,\nu_2)$ with $|\nu|=1$ i.e., $\nu=(0,1)$ be orthogonal to μ, then

$$\left\|\frac{\partial(T_\mu g)}{\partial x_2}\right\|=\left\|\frac{\partial(T_\mu g)}{\partial \nu}\right\|. \tag{6.4}$$

Recall that

$$T_\mu g(x)=\int_0^{d/r}e^{-s}g(x-s\mu)\,ds \quad\text{and}\quad \frac{\partial d}{\partial\nu}=r\frac{\nu\cdot n}{\mu\cdot n}.$$

Thus

$$\frac{\partial}{\partial\nu}(T_\mu g(x))=e^{-d/r}g\left(x-d\frac{\mu}{r}\right)\frac{\nu\cdot n}{\mu\cdot n}+\int_0^{d/r}e^{-s}\frac{\partial}{\partial\nu}g(x-s\mu)\,ds. \tag{6.5}$$

Squaring (6.5) and integrating over Ω, using the same techniques as in the proof of Lemma 4.4 we find that

$$\begin{aligned}\left\|\frac{\partial}{\partial\nu}(T_\mu g)\right\|^2 &\leqq C\left(\int_\Omega e^{-2d/r}\left[g\left(x_1-\frac{d}{r}\mu_1,x_2\right)\right]^2\left|\frac{\nu\cdot n}{\mu\cdot n}\right|^2dx_1\,dx_2\right.\\ &\qquad\left.+\int_\Omega\int_0^{d/r}e^{-2s}\left(\frac{\partial}{\partial\nu}g\right)^2ds\,dx\right)=C(I''+J''),\end{aligned}$$

$$\begin{aligned}I''&\leqq C\int_0^{\operatorname{diam}\Omega}e^{-2x_1/r}\,dx_1\int_\Gamma|g|^2\left|\frac{\nu\cdot n}{\mu\cdot n}\right|^2\frac{|\mu\cdot n|}{|\mu|}\,ds\leqq\frac{C}{|\mu||\mu\cdot n|}\int_\Gamma|g|^2|\nu\cdot n|\,ds\\ &\leqq\frac{C}{|\mu||\mu\cdot n|}\int_\Gamma|g|^2\,ds\leqq\frac{C}{|\mu||\mu\cdot n|}\|g\|_1^2,\end{aligned}$$

where we have used the fact that μ is parallel to the x_1-axis, $dx_2=(|\mu\cdot n|/|\mu|)\,ds$ and the trace estimate $\|g\|_\Gamma\leqq C\|g\|_1$.

Let now $\nu\cdot\nabla g=0$ on $\mathbb{R}^2\backslash\Omega$; then

$$J''\leqq C\int_{\mathbb{R}^2}\int_0^\infty e^{-2s}|\nabla g|^2\,ds\,dx\leqq C\|\nabla g\|^2\int_0^\infty e^{-2s}\,ds\leqq C\|\nabla g\|^2.$$

Thus, by (6.3) and (6.4) we have

$$|T_\mu g|_1 = \|\nabla(T_\mu g)\| \leqq C\left(\frac{1}{|\mu|} + \frac{1}{\sqrt{|\mu||\mu \cdot n|}}\right)\|g\|_1. \tag{6.6}$$

Using (4.3) we have

$$\begin{aligned}\sum_{\mu \in J_\varepsilon'} \frac{\omega_\mu}{|\mu|} \leqq \sum_{\mu \in \Delta} \frac{\omega_\mu}{|\mu|} = \frac{2\pi}{M} \sum_{j=1}^{M} \left(\sum_{k=1}^{N} \frac{A_k}{|\mu_{jk}|}\right) \leqq \frac{2\pi}{M} M \frac{5\pi}{4N} \sum_{k=1}^{N} \frac{r_{k+1}}{r_k} \\ = \frac{C}{N} \sum_{k=1}^{N} \frac{\sin \theta_{k+1}}{\sin \theta_k}.\end{aligned}$$

So that since $\sin \theta_{k+1}/\sin \theta_k$ is decreasing,

$$\sum_{k=1}^{N} \frac{\sin \theta_{k+1}}{\sin \theta_k} \leqq C \sum_{k=1}^{N} \frac{\sin \theta_2}{\sin \theta_1} \leqq C \sum_{k=1}^{N} \frac{\sin 4\alpha}{\sin \alpha} \leqq CN,$$

where $\alpha = \pi/(4N+2)$. Hence

$$\sum_{\mu \in J_\varepsilon'} \frac{\omega_\mu}{|\mu|} \leqq C. \tag{6.7}$$

Moreover, using the splitting (6.1) we get

$$\sum_{\mu \in J_\varepsilon'} \frac{\omega_\mu}{\sqrt{|\mu||\mu \cdot n|}} = \sum_{\mu \in J_\varepsilon'} \frac{\omega_\mu}{|\mu|\sqrt{|\sin(\mu, d^k)|}} \leqq \left(\sum_{k=1}^{N} \frac{A_k}{|\mu_k|}\right)\left(\frac{2\pi}{M} \sum_{\theta_j \geqq \pi/M} \frac{1}{\sqrt{|\sin \theta_j|}}\right).$$

It is easy to show that

$$\sum_{k=1}^{N} \frac{A_k}{|\mu_k|} \leqq C,$$

and since

$$\frac{2\pi}{M} \sum_{\theta_j \geqq \pi/M} \frac{1}{\sqrt{|\sin \theta_j|}} \leqq C \int_0^{2\pi} \frac{1}{\sqrt{|\sin \theta_j|}}\, d\theta = 4C \int_0^{\pi/2} \frac{1}{\sqrt{\sin \theta}}\, d\theta \leqq \pi C$$

we conclude

$$\sum_{\mu \in J_\varepsilon'} \frac{\omega_\mu}{\sqrt{|\mu||\mu \cdot n|}} \leqq C. \tag{6.8}$$

The inequalities (6.6)–(6.8) now yield the proof of Lemma 6.1. □

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