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ON ADAPTIVE FINITE ELEMENT METHODS FOR FREDHOLM INTEGRAL EQUATIONS OF THE SECOND KIND*

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Abstract. A posteriori and a priori error estimates are derived for a finite element discretization of a Fredholm integral equation of the second kind. A *reliable* and *efficient* adaptive algorithm is then designed for a specific computational goal with applications to potential problems. The *reliability* of the algorithm is guaranteed by the a posteriori error estimate and the *efficiency* follows from the a priori error estimate, which shows that the a posteriori error bound is sharp.

Key words. Fredholm integral equation, potential problems, adaptive finite element method, a posteriori and a priori error estimates

AMS subject classification. primary 65N15, 65N30

Introduction. We consider the problem of finding an approximate solution of a Fredholm integral equation of the second kind, i.e., of the form

$$\varphi(x) - \int_\Gamma k(x,y)\varphi(y)\, d\Gamma(y) = f(x), \quad x \in \Gamma, \tag{0.1}$$

where f and k are given data. In this paper we will have that $k(x,y) = O(|x-y|^{1-d})$, where d is the dimension of Γ, i.e., $k(x,y)$ will only be weakly singular.

There are many examples of problems that lead to an equation of this form. In this note we shall specifically consider the Neumann problem for Laplace's equation in a domain Ω in R^{d+1} with boundary Γ. This problem has a single layer potential solution that can be determined by solving an equation of the form (0.1). Similarly the corresponding Dirichlet problem can be solved by considering a similar equation for the double layer potential solution. A second-order elliptic equation in R^d with smooth coefficients can also be reduced to an equation of the form (0.1) with $k(x,y)$ the fundamental solution of Laplace's equation and $\Gamma = R^d$.

The purpose of this note is to demonstrate a method for designing *reliable* and *efficient* adaptive finite element procedures for equations of the type (0.1) based on sharp a posteriori and a priori error estimates. Finite element methods for integral equations have been studied by many authors. See, e.g., Atkinson [1], Ikebe [9], Nedelec [10], Sloan [13], and Wendland [14]. Adaptive finite element methods for integral equations have been considered more recently; c.f., e.g., [8], [11], [14], and [15].

The general idea of an adaptive mesh-refinement finite element procedure is the following; see also [3] and [8]: Given a continuous problem with exact solution φ, a norm $\|\cdot\|$, and an error tolerance δ, the adaptive algorithm should construct (typically through a sequence of successive mesh refinements and information obtained from the corresponding approximate solutions) a mesh T_h and compute the corresponding finite element solution φ_h such that

$$\|\varphi - \varphi_h\| \le \delta. \tag{0.2}$$

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Also, the total number of degrees of freedom involved in the computational process should be (nearly) minimal. The problem of designing such an algorithm thus has two main ingredients. First we want the algorithm to be *reliable* in the sense that the desired error control (0.2) is guaranteed. Secondly, we want the algorithm to be *efficient* in the sense that the constructed meshes are (nearly) optimal and nowhere overly refined for the given error tolerance. By an *optimal* mesh (given the order of approximation of the elements) we mean a mesh with a minimal number of elements for which there exists an interpolant $\tilde{\varphi} \in V_h$ such that $\|\varphi - \tilde{\varphi}\| \leq \delta$, where V_h is the corresponding finite element space.

Our refinement strategy and stopping criterion will be based on an a posteriori error estimate that will guarantee the *reliability* of the algorithm. The derivation of this estimate is one of the essential parts of the analysis. In order to demonstrate the *efficiency* of the algorithm, we also derive a corresponding a priori error estimate. The purpose of this estimate is to show that the a posteriori error estimate used for the adaptive error control is of *optimal* order and thus that the adaptive method will be efficient.

In order to be concrete, we have chosen to consider, specifically, one of the given examples leading to an equation of the form (0.1), namely, the single layer potential problem for Laplace's equation with Neumann boundary conditions. Results similar to the ones we obtain for this particular case can be derived for a fairly broad class of problems of the form (0.1) and in more general situations as well.

Motivated by our particular example, we shall consider error control in a weighted L_1-norm. Again, this should be viewed as an example; generalizing our results to error control in other norms should present no problem; cf. Remark 4.3 below.

In this note we consider the case of a weakly singular kernel $k(x, y)$. In a future paper we plan to analyze the (more interesting) case of a kernel that degenerates (at certain points) to order $O(|x-y|^{-d})$ as in potential problems for Neumann and Dirichlet problems on nonsmooth domains in R^{d+1}.

The rest of this note is organized as follows: In §1 we introduce the boundary value problem under consideration and recall the derivation of the corresponding single layer potential problem of the form (0.1). In §§2 and 3 we specify our computational goal and formulate an adaptive finite element method for the problem that is proved to be both *reliable* and *efficient*. In §§4 and 5 we derive the a posteriori and a priori error estimates underlying the design of the adaptive procedure. Finally, in §6 we present some numerical results.

1. The continuous problem. Consider the exterior Neumann problem (cf. Remarks 1.1–1.3 below)

$$
\begin{aligned}
\Delta u &= 0 \quad \text{in } \Omega', \\
\frac{\partial u}{\partial n} &= g \quad \text{on } \Gamma, \\
u, |\nabla u| &\to 0 \quad \text{as } |x| \to \infty,
\end{aligned}
\tag{1.1}
$$

where Ω' is the complement of a closed bounded, simply connected domain $\Omega \subseteq R^3$ with smooth boundary $\Gamma = \partial\Omega$, and n is the exterior (with respect to Ω, interior with respect to Ω') unit normal to Γ.

Given φ defined on Γ, we define the corresponding single layer potential

$$
u(x) = \frac{1}{4\pi} \int_\Gamma \varphi(y) \frac{1}{|x-y|} d\Gamma(y), \quad x \in R^3. \tag{1.2}
$$

Since

$$\Delta_x \frac{1}{|x-y|} = 0, \qquad x \neq y,$$

we have at once that u is harmonic, i.e., satisfies $\Delta u = 0$ in Ω'. Also, u satisfies the given conditions at infinity. Hence problem (1.1) has been reduced to the problem of finding a function φ such that $\partial u/\partial n = g$ on Γ. The normal derivative $\partial u/\partial n$ of u can be expressed in terms of φ as (cf., e.g., [7])

$$\frac{\partial u}{\partial n}(x) = -\frac{1}{2}\varphi(x) + \frac{1}{4\pi}\int_\Gamma \varphi(y)\frac{\partial}{\partial n_x}\frac{1}{|x-y|}d\Gamma(y), \qquad x \in \Gamma, \tag{1.3}$$

where $\partial/\partial n_x$ denotes differentiation with respect to x in the direction of n. Defining the integral operator T by

$$(T\varphi)(x) = \frac{1}{2\pi}\int_\Gamma \varphi(y)\frac{\partial}{\partial n_x}\frac{1}{|x-y|}d\Gamma(y),$$

our problem thus can be written in compact form as follows: Given $f = -2g$ defined on Γ, find φ defined on Γ such that

$$\varphi - T\varphi = f. \tag{1.4}$$

It is well known that the operator T is regularizing and, e.g., maps functions in $L_p(\Gamma)$ into the corresponding first-order Sobolev space $W_p^1(\Gamma)$. In particular, T is a compact operator on $L_p(\Gamma)$ for any $p \geq 1$, and so is its adjoint T^* to appear below.

Remark 1.1. The underlying problem (1.1) can be formulated more generally in R^q, of course. The fundamental solution $(1/4\pi)(1/|x-y|)$ in (1.2) is then replaced by $(1/2\pi)\log(1/|x-y|)$ for $q = 2$ and by $1/(c_q|x-y|^{(q-2)})$ for $q \geq 3$, with c_q the $q-1$-dimensional surface measure of the boundary of the unit ball in R^q. All our theoretical results can easily be obtained also for $q = 2$ (the cases $q > 3$ may not be of comparable physical interest). So far, we have implemented our adaptive algorithm only for $q = 2$. Numerical results are given in §6 below.

Remark 1.2. For the *interior* Neumann problem corresponding to (1.1), a solution exists only if $\int_\Gamma g d\Gamma = 0$, and in order to filter out a unique solution we must add a constraint such as, e.g., $\int_\Omega u d\Omega = 0$. The counterpart of (1.4) then reads

$$\varphi + T\varphi = 2g,$$
$$\int_\Gamma \varphi \, d\Gamma = 0,$$

and our analysis can easily be carried over to this case as well. (For another way of dealing with the nonuniqueness, see [2].)

Remark 1.3. The *double* layer potential problems for the interior and exterior *Dirichlet* problems also lead to Fredholm integral equations of the second kind. The solution of the Dirichlet problem, in this case, is given by the double layer potential

$$u(x) = \frac{1}{4\pi}\int_\Gamma \varphi(y)\frac{\partial}{\partial n_y}\frac{1}{|x-y|}d\Gamma(y).$$

The Dirichlet boundary condition $u = u_0$ on Γ now leads to the equation

$$\varphi - T^*\varphi = 2u_0$$

for the unknown φ, where

$$T^*\varphi(x) = \frac{1}{2\pi}\int_\Gamma \varphi(y)\frac{\partial}{\partial n_y}\frac{1}{|x-y|}d\Gamma(y).$$

It is easy to see by examining the proofs that our results can be carried over to this situation as well. In a forthcoming paper we plan to consider the latter problem in the case of a nonsmooth boundary, e.g., with re-entrant corners.

2. Finite element discretization. Equation (1.4) may be written in variational form as

$$(\varphi, v) - (T\varphi, v) = (f, v) \quad \forall v \in L_\infty(\Gamma), \tag{2.1}$$

where $(\cdot,\cdot)$ denotes the $L_2(\Gamma)$ inner product.

We consider the simplest possible (cf. Remark 3.2 below) finite element method for this problem, which is to seek $\varphi_h \in V_h$ such that

$$(\varphi_h, v) - (T\varphi_h, v) = (f, v) \quad \forall v \in V_h, \tag{2.2}$$

where $V_h = \{v \in L_\infty(\Gamma): v|_K = \text{constant for all } K \in \mathcal{C}_h\}$ and $\mathcal{C}_h = \{K\}$ is a partition of Γ into a finite number of elements K. The subscript parameter h, which usually denotes the maximal diameter of the elements K, will in this paper denote the piecewise constant *function* defined by $h|_K = h_K, K \in \mathcal{C}_h$, where h_K is the diameter of K. The approximate solution φ_h of problem (1.4) determined by (2.2), of course, defines a corresponding approximate solution of problem (1.1), namely,

$$u_h(x) = \frac{1}{4\pi}\int_\Gamma \varphi_h(y)\frac{1}{|x-y|}d\Gamma(y). \tag{2.3}$$

From (1.2) and (2.3) we easily get the error estimate

$$|(u-u_h)(x)| \le \frac{1}{4\pi}\int_\Gamma |(\varphi-\varphi_h)(y)|\,\frac{1}{|x-y|}d\Gamma(y). \tag{2.4}$$

The error analysis for $\varphi - \varphi_h$ is most often carried out in $L_2(\Gamma)$. For x bounded away from Γ, this is adequate because

$$|(u-u_h)(x)| \le \frac{1}{4\pi}\left\|\frac{1}{|x-\cdot|}\right\|_{L_2(\Gamma)} \|\varphi-\varphi_h\|_{L_2(\Gamma)}\,.$$

For x close to Γ, however, the factor $\|1/|x-\cdot|\|_{L_2(\Gamma)}$ may become arbitrary large. Below we shall therefore analyze the error $\varphi - \varphi_h$ in a more appropriate norm, which in this case, focusing on error control at a given point x, is a weighted L_1-norm with the weight $w_x(y) = 1/|x-y|$. In cases of several points x_i of particular interest we may simply change the weight function to $w(y) = \max_i 1/|x_i - y|$. For related results on error estimation at particular points, see, e.g., [4]. See also Remark 4.3 below where we consider control of $u - u_h$ in the maximum norm.

Remark 2.1. The method of approximation (2.3) is generally referred to as the *boundary element method,* because we use finite elements on the boundary of Ω' to solve problem (1.1). In this note we focus on the approximation of the solution of the resulting Fredholm integral equation, and shall therefore use the more general terminology *finite element method.* Also, there are integral equations of the form (1.1) that do not originate from boundary potential problems (cf. above).

3. An adaptive algorithm. We have adopted the computational goal to find an approximate solution u_h of problem (1.1) through (2.3) and (2.2) such that

$$|(u - u_h)(x)| \leq \delta, \tag{3.1}$$

where x is a given (fixed) point in $\Omega' \cup \Gamma$ and $\delta > 0$ is a given error tolerance. Below (Theorem 4.1) we shall show that

$$\int_\Gamma |(\varphi - \varphi_h)(y)| \frac{1}{|x-y|} d\Gamma(y) \leq C_{\text{stab}} \left\| \frac{\min\{|r_h|, C_{\text{int}} h |\nabla_h r_h|\}}{|x - \cdot|} \right\|^*_{L_1(\Gamma)} \tag{3.2}$$

where r_h is the *residual* of the approximate solution φ_h of problem (1.4), i.e., $r_h = \varphi_h - T\varphi_h - f$. Moreover, ∇_h is a mesh dependent gradient defined on the surface manifold Γ (cf. below), C_{stab} and C_{int} are constants only depending on Γ, and

$$\left\| \frac{\min\{|r_h|, C_{\text{int}} h |\nabla_h r_h|\}}{|x - \cdot|} \right\|^*_{L_1(\Gamma)} := \sum_{K \in \mathcal{C}_h} \min \left\{ \left\| \frac{r_h}{|x - \cdot|} \right\|_{L_1(K)}, C_{\text{int}} \left\| \frac{h \nabla_h r_h}{|x - \cdot|} \right\|_{L_1(K)} \right\}.$$

In particular, from (3.2) we have that

$$\int_\Gamma |(\varphi - \varphi_h)(y)| \frac{1}{|x-y|} d\Gamma(y) \leq C_{\text{stab}} \left\| \frac{r_h}{|x - \cdot|} \right\|_{L_1(\Gamma)}. \tag{3.3}$$

From now on we shall use the letter C to denote various positive constants but not necessarily the same at each occurrence. Thus, recalling (2.4), we find that (3.1) is guaranteed if

$$C \left\| \frac{r_h}{|x - \cdot|} \right\|_{L_1(\Gamma)} \leq \delta, \tag{3.4}$$

where $C = C_{\text{stab}}/(4\pi)$. In practice, in order to attain (3.4) we proceed as follows. Clearly, (3.4) holds if

$$C \sum_K h_K^2 \left\| \frac{r_h}{|x - \cdot|} \right\|_{L_\infty(K)} \leq \delta.$$

For each element K in the partition we would thus like to have

$$h_K^2 \left\| \frac{r_h}{|x - \cdot|} \right\|_{L_\infty(K)} \leq \frac{\delta}{CN}, \tag{3.5}$$

where N is the number of elements in $\mathcal{C}_h$. That is, our strategy is to adapt the mesh size h_K to the size of r_h so as to equidistribute the element contributions to the global error. We then note that (3.5) may be impossible to obtain for some K when $x \in \Gamma$. To get around this technical complication we note that (3.4) also follows from

$$C \sum_K h_K^2 \left\| \frac{r_h}{|x - \cdot| + \varepsilon} \right\|_{L_\infty(K)} \leq \delta,$$

provided ε (with $\varepsilon = O(h_K)$) and $\max_{K \in \mathcal{C}_h} h_K$ are sufficiently small. This follows easily from the fact that for plane elements

$$\left\| \frac{|x - \cdot| + \varepsilon}{|x - \cdot|} \right\|_{L_1(K)} \leq \frac{\pi}{4} h_K^2 + \pi \varepsilon h_K.$$

Thus we may replace (3.5) by

$$h_K^2 \left\| \frac{r_h}{|x-\cdot|+\varepsilon} \right\|_{L_\infty(K)} \le \frac{\delta}{CN}, \tag{3.6}$$

e.g., with $\varepsilon = ch_K, c$ a small constant.

An algorithm designed to accomplish (3.1) through the control (3.6) would then look, roughly, as follows:

1°. Start with a (coarse, quasi-uniform) mesh $\mathcal{C}_h^{(0)}$.

2°. Given a mesh $\mathcal{C}_h^{(j)}$ compute the corresponding $\varphi_h^{(j)}$ and its residual

$$r_h^{(j)} = \varphi_h^{(j)} - T\varphi_h^{(j)} - f.$$

3°. If

$$h_K \le \left[\frac{\delta}{CN} \inf_{y\in K} \frac{|x-y|+\varepsilon}{\left|r_h^{(j)}(y)\right|} \right]^{1/2} \qquad \forall K \in \mathcal{C}_h^{(j)}, \tag{3.7}$$

then stop and accept $\varphi_h^{(j)}$ (and the corresponding $u_h^{(j)}$ defined by (2.3)). Otherwise, refine all elements K for which (3.7) does not hold (or remesh completely) in order to obtain a new partition $\mathcal{C}_h^{(j+1)}$ for which the inequality in (3.7) holds for all K in the new mesh (given $r_h^{(j)}$ and ε), and then go back to step 2° and recompute with j replaced by $j+1$.

Following the program in [5] (see also [6]) we would now like to investigate the performance of the proposed algorithm. First we note that the method will be *reliable* in the sense that once the algorithm has reached a successful stop then we know from the a posteriori error estimate, on which the method is based (Theorem 4.1), that (3.1) will hold for the corresponding u_h. Concerning the *efficiency* of the method we would like to know first of all that it is *operational* in the sense that (3.1) can be realized through mesh refinement. Note that r_h depends on h so that theoretically it could happen that min $\{|r_h|, C_{\text{int}}h|\nabla_h r_h|\}$ increases as h decreases in such a way that the quantity $C_{\text{stab}}\|\min\{|r_h|, C_{\text{int}}h|\nabla_h r_h|\}/|x-\cdot|\|_{L_1}^*$, which we try to control, never gets small. Below (Theorem 5.1) we shall prove that

$$\left\| \frac{r_h}{|x-\cdot|} \right\|_{L_1(\Gamma)} \le C \left\| \frac{h\nabla\varphi}{|x-\cdot|} \right\|_{L_1(\Gamma)}.$$

This estimate shows not only that the method is operational, but also that it is *efficient* in the sense that large values of $\nabla\varphi$ can be locally compensated for by choosing h locally small and that, indeed, the a posteriori error estimate in (3.2) is of *optimal* order since, for a general class of functions φ, we cannot expect to have a bound for the error better than $C\|h\nabla\varphi/|x-\cdot|\|_{L_1(\Gamma)}$. In fact, our proofs show that

$$\left\| \frac{r_h}{|x-\cdot|} \right\|_{L_1(\Gamma)} \le C \inf_{\tilde\varphi\in V_h} \left\| \frac{\varphi-\tilde\varphi}{|x-\cdot|} \right\|_{L_1(\Gamma)}.$$

Concerning the efficiency of the method we also refer to the numerical examples in §6, which show that

$$\frac{\text{estimated error}}{\text{exact error}} \le 1 + \text{small constant}.$$

Remark 3.1. To have the algorithm based on the more precise estimate (3.2) rather than on (3.4) we simply replace (3.6) by

$$h_K \leq \left[\frac{\delta}{CN} \inf_{y \in K} \frac{|x-y|+\varepsilon}{\min\{|r_h|, C_{\text{int}} h |\nabla_h r_h|\}}\right]^{1/2} \quad \forall K \in \mathcal{C}_h^{(j)}. \tag{3.8}$$

See also Remark 4.1 below.

Remark 3.2. The generalization of the results of this paper to methods using higher-order polynomial approximation is straightforward.

4. The a posteriori error estimate. This section is devoted to the proof of the a posteriori error estimate (3.2). For simplicity we shall assume that each element $K \in \mathcal{C}_h$ can be equipped with a local coordinate system $(\hat{x}_1, \hat{x}_2)$, i.e., a one-to-one mapping $F_K : (\hat{x}_1, \hat{x}_2) \to (x_1, x_2, x_3)$ from the plane reference element

$$\hat{K} = \left\{(\hat{x}_1, \hat{x}_2) \in R^2 : \hat{x}_1 \geq 0, \hat{x}_2 \geq 0, \hat{x}_1 + \hat{x}_2 \leq 1\right\},$$

onto K such that

$$\left|\frac{\partial x_i}{\partial \hat{x}_j}\right| \leq C h_K, \qquad i = 1, 2, 3, \quad j = 1, 2, \tag{4.1}$$

and

$$J_K^{-1} \leq C h_K^{-2} \quad \text{on } K, \tag{4.2}$$

where (x_1, x_2, x_3) are the Cartesian coordinates on K, h_K is the diameter of K, and

$$J_K = \left|\left(\frac{\partial x_1}{\partial \hat{x}_1}, \frac{\partial x_2}{\partial \hat{x}_1}, \frac{\partial x_3}{\partial \hat{x}_1}\right) \times \left(\frac{\partial x_1}{\partial \hat{x}_2}, \frac{\partial x_2}{\partial \hat{x}_2}, \frac{\partial x_3}{\partial \hat{x}_2}\right)\right|. \tag{4.3}$$

With any function w defined on K we associate the function $\hat{w} = w \circ F_K$ defined on the reference element $\hat{K}$.

THEOREM 4.1. *There are constants C_{stab} and C_{int} independent of x and h such that*

$$\int_\Gamma |(\varphi - \varphi_h)(y)| \frac{1}{|x-y|} d\Gamma(y) \leq C_{\text{stab}} \left\|\frac{\min\{|r_h|, C_{\text{int}} h |\nabla_h r_h|\}}{|x - \cdot|}\right\|^*_{L_1(\Gamma)},$$

where $\nabla_h r_h|_K = h_K^{-1}(\nabla \hat{r}_h) F_K^{-1}$.

Proof. For the error $e = \varphi - \varphi_h$, we have from (2.1) and (2.2) that

$$(e, \chi) - (Te, \chi) = 0 \quad \forall \chi \in V_h. \tag{4.4}$$

Now let z be the solution of

$$(v, z) - (Tv, z) = \left(v, \frac{w}{|x - \cdot|}\right) \quad \forall v \in L_\infty(\Gamma), \tag{4.5}$$

or, equivalently,

$$z - T^* z = \frac{w}{|x - \cdot|} \quad \text{a.e. on } \Gamma,$$

where T^* is the adjoint of the operator T and $|w| \leq 1$ on Γ. Here different points x and functions w give rise to different solutions z. From (4.5) with $v = e$ and using (4.4) we obtain

$$\left(e, \frac{w}{|x-\cdot|}\right) = (e, z) - (Te, z) = (e, z - z_i) - (Te, z - z_i), \tag{4.6}$$

where $z_i \in V_h$ is any interpolant of z. We now choose $w = \operatorname{sign}(e) = e/|e|$ and write (4.6) as

$$\begin{aligned}\int_\Gamma \frac{|e|}{|x-\cdot|} d\Gamma &= \left(e, \frac{w}{|x-\cdot|}\right) \\ &= (\varphi - T\varphi, z - z_i) - (\varphi_h - T\varphi_h, z - z_i) \\ &= (f - \varphi_h + T\varphi_h, z - z_i) = (r_h, z_i - z).\end{aligned}$$

The proof is then a direct consequence of the following two lemmas. □

LEMMA 4.1. *There exists an interpolant* $z_i \in V_h$ *of* z *such that*

$$|(r_h, z - z_i)| \leq \left\| \frac{\min\{|r_h|, C_{\text{int}} h |\nabla_h r_h|\}}{|x-\cdot|} \right\|^*_{L_1(\Gamma)} \| |x-\cdot| \, z \|_{L_\infty(\Gamma)},$$

where C_{int} *only depends on* Γ *and the constants in* (4.1) *and* (4.2).

LEMMA 4.2. *There is a constant* C_{stab} *independent of* x *and* w *such that*

$$\| |x-\cdot| \, z \|_{L_\infty(\Gamma)} \leq C_{\text{stab}}.$$

Proof of Lemma 4.1. Let $F_K : \hat{K} \to K$ be the bijective mapping from the reference element $\hat{K}$ in R^2 onto K defining the local coordinate system $(\hat{x}_1, \hat{x}_2)$ on K. With $\hat{w} = w \circ F_K$ and J_K as above, we then define the piecewise constant interpolant z_i of z by

$$z_i|_K = \hat{z}_i|_{\hat{K}} = \text{constant} = \int_{\hat{K}} \hat{z} J_K \, d\hat{K} \Big/ \int_{\hat{K}} J_K \, d\hat{K}.$$

Now let ψ be the solution of

$$\begin{cases} -\Delta\psi = (\hat{z} - \hat{z}_i) J_K & \text{in } \hat{K}, \\ \dfrac{\partial \psi}{\partial n} = 0 & \text{on } \partial\hat{K}, \\ \int_{\hat{K}} \psi \, d\hat{K} = 0. \end{cases} \tag{4.7}$$

The existence of a unique solution ψ of this problem is guaranteed by the fact that $\int_{\hat{K}} (\hat{z} - \hat{z}_i) J_K \, d\hat{K} = 0$. We find, using (4.7) and Green's formula, that

$$\begin{aligned}\int_K r_h (z - z_i) \, dK &= \int_{\hat{K}} \hat{r}_h (\hat{z} - \hat{z}_i) J_K \, d\hat{K} \\ &= \int_{\hat{K}} \hat{r}_h (-\Delta\psi) \, d\hat{K} = \int_{\hat{K}} \nabla \hat{r}_h \cdot \nabla\psi \, d\hat{K}.\end{aligned} \tag{4.8}$$

The solution ψ of problem (4.7) may be represented in terms of the associated Green's function and the data $\eta := (\hat{z} - \hat{z}_i) J_K$ in the usual way. Differentiating this representation and using well-known bounds for the derivatives of the logarithmic singularity of the Green's function for a two-dimensional problem, we get, for any y in $\hat{K}$,

$$|\nabla\psi(y)| \leq C \int_{\hat{K}} \frac{|\eta(s)|}{|y - s|} ds,$$

and consequently with $q(y) = |x - F_K(y)|$ (cf. (5.4)),

$$q(y)\,|\nabla\psi(y)| \le Cq(y) \int_{\hat K} \frac{ds}{|y-s|\,q(s)}\,\|q\eta\|_{L_\infty(\hat K)} \le C\,\|q\eta\|_{L_\infty(\hat K)}\,.$$

Thus, also using (4.1) we have

$$\begin{aligned} \|q\nabla\psi\|_{L_\infty(\hat K)} &\le C\,\|q\,(\hat z - \hat z_i)\,J_K\|_{L_\infty(\hat K)} \\ &\le Ch_K^2\,\||x-\cdot|\,(z-z_i)\|_{L_\infty(K)} \\ &\le Ch_K^2\,\||x-\cdot|\,z\|_{L_\infty(K)}\,. \end{aligned} \tag{4.9}$$

Here, in the last step, we have used the fact that $z_i|_K$ is constant and

$$\begin{aligned} \||x-\cdot|\,z_i\|_{L_\infty(K)} &= \||x-\cdot|\|_{L_\infty(K)} \left| \int_{\hat K} \hat z J_K\,d\hat K \Big/ \int_{\hat K} J_K\,d\hat K \right| \\ &\le \||x-\cdot|\|_{L_\infty(K)}\,\|q\hat z\|_{L_\infty(\hat K)} \left\|\frac{J_K}{q}\right\|_{L_1(\hat K)} \Big/ \int_{\hat K} J_K\,d\hat K \\ &\le C\,\||x-\cdot|\,z\|_{L_\infty(K)}\,. \end{aligned}$$

Furthermore, we have that

$$\left\|\frac{\nabla \hat r_h}{q}\right\|_{L_1(\hat K)} \le Ch_K^{-1} \left\|\frac{\nabla_h r_h}{|x-\cdot|}\right\|_{L_1(K)}. \tag{4.10}$$

Now (4.8)–(4.10) give

$$\begin{aligned} \left|\int_K r_h\,(z-z_i)\,dK\right| &\le \left\|\frac{\nabla \hat r_h}{q}\right\|_{L_1(\hat K)} \|q\nabla\psi\|_{L_\infty(\hat K)} \\ &\le Ch_K \left\|\frac{\nabla_h r_h}{|x-\cdot|}\right\|_{L_1(K)} \||x-\cdot|\,z\|_{L_\infty(K)} \quad \forall K \in \mathcal{C}_h. \end{aligned} \tag{4.11}$$

Clearly we also have that

$$\left|\int_K r_h\,(z-z_i)\,dK\right| \le \left\|\frac{r_h}{|x-\cdot|}\right\|_{L_1(K)} \||x-\cdot|\,z\|_{L_\infty(K)} \quad \forall K \in \mathcal{C}_h.$$

Together our estimates now show that

$$\left|\int_K r_h\,(z-z_i)\,dK\right| \le \min\left\{ \left\|\frac{r_h}{|x-\cdot|}\right\|_{L_1(K)}, C\left\|\frac{h\nabla_h r_h}{|x-\cdot|}\right\|_{L_1(K)} \right\} \||x-\cdot|\,z\|_{L_\infty(K)}\,.$$

By summation over all K the desired result now follows. □

Remark 4.1. Obviously, one can have that $|r_h| \ll h|\nabla_h r_h|$, for instance, if f is highly oscillatory. Locally one can also have that $h|\nabla_h r_h| \ll |r_h|$. On the other hand, it is not clear that the latter inequality can hold globally. In practice, it is probably enough to consider the simple estimate (set $z_i \equiv 0$ above)

$$\left\|\frac{\varphi - \varphi_h}{|x-\cdot|}\right\|_{L_1(\Gamma)} \le C \left\|\frac{r_h}{|x-\cdot|}\right\|_{L_1(\Gamma)}.$$

Note also that computing $\nabla_h r_h$ is somewhat more involved than just computing r_h. In our numerical tests below we have used an algorithm based on the latter simplified estimate.

We now return to the proof of the stability Lemma 4.2.

Proof of Lemma 4.2. Recall the dual equation

$$z - T^* z = \frac{w}{|x - \cdot|} \quad \text{a.e. on } \Gamma,$$

where

$$(T^* z)(y) = \frac{1}{2\pi} \int_\Gamma z(s) \frac{\partial}{\partial n_s} \frac{1}{|y - s|} d\Gamma(s).$$

Let $\bar{y} \in \Gamma$ and M be given by

$$M = \sup_{y \in \Gamma} |x - y| \, |z(y)| = |x - \bar{y}| \, |z(\bar{y})| .$$

Furthermore, let $d > 0$ be the largest constant such that

$$Cs \left(|\ln s| + 1 \right) \leq \frac{1}{2} \quad \text{for } 0 < s \leq d,$$

where C is the final constant (only depending on Γ) in the estimate

$$\begin{aligned} |T^* z(\bar{y})| &\leq \frac{1}{2\pi} \int_\Gamma |z(s)| \left| \frac{\partial}{\partial n_s} \frac{1}{|\bar{y} - s|} \right| d\Gamma(s) \\ &\leq \frac{M}{2\pi} \int_\Gamma \frac{1}{|x - s|} \frac{C}{|\bar{y} - s|} d\Gamma(s) \\ &\leq CM \left(\left| \ln |x - \bar{y}| \right| + 1 \right). \end{aligned}$$

Then, if $|x - \bar{y}| < d$, we have that

$$\begin{aligned} M = |x - \bar{y}| \, |z(\bar{y})| &\leq |w(\bar{y})| + |x - \bar{y}| \, |(T^* z)(\bar{y})| \\ &\leq 1 + CM |x - \bar{y}| \left(\left| \ln |x - \bar{y}| \right| + 1 \right) \leq 1 + \frac{M}{2}. \end{aligned} \tag{4.12}$$

On the other hand, if $|x - \bar{y}| \geq d$, we set $M_B = \sup_{y \in B} |z(y)|$, where

$$B = \{ y \in \Gamma : |y - \bar{y}| \leq d' \}, \qquad d' \leq \frac{d}{2}.$$

We have that

$$\begin{aligned} M_B &= \sup_{y \in B} \frac{|x - y| \, |z(y)|}{|x - y|} \\ &\leq \frac{|x - \bar{y}| \, |z(\bar{y})|}{\min_{y \in B} |x - y|} \leq \frac{2}{d} \max_{y \in \Gamma} |x - y| \, |z(\bar{y})| . \end{aligned}$$

Moreover,

$$|z(\bar{y})| \leq \left| \frac{w(\bar{y})}{|x - \bar{y}|} \right| + \frac{1}{2\pi} \int_\Gamma |z(s)| \left| \frac{\partial}{\partial n_s} \frac{1}{|\bar{y} - s|} \right| d\Gamma(s)$$

$$\le \frac{1}{|x-\bar{y}|} + \frac{1}{2\pi}\left(\int_B + \int_{\Gamma\setminus B}\right)|z(s)|\left|\frac{\partial}{\partial n_s}\frac{1}{|\bar{y}-s|}\right| d\Gamma(s)$$
$$\le \frac{1}{|x-\bar{y}|} + \frac{M_B}{2\pi}\int_B \frac{C}{|\bar{y}-s|}d\Gamma(s) + \frac{C}{2\pi}\|z\|_{L_1(\Gamma\setminus B)} \max_{s\in\Gamma\setminus B}\left|\frac{\partial}{\partial n_s}\frac{1}{|\bar{y}-s|}\right|$$
$$\le \frac{1}{|x-\bar{y}|} + CM_B d' + C\|z\|_{L_1(\Gamma)}\frac{1}{d'}.$$

For d' sufficiently small this gives

$$|z(\bar{y})| \le \frac{1}{|x-\bar{y}|} + \frac{1}{2}|z(\bar{y})| + \frac{C}{d'}\|z\|_{L_1(\Gamma)},$$

and hence

$$|x-\bar{y}|\,|z(\bar{y})| \le 2 + \frac{C|x-\bar{y}|}{d'}\|z\|_{L_1(\Gamma)}. \tag{4.13}$$

We now claim that there is a constant C independent of x such that

$$\|z\|_{L_1(\Gamma)} \le C. \tag{4.14}$$

In fact, since $\|w/|x-\cdot|\|_{L_1(\Gamma)} \le C$, this follows from the more general estimate

$$\|z\|_{L_p(\Gamma)} \le C\left\|\frac{w}{|x-\cdot|}\right\|_{L_p(\Gamma)}, \quad 1 \le p \le \infty. \tag{4.15}$$

For x bounded away from infinity, the desired stability estimate of Lemma 4.2 now follows from (4.12) and (4.13)–(4.14). For $|x|$ large (x away from Γ), the desired estimate follows directly from (4.15) with $p=\infty$. Thus it suffices to prove (4.15). We then note that $(I-T^*)z = w/|x-\cdot|$, so that (4.15) follows from the fact that $I-T^*$ has a bounded inverse as an operator on $L_p(\Gamma), 1\le p\le\infty$. In fact, since the integral kernels of T and T^* are weakly singular, both operators are compact on $L_p(\Gamma)$. Thus if we can show that $I-T^*$ is one-to-one it follows from Fredholm theory that it is also onto; consequently $I-T^*$ is invertible with $(I-T^*)^{-1} : L_p(\Gamma)\to L_p(\Gamma)$ bounded and (4.15) follows.

To prove that $I-T^*$ is one-to-one it suffices to show that $I-T$ is one-to-one (see, e.g., [12], Theorem 4.25), i.e., we want to prove that

$$(I-T)\varphi = 0 \quad \text{implies } \varphi = 0.$$

We recall that with u defined by

$$u(x) = \frac{1}{4\pi}\int_\Gamma \varphi(y)\frac{1}{|x-y|}d\Gamma(y),$$

we have that $(I-T)\varphi = -2g$ implies

$$\frac{\partial u^e}{\partial n} = g.$$

Thus, if $(I-T)\varphi = 0$ (i.e., $g=0$) we have that

$$\begin{cases} \Delta u = 0 & \text{in } \Omega', \\ \dfrac{\partial u^e}{\partial n} = 0 & \text{on } \Gamma, \\ u \to 0 & \text{as } |x|\to\infty, \end{cases}$$

which implies that $u = 0$ in Ω'. In particular, u vanishes on Γ, and since u is also harmonic in the interior of Ω it follows that $u = 0$ in all of R^3. Consequently, $\varphi = (\partial u^e/\partial n) - (\partial u^i/\partial n) = 0$. We have thus demonstrated that $I - T$ is one-to-one and the proof is complete. □

Remark 4.2. Implementation of the adaptive algorithm outlined in §3 involves, in particular, finding an appropriate numerical value (the smaller the better) for the stability constant C_{stab} from Lemma 4.2. One could try to backtrack C_{stab} through the proof of Lemma 4.2, but this procedure would probably lead to a rather pessimistic (too large) value. A more practical approach would be to try to estimate C_{stab} from an approximate solution of the dual problem (4.5). One complication then is that the data in this problem depends on the unknown error; recall from the proof of Theorem 4.1 that we consider (4.5) with $w = \text{sign}(\varphi - \varphi_h)$ where φ is unknown. In the present situation it is reasonable to believe that $\||x - \cdot|z\|_{L_\infty(\Gamma)}$ should be more or less independent of w for $|w| = 1$ (cf. below and the corresponding discussion in [6]). It would then suffice to solve the dual problem numerically with one or a few different choices of w to get a reasonable estimate of the constant C_{stab}. In fact, it may not even be necessary to estimate C_{stab} at all since $\||x - \cdot|z\|_{L_\infty(\Gamma)}$ should be close to one for small h. To justify this statement we consider for simplicity the two-dimensional counterpart of our given potential problem with Γ the unit circle. In this case it is possible to solve the dual problem analytically. We find that $T^*\varphi(y) = -1/2\pi \int_\Gamma \varphi(s)\, ds =$ constant and from this that

$$z(y) = \frac{1}{2\pi}\,|\log(|x-y|)|\,w(y) - \frac{1}{8\pi^2}\int_\Gamma w(s)\,|\log(|x-s|)|\,ds.$$

Here, for $w = \text{sign}(\varphi - \varphi_h)$, the integral term should be much smaller than the first term because w is oscillatory, and thus we expect to have $2\pi z(y)/|\log\,|x-y|| \approx w(y)$ for small h in this case. We believe that this should not be specifically related to the particular situation we have considered here but hold fairly much in general as long as the kernel is weakly singular and Γ is smooth.

In the general case of an algorithm based on (3.8), we also need to find an appropriate numerical value for C_{int}. This constant is basically an interpolation error constant (although this is not quite transparent from the derivation; cf. also [6]). A reasonable estimate for C_{int} can thus be obtained by seeking the smallest possible constant for which $\|\hat{z} - \hat{z}_i\|_{L_1(\hat{K})} \le C\|\nabla\hat{z}\|_{L_1(\hat{K})}$, where $\hat{z}_i$ is the mean value of $\hat{z}$, i.e., for $q = 2$ one may put $C_{\text{int}} = 1/2$ whereas for $q = 3$ (and triangular elements) simple calculations indicate that it should be possible to put $C_{\text{int}} = 1/4$. It should be clear that these values are based on somewhat heuristic arguments; to obtain a full proof value we have to backtrack the constant through the proof of Lemma 4.1, which may lead to a somewhat more pessimistic estimate.

Remark 4.3. Above we have considered control of $u - u_h$ at a particular point x through control of $\varphi - \varphi_h$ in a weighted L_1-norm. A posteriori estimates for error control in other norms can be obtained similarly. For example, let us consider maximum norm control of $u - u_h$, i.e., as our computational goal, to find an approximate solution u_h of (1.1) such that

$$\|u - u_h\|_{L_\infty(\Omega')} \le \delta. \tag{4.16}$$

From (1.2) and (2.3) we obtain for any $p \in (2, \infty]$ and $q = p/(p-1)$,

$$\|u - u_h\|_{L_\infty(\Omega')} \le \frac{1}{4\pi}\sup_{x\in\Omega'}\left\|\frac{1}{|x-\cdot|}\right\|_{L_q(\Gamma)} \|\varphi - \varphi_h\|_{L_p(\Gamma)} \le C_p\,\|\varphi - \varphi_h\|_{L_p(\Gamma)}\,.$$

In order to obtain an estimate for $\|\varphi-\varphi_h\|_{L_p(\Gamma)}$, we consider now for a given $w \in L_q(\Gamma)$ the dual problem

$$z - T^* z = w \quad \text{a.e. on } \Gamma.$$

We recall from the discussion in the proof of Lemma 4.2 that $I - T^*$ has a bounded inverse on $L_q(\Gamma)$, so that, in particular,

$$\|z\|_{L_q(\Gamma)} \leq C \|w\|_{L_q(\Gamma)} .$$

We thus obtain with $e = \varphi - \varphi_h$,

$$\begin{aligned}\int_\Gamma e\ w\ ds = \int_\Gamma e\,(z - T^* z)\,ds = \int_\Gamma (e - Te) z\ ds = \int_\Gamma r_h z\ ds \\ \leq \|r_h\|_{L_p(\Gamma)} \|z\|_{L_q(\Gamma)} \leq C \|r_h\|_{L_p(\Gamma)} \|w\|_{L_q(\Gamma)} ,\end{aligned}$$

and consequently, by duality,

$$\|e\|_{L_p(\Gamma)} \leq C \|r_h\|_{L_p(\Gamma)} .$$

If we summarize, we thus have

$$\|u - u_h\|_{L_\infty(\Omega')} \leq C_p \|r_h\|_{L_p(\Gamma)} , \qquad p \in (2, \infty].$$

From this estimate we can easily set up an adaptive scheme similar to that in §3 for obtaining (4.16).

5. Efficiency and an optimal a priori error estimate. In this section we shall verify the efficiency of the constructed adaptive algorithm. As a by-product we will also get an optimal a priori error estimate for the finite element method under consideration. Our main result is the following.

THEOREM 5.1. *There is a constant C such that for h sufficiently small,*

$$\left\| \frac{r_h}{|x - \cdot|} \right\|_{L_1(\Gamma)} \leq C \left\| \frac{h \nabla \varphi}{|x - \cdot|} \right\|_{L_1(\Gamma)} .$$

This estimate shows, in particular, that the a posteriori error bound in Theorem 4.1 is of optimal order and thus indicates that our adaptive method will be *efficient* (cf. the discussion in §3 above). Combining the a posteriori error estimate of Theorem 4.1 with Theorem 5.1 we get as a corollary the following a priori error estimate.

THEOREM 5.2. *There is a constant C such that for h sufficiently small,*

$$\left\| \frac{\varphi - \varphi_h}{|x - \cdot|} \right\|_{L_1(\Gamma)} \leq C \left\| \frac{h \nabla \varphi}{|x - \cdot|} \right\|_{L_1(\Gamma)} .$$

To prove Theorem 5.1 we shall need a few technical lemmas.

LEMMA 5.1. *The residual r_h can be represented as*

$$r_h = (I - P_h)\, T\,(\varphi - \varphi_h) - (I - P_h)\,\varphi,$$

where P_h is the usual L_2-projection onto V_h.

Proof. The variational equation

$$(\varphi_h, v) - (T\varphi_h, v) = (f, v) \quad \forall v \in V_h$$

can equivalently be written in the form

$$\varphi_h = P_h T \varphi_h + P_h f.$$

Thus

$$\begin{aligned} r_h &= \varphi_h - T\varphi_h - f = (P_h - I)(f + T\varphi_h) \\ &= (P_h - I)(\varphi - T\varphi + T\varphi_h) \\ &= (I - P_h) T(\varphi - \varphi_h) - (I - P_h)\varphi, \end{aligned}$$

which is the desired result. □

Now we give a weighted L_1 estimate for the L_2-projection, which we shall use to estimate each term of the above representation.

LEMMA 5.2. *There is a constant C such that*

$$\left\| \frac{P_h\psi}{|x - \cdot|} \right\|_{L_1(\Gamma)} \leq C \left\| \frac{\psi}{|x - \cdot|} \right\|_{L_1(\Gamma)}.$$

Proof. For the L_2-projection P_h we have

$$(P_h\psi, v) = (\psi, v) \quad \forall v \in V_h.$$

Thus, setting $v \equiv 1$ on $K, v \equiv 0$ elsewhere, we get

$$P_h\psi|_K m(K) = \int_K \psi \, d\Gamma,$$

where $m(K) = \mathcal{O}(h_K^2)$ is the surface measure of K. Using this we get

$$\int_K \frac{|P_h\psi|}{|x - \cdot|} d\Gamma = \left| P_h\psi|_K \right| \int_K \frac{1}{|x - \cdot|} d\Gamma \leq \frac{1}{m(K)} \int_K |\psi| \, d\Gamma \int_K \frac{1}{|x - \cdot|} d\Gamma.$$

Now,

$$\int_K |\psi| \, d\Gamma \leq \max_K |x - \cdot| \int_K \frac{|\psi|}{|x - \cdot|} d\Gamma$$

and, for $\operatorname{dist}(x, K) \geq Ch_K, C > 0$,

$$\int_K \frac{1}{|x - \cdot|} d\Gamma \leq \frac{m(K)}{\min_K |x - \cdot|} \leq C \frac{m(K)}{\max_K |x - \cdot|},$$

whereas, for $\operatorname{dist}(x, K) = \mathcal{O}(h_K)$, $\int_K 1/|x - \cdot| \, d\Gamma = \mathcal{O}(h_K)$, and therefore the proof is complete. □

We shall also need the following error estimate in L_1.

LEMMA 5.3. *There is a constant C such that for h sufficiently small,*

$$\|\varphi - \varphi_h\|_{L_1(\Gamma)} \leq C \|h\nabla\varphi\|_{L_1(\Gamma)}.$$

Proof. From the equations

$$(I - T)\varphi = f$$

and

$$(I - P_h T)\varphi_h = P_h f,$$

we have that

$$(T - P_h T)(\varphi - \varphi_h) = (I - P_h)(f + T\varphi) = (I - P_h)\varphi.$$

Thus if we can show that

$$\| (I - P_h T)^{-1} \|_{L_1(L_1)} \leq C, \tag{5.1}$$

then we will have

$$\|\varphi - \varphi_h\|_{L_1(\Gamma)} \leq C \|(I - P_h)\varphi\|_{L_1(\Gamma)} \leq C \|h\nabla\varphi\|_{L_1(\Gamma)},$$

and the proof will be complete. □

Thus it only remains to prove (5.1).

LEMMA 5.4. *There is a constant C such that for h sufficiently small,*

$$\|(I - T_h)^{-1}\|_{L_1(L_1)} \leq C,$$

where $T_h = P_h T$.

Proof. In the proof of Lemma 4.2 we demonstrated that

$$\|(I - T)^{-1}\|_{L_1(L_1)} \leq C. \tag{5.2}$$

Assume now that

$$(I - T_h)\psi = g.$$

We then have

$$(I - T)\psi = (I - T_h)\psi + (T_h - T)\psi = g + (T_h - T)\psi,$$

and using (5.2) we obtain

$$\|\psi\|_{L_1} \leq C\left(\|g\|_{L_1} + \|(T_h - T)\psi\|_{L_1}\right). \tag{5.3}$$

Recalling the L_1-stability of the L_2-projection, we have that

$$\|(T - T_h)\psi\|_{L_1(\Gamma)} = \|(I - P_h)T\psi\|_{L_1(\Gamma)} \leq C\|T\psi - \widetilde{T\psi}\|_{L_1(\Gamma)},$$

where $\widetilde{T\psi} \in V_h$ is an interpolant of $T\psi$ determined by $\widetilde{T\psi}|_K = T\psi(x_k)$ for some $x_k \in K$.

On the other hand, with $k(x, y) = (1/2\pi)(\partial/\partial n_x)(1/|x - \cdot|)$,

$$\begin{aligned}\int_\Gamma |T\psi - \widetilde{T\psi}| d\Gamma(x) &= \sum_i \int_{K_i} |T\psi(x) - \widetilde{T\psi}(x_i)| d\Gamma(x) \\ &\leq \sum_i \int_{K_i} \left| \int_\Gamma \psi(y)\left[k(x, y) - k(x_i, y)\right] d\Gamma(y) \right| d\Gamma(x) \\ &\leq \int_\Gamma |\psi(y)| \left(\sum_i \int_{K_i} |k(x, y) - k(x_i, y)|\, d\Gamma(x) \right) d\Gamma(y).\end{aligned}$$

For a fixed y we let $B_d(y) = \{x : |x-y| \leq d\}$, where d is a sufficiently small and fixed positive constant. Recall that $k(x,y) = \mathcal{O}(|x-y|^{-1})$; hence for $x \in B_d(y)$ using polar coordinates we have that

$$\begin{aligned}\sum_i \int_{K_i \cap B_d(y)} |k(x,y) - k(x_i,y)|\, d\Gamma(x) &\leq \sum_i \int_{K_i \cap B_d(y)} (|k(x,y)| + |k(x_i,y)|)\, d\Gamma(x) \\ &= \mathcal{O}(d),\end{aligned}$$

whereas for other points x, i.e., $x \in \Gamma / B_d(y)$, we have that

$$\begin{aligned}&\sum_i \int_{K_i \setminus B_d(y)} |k(x,y) - k(x_i,y)|\, d\Gamma(x) \\ &\quad \leq \int_{\Gamma \setminus B_d(y)} h(x) \frac{C}{|x-y|^2} d\Gamma(x) \leq C\bar{h}\left(|\ln d| + 1\right),\end{aligned}$$

where $\bar{h} = \max_\Gamma\ h$. Since d is assumed to be sufficiently small, we get from (5.3) for $\bar{h}$ sufficiently small that

$$\|\psi\|_{L_1} \leq C\, \|g\|_{L_1} + C\left(d + \bar{h}|\ln d| + \bar{h}\right) \int_\Gamma |\psi|\, d\Gamma \leq C\, \|g\|_{L_1} + \frac{1}{2}\, \|\psi\|_{L_1}.$$

This gives the desired result. □

We may now give the following.

Proof of Theorem 5.1. Recall from Lemma 5.1 the representation of the residual r_h:

$$r_h = (I - P_h)\, Te - (I - P_h)\, \varphi.$$

Let $\tilde{\varphi} \in V_h$ be an interpolant of φ. Then, since

$$(I - P_h)\, \varphi = I\,(\varphi - \tilde{\varphi}) + P_h\,(\tilde{\varphi} - \varphi),$$

we have, using Lemma 5.2,

$$\left\| \frac{(I-P_h)\,\varphi}{|x-\cdot|} \right\|_{L_1(\Gamma)} \leq \left\| \frac{\varphi - \tilde{\varphi}}{|x-\cdot|} \right\|_{L_1(\Gamma)} + \left\| \frac{P_h\,(\varphi - \tilde{\varphi})}{|x-\cdot|} \right\|_{L_1(\Gamma)} \leq C \left\| \frac{\varphi - \tilde{\varphi}}{|x-\cdot|} \right\|_{L_1(\Gamma)}.$$

Moreover,

$$\left\| \frac{(I-P_h)\,Te}{|x-\cdot|} \right\|_{L_1(\Gamma)} \leq C \left\| \frac{Te}{|x-\cdot|} \right\|_{L_1(\Gamma)}.$$

Here

$$\left\| \frac{Te}{|x-\cdot|} \right\|_{L_1(\Gamma)} = \int_\Gamma \frac{|Te(y)|}{|x-y|} d\Gamma(y) = \int_\Gamma \frac{1}{|x-y|} \left| \int_\Gamma e(z) \frac{\partial}{\partial n_y} \frac{1}{|y-z|} d\Gamma(z) \right| d\Gamma(y).$$

Now, since $(\partial/\partial n_y)(1/|y-z|) = \mathcal{O}(1/|y-z|)$, we obtain, using Fubini's Theorem,

$$\begin{aligned}\left\| \frac{Te}{|x-\cdot|} \right\|_{L_1(\Gamma)} &\leq C \int_\Gamma |e(z)| \int_\Gamma \frac{1}{|x-y|} \cdot \frac{1}{|y-z|} d\Gamma(y)\ d\Gamma(z) \\ &\leq C \int_\Gamma |e(z)|\, (|\ln|x-z|| + 1)\, d\Gamma(z).\end{aligned} \tag{5.4}$$

Finally we have that

$$\begin{aligned}\left\|\frac{e}{|x-\cdot|}\right\|_{L_1(\Gamma)}+\left\|\frac{r_h}{|x-\cdot|}\right\|_{L_1(\Gamma)} &\leq C\left\|\frac{r_h}{|x-\cdot|}\right\|_{L_1(\Gamma)}\\ &\leq C\left\|\frac{h\nabla\varphi}{|x-\cdot|}\right\|_{L_1(\Gamma)}+C\int_\Gamma |e(z)|\,(|\ln|x-z||+1)\,d\Gamma(z)\\ &\leq C\left\|\frac{h\nabla\varphi}{|x-\cdot|}\right\|_{L_1(\Gamma)}+\frac{1}{2}\left\|\frac{e}{|x-\cdot|}\right\|_{L_1(\Gamma)}+C\,\|e\|_{L_1(\Gamma)}.\end{aligned}$$

It follows that

$$\left\|\frac{r_h}{|x-\cdot|}\right\|_{L_1(\Gamma)}\leq C\left(\left\|\frac{h\nabla\varphi}{|x-\cdot|}\right\|_{L_1(\Gamma)}+\|e\|_{L_1(\Gamma)}\right).$$

Using Lemma 5.3 we have that

$$\|e\|_{L_1(\Gamma)}\leq C\,\|h\nabla\varphi\|_{L_1(\Gamma)}\leq C\left\|\frac{h\nabla\varphi}{|x-\cdot|}\right\|_{L_1(\Gamma)},$$

and consequently,

$$\left\|\frac{r_h}{|x-\cdot|}\right\|_{L_1(\Gamma)}\leq C\left\|\frac{h\nabla\varphi}{|x-\cdot|}\right\|_{L_1(\Gamma)},$$

which proves the efficiency (Theorem 5.1) and also the a priori error estimate

$$\left\|\frac{e}{|x-\cdot|}\right\|_{L_1(\Gamma)}\leq C\left\|\frac{h\nabla\varphi}{|x-\cdot|}\right\|_{L_1(\Gamma)}$$

of Theorem 5.2. □

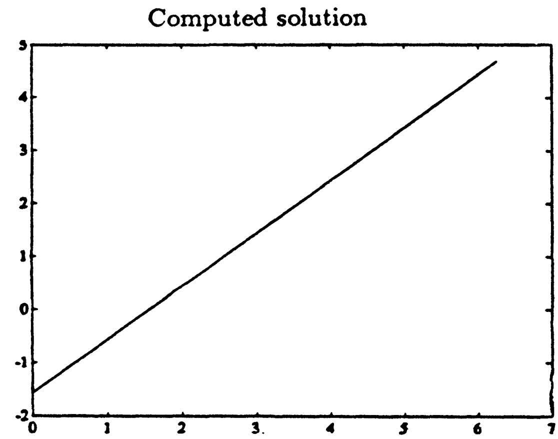
6. Numerical experiments. For simplicity we consider here the two-dimensional counterpart of problem (1.1) and seek, for a simple closed curve Γ in the plane, the single layer potential

$$u_h(x)=\frac{1}{2\pi}\int_\Gamma \varphi_h(y)\log\left(\frac{1}{|x-\cdot|}\right)d\Gamma(y),$$

where $\varphi_h\in V_h$ is the solution of (2.2) now with

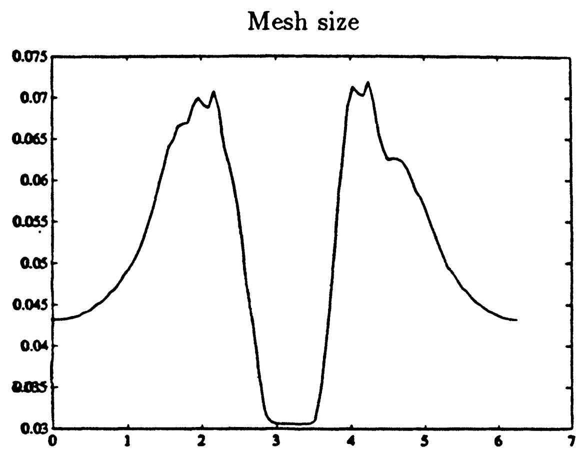
$$T\varphi(y)=\frac{1}{\pi}\int_\Gamma \varphi(s)\frac{\partial}{\partial n_y}\log\left(\frac{1}{|y-s|}\right)d\Gamma(s),$$

and V_h consisting of all piecewise constant functions on a partition of Γ into curve segments of length proportional to h. As before, x is a fixed point in which we want to control the error in u_h and δ is the given error tolerance. The counterpart of the error estimate (3.3) for the present two-dimensional case shows that the desired error control will follow if $C_{\text{stab}}\|w_x r_h\|_{L_1(\Gamma)}\leq\delta$, where $r_h=\varphi_h-T\varphi_h-f$ is the residual and $w_x(y)=(1/2\pi)|\log(1/|x-y|)|$ is the weight function originating from the single layer potential representations of u and u_h in two dimensions. Unless otherwise stated the tolerance δ was set to 0.01. The stability constant C_{stab} was put to one (cf. Remark 4.2).



$f = v.$

(a)



$x = (-1, 0) \quad f = v \quad \delta = 0.01$

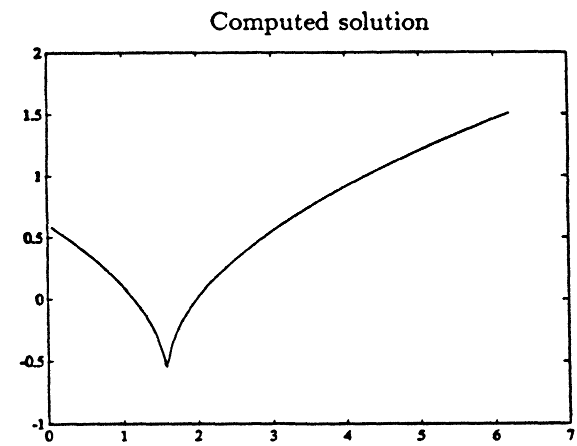
$n = 130 \quad ex = 0.0069 \quad e = 0.0070.$

(b)

FIG. 1.

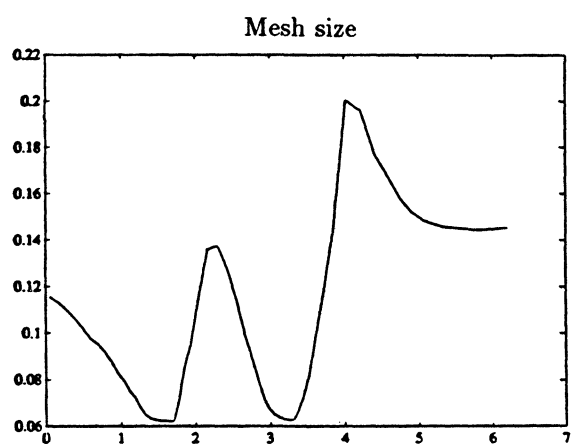
In the first set of experiments discussed in Examples 1–3 below, we have taken Γ to be the unit circle, parameterized in the usual way using the polar angle $v, 0 \leq v \leq 2\pi$. In this case it is possible to solve the problem analytically (cf. Remark 4.2) so that we may compare computed errors with exact errors. The exact error will be denoted by ex and the estimated error by e. A comparison of the two on the accepted meshes (see Figs. 1–3) indicates that our adaptive method is both reliable and efficient. Note, however, that the difference between e and ex may be larger in the early stages of the adaptive process when the meshes are not yet properly refined. For definiteness, note also that by "the exact error" we here mean $\|(\varphi - \varphi_h)w_x\|_{L_1(\Gamma)}$ (rather than $|(u - u_h)(x)|$).

In each case the algorithm started from a uniform partition with mesh size of order one. The number of iterations (successive remeshings) required to reach the



$f = \sqrt{|v - \pi/2|}$.

(a)



$x = (-1, 0)$ $f = \sqrt{|v - \pi/2|}$ $\delta = 0.01$
$n = 60$ $ex = 0.0067$ $e = 0.0067$.

(b)

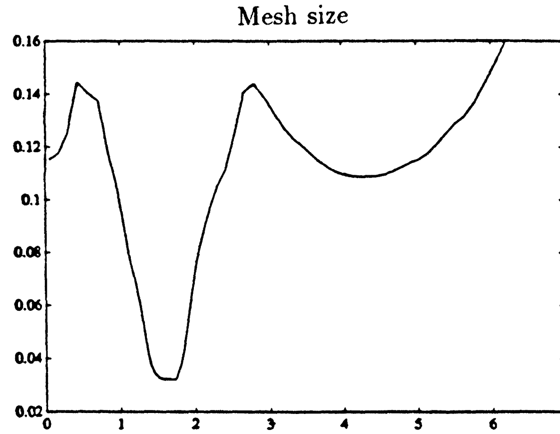
FIG. 2.

desired error control ranged from five to thirteen. The number of elements on the final mesh is denoted by n.

In our first example we have isolated the effect on the mesh size due to the singularities in the weight function w_x. In the following examples we will see combined effects of the singularities in w_x and in the solution.

Example 1. We first consider the case $f(v) = v$ and $x = (-1, 0)$ with a singularity in w_x at $v = \pi$. Note that the discontinuity in the data f at (1,0) is not important here, because it is located at a mesh point and we use piecewise *discontinuous* polynomial approximation. Thus, here the solution is (uniformly) smooth on each mesh subinterval and only the singularities in w_x will influence the variation in the local mesh size.

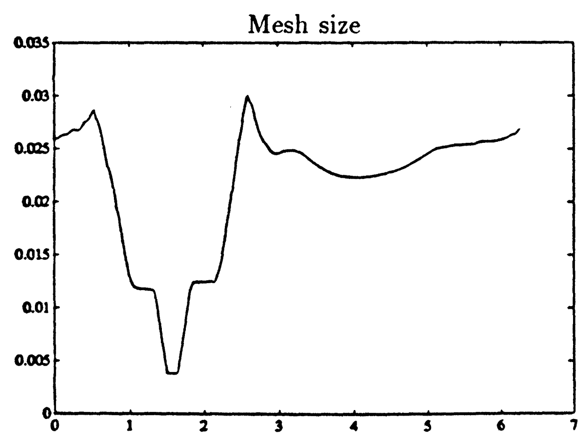
Figure 1a shows the computed solution φ_h. Note that, for convenience, we have not



$x = (0,1) \quad f = \sqrt{|v - \pi/2|} \quad \delta = 0.01$

$n = 66 \quad ex = 0.0090 \quad e = 0.0095.$

(c)



$x = (0,1) \quad f = \sqrt{|v - \pi/2|} \quad \delta = 0.002$

$n = 358 \quad ex = 0.0015 \quad e = 0.0015.$

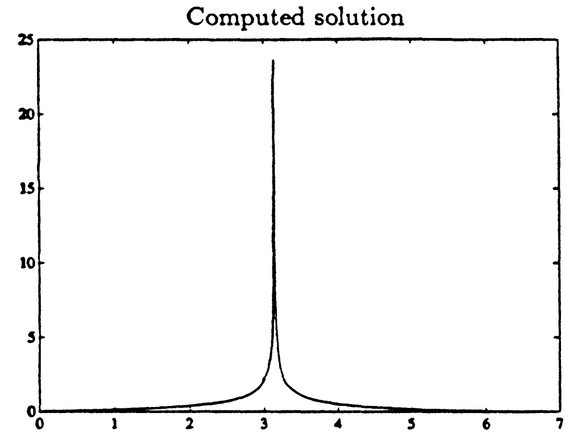
(d)

FIG. 2. (*continued*).

plotted the piecewise constant function φ_h but rather its piecewise linear interpolant. Figure 1b shows the corresponding mesh size h similarly represented by a piecewise linear interpolant as a function of v. We note that the mesh has been refined near $v = \pi$ to compensate for the singularity in the weight function $w_x = w_x(y(v))$ at $v = \pi$. (One can also see that the mesh size is comparatively large near $v = 2\pi/3$ and $v = 4\pi/3$ due to the fact that the weight function w_x happens to be small near these two points.)

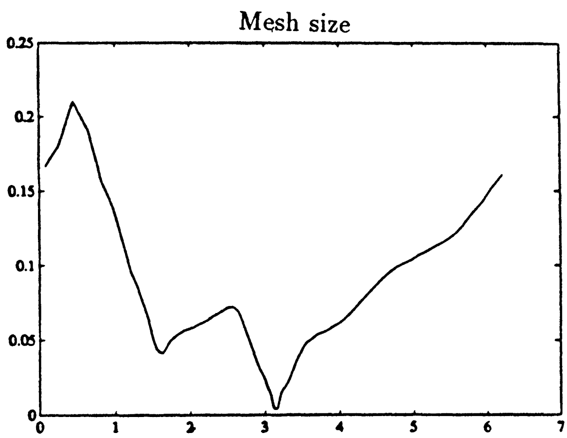
Example 2. We shall now consider two examples with (genuine) singularities also in the data f to see the combined effect on the local mesh size of adaptation with respect to singularities both in w_x and f.

We first take $f(v) = \sqrt{|v - \pi/2|}$. Figure 2a shows the computed solution. We note in particular the singularity at $v = \pi/2$. Figures 2b–c show the mesh function h



(a)

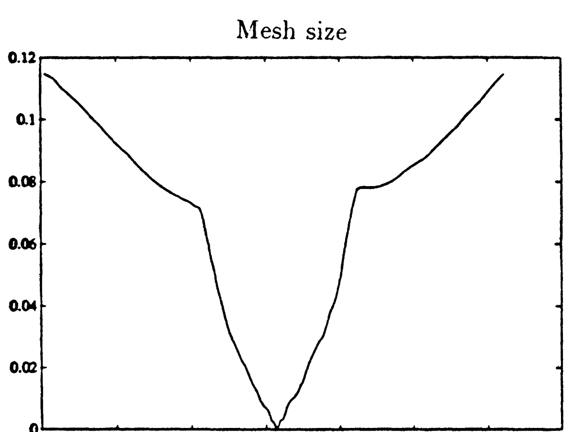
$$f = \frac{1}{\sqrt{|v-\pi|+0.001}}.$$



(b)

$x = (0,1) \quad f = \frac{1}{\sqrt{|v-\pi|+0.001}} \quad \delta = 0.01$

$n = 109 \qquad ex = 0.0076 \qquad e = 0.0086.$



(c)

$x = (-1,0) \quad f = \frac{1}{\sqrt{|v-\pi|+0.001}} \quad \delta = 0.01$

$n = 290 \qquad ex = 0.0097 \qquad e = 0.0100.$

FIG. 3.

for two different locations of the point x. In Fig. 2b we have put $x = (-1, 0)$. We note the refinement near the singularity in f at $v = \pi/2$ and the refinement near $v = \pi$ due to the singularity in w_x. In Fig. 2c we have chosen $x = (0, 1)$ so that we now have two singularities to account for at $v = \pi/2$. Additional refinement is now required to compensate for the combined singularities of f and w_x. Figure 2d shows the mesh size for the same problem setting the error tolerance somewhat lower ($\delta = 0.002$). We now see two levels of refinements reflecting the fact that two different singularities are involved in the refinement process.

Example 3. This example is similar to Example 2. Here we consider the case $f(v) = 1/\sqrt{|v - \pi| + 0.001}$. Figure 3a shows the computed solution. In Fig. 3b we have taken $x = (0, 1)$. We note the refinement near $v = \pi$ caused by the singularity in f and the somewhat less pronounced refinement near $v = \pi/2$ caused by the singularity in w_x. In Fig. 3c we have taken $x = (-1, 0)$ and again we see that the mesh has been additionally refined near the point of combined singularities.

In our second set of experiments we shall take into account also the effect of Γ being nonsmooth, and will thus have up to three sources of singularities influencing the adaptively chosen local mesh size. We consider the case when Γ consists of the two circular arcs bounding the lens-shaped region:

$$\Omega = \left\{ (x_1, x_2) : 0.5 - \sqrt{1 - x_1^2} \leq x_2 \leq -0.5 + \sqrt{1 - x_1^2}, \sqrt{3}/2 \leq x_1 \leq \sqrt{3}/2 \right\}.$$

Note that our theoretical results do not cover this case, since in our analysis we have assumed Γ to be smooth. We plan to return to this problem for a detailed study in a forthcoming paper.

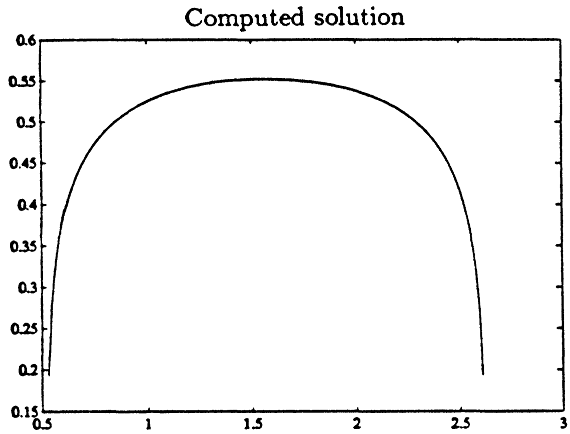
In our two examples below we have assumed all relevant data to be symmetric with respect to the x_2 variable. Consequently, the mesh size and solution will show the same type of symmetry. We may thus reformulate the given problem as a problem on the upper circular arc segment $x_1 = \cos(v), x_2 = -0.5 + \sin(v), \pi/6 \leq v \leq 5\pi/6$.

Example 4. In this example we take $f = 1$ and $\delta = 0.002$. We first consider the case $x = (0, 0)$. Figure 4a shows the computed solution. Note the singularities in the solution at both endpoints corresponding to the two points of nonsmoothness on Γ. Figure 4b shows the mesh-size h. We note the refinements at both endpoints, which are now caused by the singularity in Γ alone. Figure 4c concerns the case $x = (\sqrt{3}/2, 0)$. In this case we have again a combined effect of two sources of singularities and additional refinement is required near $v = \pi/6$ because of the singularity in the weight function w_x.

Example 5. We now consider the case $f(v) = \log(|v - \pi/6|)$ and $\delta = 0.005$. Figure 5a shows the computed solution. In Fig. 5b we have $x = (0, 0)$. We note the refinements at both end-points and the somewhat more pronounced refinement to the left near $v = \pi/6$ from the combined effect of the singularity in f and the singularity in Γ. Finally, in Fig. 5c we consider the effects of combining all three sources of singularities in Γ, f and w_x by choosing $x = (\sqrt{3}/2, 0)$ requiring even further refinement near $v = \pi/6$.

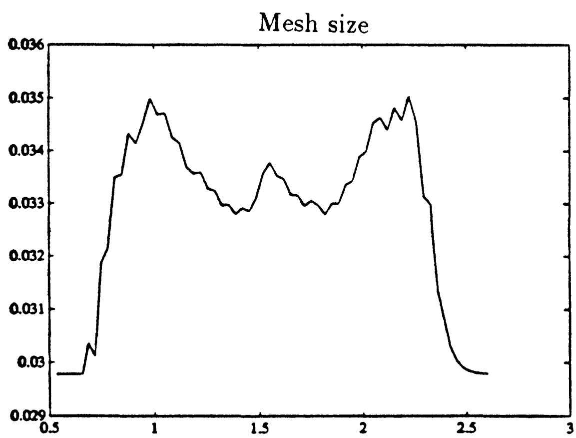
Remark 6.1. Our simple examples have been intended to illustrate the general ideas, especially for three-dimensional problems; note for instance that for planar problems and Γ smooth, the kernel of the integral operator T is in fact smooth (constant in our case) in which case the given potential problem may be solved more easily by other methods.

(a)



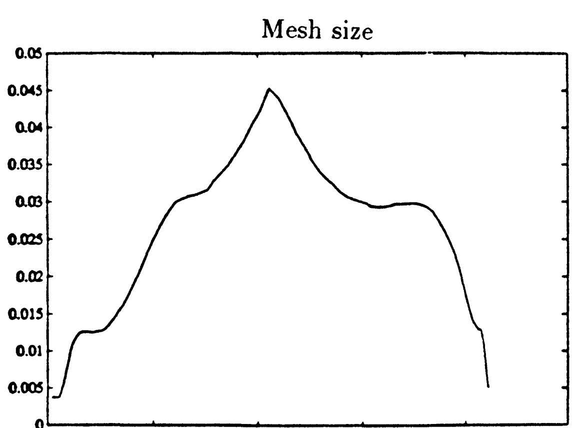
$f = 1.$

(b)



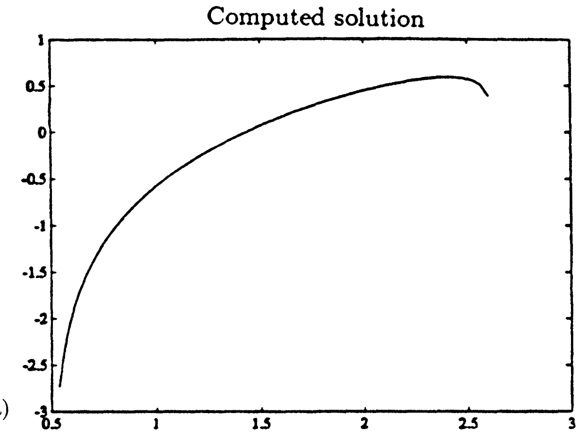
$x = (0, 0)$ $f = 1$ $\delta = 0.002$
$n = 64$ $e = 0.0014.$

(c)

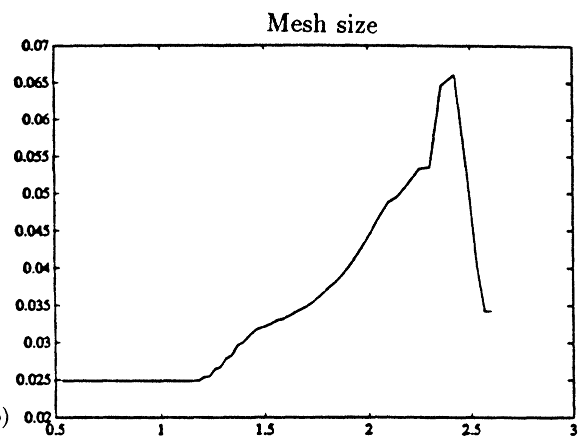


$x = (\sqrt{3}/2, 0)$ $f = 1$ $\delta = 0.002$
$n = 102$ $e = 0.0016.$

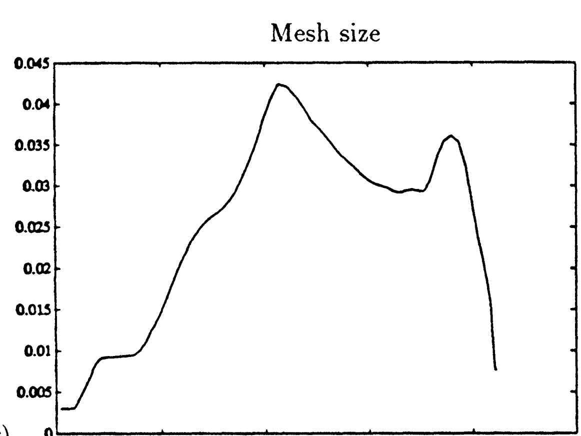
FIG. 4.



(a) $f = \log|v - \pi/6|$.



(b) $x = (0, 0)$ $\quad f = \log|v - \pi/6|$ $\quad \delta = 0.005$
$n = 64$ $\quad e = 0.0039$.



(c) $x = (\sqrt{3}/2, 0)$ $\quad f = \log|v - \pi/6|$ $\quad \delta = 0.005$
$n = 129$ $\quad e = 0.0047$.

FIG. 5.

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