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A FINITE ELEMENT METHOD FOR THE NEUTRON TRANSPORT EQUATION IN AN INFINITE CYLINDRICAL DOMAIN*

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Abstract. We study the spatial discretization, in a fully discrete scheme, for the numerical solution of a model problem for the neutron transport equation in an infinite cylindrical domain. Based on using an interpolation technique in the discontinuous Galerkin finite element procedure, we derive an *almost optimal* error estimate for the scalar flux in the L_2-norm. Combining a duality argument applied to the above result together with the previous semidiscrete error estimates for the velocity discretizations, we also obtain *globally optimal* error bounds for the critical eigenvalues.

Key words. neutron transport equation, spatial discretization, finite element, convergence rate, Besov spaces, interpolation spaces, scalar flux, duality algorithm, critical eigenvalue

AMS subject classifications. 65N15, 65N30

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1. Introduction.

We consider a fully discrete scheme for the numerical solution of the stationary, isotropic, one-velocity neutron transport equation in an infinite cylindrical domain in $\mathbb{R}^3$ with a polygonal cross section Ω. The restriction to the one-velocity case means that the velocity domain is assumed to be the unit sphere $S^2 \subset \mathbb{R}^3$. The cylindrical symmetry reduces the problem to $\mathbb{R}^2$ by projecting along the axis of the cylinder. Thus we study the neutron transport equation in a bounded polygonal domain $\Omega \subset \mathbb{R}^2$ with the velocity space being the unit disc $\mathbf{D} \subset \mathbb{R}^2$.

We analyze the discontinuous Galerkin finite element method, with piecewise linear trial functions, for the space discretization, by means of a quasi-uniform triangulation of the space domain Ω with the mesh size h. In order to obtain *sharp* error bounds, we use embedding relations between Sobolev and Besov spaces and interpolate up to the maximal available regularity of the exact solution. For this method we give an L_2 error estimate for the scalar flux of order $h^{1-\varepsilon}$ and a *globally optimal* error bound for the largest (critical) eigenvalue of order $h^{3-\varepsilon}$. Our motivation has been to improve the previous convergence rates of [1]. For the approximate solutions of the hyperbolic problems with the discontinuous Galerkin method, an L_2 error estimate of the form

$$\|\varphi - \varphi_h\|_{L_2(\Omega)} \le h^{s-1/2}\|\varphi\|_{H^s(\Omega)} \tag{1.1}$$

is optimal; see [7]. This requires that the exact solution φ is in $H^s(\Omega)$ (where $H^s(\Omega)$, $s > 0$ is the usual Sobolev space and for noninteger s, $H^s(\Omega)$ is defined by the interpolation; see [5]). Loosely speaking, this means that φ has s derivatives in $L_2(\Omega)$. Since, for the neutron transport equation the exact scalar flux is at most in $H^{3/2-\varepsilon}(\Omega)$ (see [9]), by (1.1) a convergence rate of order $\mathcal{O}(h^{1-\varepsilon})$ is sharp. However, as a consequence of embeddings, our final rate of convergence will be of order $\mathcal{O}(h^{1-\varepsilon'})$ with $\varepsilon' > \varepsilon$. As a completion of the semidiscrete analysis of [4] we continue using the Besov space norms in here, although it might be possible to obtain exact optimal rate of convergence $\mathcal{O}(h^{1-\varepsilon})$ by using Hölder space techniques of [9].

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As for the fully discrete eigenvalue estimates, we combine our results here with the semidiscrete error estimates of [1] and [4] for the discretization of the velocity domain $\mathbf{D}$ by the discrete ordinates method. The latter method is based on using an N-point Gaussian quadrature in the radial variable and a uniform M-point quadrature rule in the angular variable. For the fully discrete scheme, we obtain an error estimate of order $(N^{-4} + M^{-2+\varepsilon_1}) + h^{3-\varepsilon'}$ for the largest (critical) eigenvalue.

Problems of this type have been studied in various settings by several authors. The slab geometry $\Omega \subset \mathbb{R}$ and velocity space $[-1, 1]$ were considered by Pitkäranta and Scott [10], where L_p and eigenvalue error estimates have been carried out for both semidiscrete and fully discrete schemes. The two-dimensional geometry, $\Omega \subset \mathbb{R}^2$ and velocities in the unit circle S^1 were considered by Johnson and Pitkäranta [8] and Asadzadeh [2]. In [8] semidiscrete and fully discrete schemes were analyzed in L_2, whereas [2] contains $L_p, 1 \leq p \leq \infty$, and eigenvalue estimates for the discrete ordinates method.

In Asadzadeh [3] the discrete ordinates method was studied, in L_2, in a fully three-dimensional setting $\Omega \subset \mathbb{R}^3$ and velocity space S^2. In Asadzadeh [1] and Asadzadeh, Kumlin, and Larsson [4] the geometry is the same as in the present work. In [1], as in [8], L_2 error estimates are proved for both semidiscrete and fully discrete problems. In [4] the semidiscrete problem is studied in the L_1-norm, which is the most relevant norm from a physical point of view, since the scalar flux represents a particle density. Also, because of the limited regularity of the exact solution, error estimates in the L_1 norm for eigenfunctions yield the sharpest error bound for the eigenvalues. However, in our case here, i.e., for the spatial discretization, based on the finite element method, the L_2-norm is more suitable. For instance, in estimations in L_2, using a duality argument, the error introduced by the interpolant of the exact solution coincides with the L_2-projection which, in combination with the error for the scalar flux, gives error estimates for the critical eigenvalues sharper than that of the optimal L_1 case. This improves the convergence rate for the eigenvalues (which, in general, is the same as that for the scalar flux) *more than* three times $(\mathcal{O}(h^{3-\varepsilon'}))$ as that we obtain for the pointwise scalar flux $(\mathcal{O}(h^{1-\varepsilon'}))$.

The remainder of the paper is organized as follows: in section 2 we introduce the model problem and drive the governing integral equation. Section 3 contains some previously known error estimates for the velocity discretizations and the embedding relations which are relevant to our purpose. In section 4 we give error estimates for the space discretization and prove the main result, Theorem 4.1. Our concluding section 5 is devoted to a duality argument leading to globally optimal eigenvalue estimates.

2. A model problem. We shall consider the following model problem for mono-energetic transport of neutrons in an infinite cylindrical media $\tilde{\Omega} \subset \mathbb{R}^3$: given the source function f and the coefficient λ, find $u(x, \mu)$ such that for $\mu \in S^2$,

$$\begin{aligned} \mu \cdot \nabla_x u(x, \mu) + u(x, \mu) &= \lambda \int_{S^2} u(x, \eta) d\eta + f(x) \quad \text{for } x \in \tilde{\Omega}, \\ u(x, \mu) &= 0 \quad \text{for } x \in \tilde{\Gamma}_\mu^-, \end{aligned} \tag{2.1}$$

where $\mu \cdot \nabla_x = \sum_{i=1}^3 \mu_i (\partial / \partial x_i)$. The problem corresponds to the case of an infinite cylindrical domain with the isotropic source and scattering. Here λ is a real parameter and $u(x, \mu)$ is the density of neutrons at the point $x \in \tilde{\Omega}$ moving in the direction $\mu \in S^2 = \{\mu \in \mathbb{R}^3 : |\mu| = 1\}$. The boundary condition is specified on the inflow boundary:

$$\tilde{\Gamma}_\mu^- = \{x \in \tilde{\Gamma} : \mu \cdot \hat{n}(x) < 0\},$$

where $\tilde{\Gamma}$ is the boundary of $\tilde{\Omega}$ and $\hat{n}(x)$ is the outward unit normal to $\tilde{\Gamma}$ at $x \in \tilde{\Gamma}$.

We assume that the cross section Ω of the cylinder $\tilde{\Omega}$ is a bounded convex polygonal domain in $\mathbb{R}^2$ with the boundary Γ. Assuming also that the source term f is constant along the axial direction of the cylinder we may project the integro-differential equation (2.1) on the cross section Ω to obtain for μ in the unit disc $\mathbf{D} = \{\mu \in \mathbb{R}^2 : |\mu| \le 1\}$,

$$\begin{aligned} \mu \cdot \nabla_x u(x,\mu) + u(x,\mu) &= \lambda \int_{\mathbf{D}} u(x,\eta)(1-|\eta|^2)^{-1/2} d\eta + f(x) \quad x \in \Omega, \\ u(x,\mu) = 0 \quad &\text{for } x \in \Gamma_\mu^- := \{x \in \Gamma = \partial\Omega : \mu \cdot \hat{n}(x) < 0\}, \end{aligned} \tag{2.2}$$

where Γ_μ^- is the inflow boundary of Ω with respect to μ, with $\hat{n}(x)$, this time, being the outward unit normal to Γ at $x \in \Gamma$ and $\mu \cdot \nabla_x = \sum_{i=1}^2 \mu_i(\partial/\partial x_i)$.

We introduce the *scalar flux* U defined by

$$U(x) = \int_{\mathbf{D}} u(x,\mu)(1-|\mu|^2)^{-1/2} d\mu. \tag{2.3}$$

Now consider the following hyperbolic partial differential equation: given $g \in L_p(\Omega), 1 \le p \le \infty$, find $w(x,\mu)$ such that for $\mu \in \mathbf{D} \setminus \{0\}$,

$$\begin{aligned} \mu \cdot \nabla w + w &= g \qquad \text{in } \Omega, \\ w &= 0 \qquad \text{on } \Gamma_\mu^-. \end{aligned} \tag{2.4}$$

The solution of this problem is given by

$$w(x,\mu) = T_\mu g(x) = \int_0^{d(x,\mu)/|\mu|} e^{-s} g(x - s\mu) ds, \tag{2.5}$$

where T_μ is the solution operator and $d(x,\mu)$ is the distance from $x \in \Omega$ to the inflow boundary in the direction $-\mu$:

$$d(x,\mu) = \inf\{s > 0 : (x - s\mu/|\mu|) \notin \Omega\}.$$

Let $g = \lambda U + f$; then, using equations (2.4) and (2.5), our model problem (2.2) has a solution of the form

$$u(x,\mu) = T_\mu(\lambda U + f)(x), \qquad x \in \Omega, \quad \text{and } \mu \in \mathbf{D}; \tag{2.6}$$

consequently, we have the following integral equation, for the scalar flux U:

$$(I - \lambda T)U = Tf, \tag{2.7}$$

where

$$Tg(x) = \int_{\mathbf{D}} T_\mu g(x)(1-|\mu|^2)^{-1/2} d\mu.$$

T is an integral operator with weakly singular kernel, i.e., $T : L_p(\Omega) \to W_p^1(\Omega), 1 \le p \le \infty$ (see [1, Lemma 1.1] or [6]). In particular, $T : L_p(\Omega) \to L_p(\Omega)$ is compact. Thus

(2.7) is a Fredholm integral equation of the second kind, hence if $\lambda^{-1} \notin \sigma(T)$, where σ is the spectrum of the operator T, then there is a unique $U \in L_p(\Omega), 1 \leq p \leq \infty$, satisfying (2.7).

Remark 2.1. Here are some restrictions strongly affecting the error analysis.

i) We know that the scalar flux U (no matter how smooth the given data f is) has a limited regularity; in fact, we have at most $U \in H^{3/2-\varepsilon}(\Omega), 0 < \varepsilon << 1$; see, e.g., [9].

ii) In the error analysis it appears that singularities arise from small $r = |\mu|$ values as well as from the *closeness* of the directions of the velocity variable μ to the directions of the sides of the polygonal domain Ω. Therefore we split the discrete velocity directions into the so-called *"good ones"* (many) and *"bad ones"* (a few) so that each split part contributes to the same order of convergence.

Throughout the paper $\|.\|$ will denote the $L_2(\Omega)$-norm and C is a positive constant not necessarily the same at each occurrence and independent of all the involved parameters, unless otherwise explicitly stated.

3. The preliminaries.

3.1. The semidiscrete problem. We introduce the semidiscrete analogue of the model problem (2.6): given a function f, find $u_n(x, \mu)$ such that for $\mu \in \mathbf{D}$,

$$u_n(x, \mu) = T_\mu(\lambda U_n + f)(x) \qquad \text{for } x \in \Omega, \tag{3.1}$$

where U_n is the quadrature approximation of the scalar flux U, i.e.,

$$U_n(x) = \sum_{\mu \in \Delta} u_n(x, \mu)\omega_\mu \cong \int_{\mathbf{D}} u(x, \mu)(1 - |\mu|^2)^{-1/2}\, d\mu,$$

with $\Delta = \{\mu^1, \mu^2, ..., \mu^n\}$ being a discrete set of quadrature points $\mu^i \in \mathbf{D}$, $i = 1, ..., n$, with the corresponding positive weights $\omega_\mu, \mu \in \Delta$. We assume that Δ has an even number of points by letting both μ and $-\mu \in \Delta$. We have $n = MN$, where M is an even number of (equidistributed) discrete points on the unit circle and N is the number of Gauss points on $[0, 1]$, chosen according to the special quadrature structure below. Using (3.1), we obtain the following semidiscrete analogue of (2.7): find U_n, such that

$$(I - \lambda T_n)U_n = T_n f, \tag{3.2}$$

where $T_n g(x) = \sum_{\mu \in \Delta} \omega_\mu T_\mu g(x)$. For convenience we introduce the notation

$$\bar{n} = \min(M, N).$$

If $\lambda^{-1} \notin \sigma(T)$ and $\bar{n}$ is sufficiently large, then $(I - \lambda T_n)$ is invertible and (3.2) has a unique solution $U_n \in L_2(\Omega)$; see [1, section 4]. We construct a quadrature rule

$$\int_{\mathbf{D}} v(\mu)(1 - |\mu|^2)^{-1/2}\, d\mu \cong \sum_{\mu \in \Delta} v(\mu)\omega_\mu, \tag{3.3}$$

where we use the polar coordinates $\mu = r\hat{\mu}(\varphi)$, $\hat{\mu}(\varphi) = (\cos\varphi, \sin\varphi)$, and rewrite the discrete set Δ as

$$\Delta = \{r_k \hat{\mu}(\varphi_j)\}_{k=1}^{N}{}_{j=1}^{M}, \qquad \omega_{kj} = A_k W_j,$$

with

$$\varphi_j = \frac{2\pi j}{M}, \qquad W_j = \frac{2\pi}{M}, \qquad j = 1, ..., M,$$
$$r_k = \sin\theta_k, \qquad A_k = \alpha(s_k) - \alpha(s_{k-1}), \qquad k = 1, ..., N,$$

where $\alpha(r) = -\sqrt{1-r^2}$ and θ_k and s_k are certain points satisfying

$$\theta_k \in \left[\frac{(2k-1)\pi}{4N+2}, \frac{2k\pi}{4N+2}\right], \qquad s_k \in (r_k, r_{k+1}), \quad s_0 = 0, \quad s_N = 1;$$

for further details, see [1]. Then we have the following semidiscrete error estimates.

PROPOSITION 3.1. *Assume that λ^{-1} is not an eigenvalue of T. For each $\varepsilon_1 > 0$ there is a constant C such that, for sufficiently large $\bar{n}$,*

$$\|U - U_n\|_{L_1(\Omega)} \le C\left[\frac{1}{N^4} + \frac{1}{M^{2-\varepsilon_1}}\right]\left(\|U\|_{L_1(\Omega)} + \|f\|_{W_1^1(\Omega)}\right), \tag{3.4}$$

and for $e_n := (T - T_n)(\lambda U + f)$, there is a constant C such that for $M \sim N$,

$$\|e_n\|_{L_2(\Omega)} \le Cn^{-1/2}\|\lambda U + f\|_{H^1(\Omega)}. \tag{3.5}$$

The estimate (3.4) is the main result of [4] and (3.5) is the matter of Lemma 4.3 in [1].

3.2. Some function spaces. Below, for the convenience of the reader, we recall the definitions of some function spaces and also include the embedding relations between them which are frequently used in our error analysis.

For k a nonnegative integer, $H_p^k(\Omega)$ is the usual Sobolev space with the norm $\|f\|_{H_p^k(\Omega)}$ and the corresponding seminorm $|f|_{H_p^k(\Omega)}$:

$$\|f\|^p_{H_p^k(\Omega)} = \sum_{|\alpha|\le k}\|\mathcal{D}^\alpha f\|^p_{L_p(\Omega)}, \qquad |f|^p_{H_p^k(\Omega)} = \sum_{|\alpha|=k}\|\mathcal{D}^\alpha f\|^p_{L_p(\Omega)}, \quad p \ge 1.$$

For $s \in \mathbb{R}$, consider the generalized Sobolev space $H_p^s(\mathbb{R}^n)$ with the norm

$$\|f\|_{H_p^s} = \|\mathcal{F}^{-1}\{(1 + |\cdot|^2)^{s/2}\mathcal{F}f\}\|_{L_p},$$

where $\mathcal{F}$ denotes the Fourier transform. By $H_p^s(\Omega)$, $s \in \mathbb{R}$, we simply mean the following restriction of $H_p^s(\mathbb{R}^n)$ to Ω:

$$\|f\|_{H_p^s(\Omega)} = \inf_{\mathcal{G}_f}\|g\|_{H_p^s(\mathbb{R}^n)}, \qquad \mathcal{G}_f = \{g \in H_p^s(\mathbb{R}^n) : g|_\Omega = f\}.$$

In what follows and if necessary, we think of a function in $L_p(\Omega)$, $p \ge 1$, $\Omega \subset \mathbb{R}^n$, as being defined on $\mathbb{R}^n$ (extended by 0 on $\mathbb{R}^n \setminus \Omega$). We write $L_p = L_p(\mathbb{R}^n)$, $B^s_{p,q} = B^s_{p,q}(\mathbb{R}^n)$, etc., although the final results refer to $L_p(\Omega)$, $B^s_{p,q}(\Omega)$. The justification for this is the existence of continuous linear extension operators

$$H_p^s(\Omega) \to H_p^s(\mathbb{R}^n), \qquad s \in \mathbb{N}, \quad 1 \le p < \infty$$

for the bounded Lipschitz domains Ω, see Stein [11, Theorems 5 and 5', p. 181]. For $p = 2$, the notation $\|\cdot\|_s$, $s \in \mathbb{R}$, is the usual H^s-norm over the domain Ω or $\mathbb{R}^n$, which is obvious from the context. We shall use embedding relations and interpolation properties between the generalized Sobolev spaces H_p^s, $s \in \mathbb{R}$, and Besov spaces $B_{p,q}^s$, $s \in \mathbb{R}$, $1 \le p\,,\, q \le \infty$, with the norm

$$\|g\|_{B_{p,q}^s(\Omega)} = \sum_{|\alpha|\le[s]} \|\mathcal{D}^\alpha g\|_{L_p(\Omega)} + \sum_{|\alpha|=[s]} \left(\int_0^1 \left(t^{[s]-s}\omega_p(\mathcal{D}^\alpha g)(t)\right)^q \frac{dt}{t}\right)^{1/q} \tag{3.6}$$

for $q \in [1,\infty]$ and $s > 0$, where $[s]$ denotes the integer part of s and

$$\omega_p(\varphi)(t) = \sup_{|\eta|\le t} \|\varphi(\cdot+\eta) - \varphi(\cdot)\|_{L_p(\Omega_{\eta,l})}, \qquad \Omega_{\eta,l} = \bigcap_{j=0}^{l}\{x : x + j\eta \in \Omega\},$$

where $l = [s]$, and with the usual modification for $q = \infty$; see [5] and [12]. In our case we have $[s] = 1$ and we shall concisely denote $\Omega_{\eta,1}$ by Ω_η. The definition (3.6) remains true for both bounded and unbounded domains of the cone type, and Ω being a convex polygonal domain satisfies this property. For simplicity, we may drop Ω from the relations below knowing that they hold for both $\mathbb{R}^n$ and the Lipschitz domains Ω in our case, i.e., for $p = 2 > 1$. Besov spaces are interpolation spaces satisfying

$$\left(H_p^{s_1}, H_p^{s_2}\right)_{\theta,q} = B_{p,q}^s\,, \qquad s = (1-\theta)s_1 + \theta s_2, \quad 0 < \theta < 1 \tag{3.7}$$

for $s_1 < s_2$, $H_p^{s_2} \subset H_p^{s_1}$, and

$$H_p^{s_2} \subset B_{p,q}^s \subset H_p^{s_1}, \quad s_1 < s < s_2. \tag{3.8}$$

(3.6)–(3.8) hold for weighted norms as well. We recall the Sobolev embeddings

$$B_{p,1}^s \subset L_{p_1}, \qquad s - \frac{n}{p} \ge -\frac{n}{p_1}, \tag{3.9}$$

$$H_p^s \subset B_{q,p}^t, \qquad s - \frac{n}{p} \ge t - \frac{n}{q}, \quad 1 < p < q < \infty. \tag{3.10}$$

Moreover, for $s \in \mathbb{R}$, $\varepsilon > 0$, $1 < p < \infty$, $1 \le q_1 \le q_2 \le \infty$, we have

$$B_{p,\infty}^{s+\varepsilon} \subset B_{p,1}^s \subset B_{p,q_1}^s \subset B_{p,q_2}^s \subset B_{p,\infty}^s \subset B_{p,1}^{s-\varepsilon}. \tag{3.11}$$

Finally, for arbitrary domain $\Omega \subset \mathbb{R}^n$ and if $1 < p < \infty$, $1 \le q \le \infty, s > 1/p$, then we have the trace inequality

$$\|\cdot\|_{B_{p,q}^{s-1/p}(\Gamma)} \le \|\cdot\|_{B_{p,q}^s(\Omega)}. \tag{3.12}$$

For further details, see [5] and [12]. We shall use the above relations for the parameters $n = 1, 2$, $q = 1$, and $p = 2$ except when applying (3.10), where q is replaced by $p > 1$.

4. The fully discrete problem. We denote by $\{\mathcal{C}_h\}$ a family of quasi-uniform triangulation $\mathcal{C}_h = \{K\}$ of Ω indexed by the parameter h, the maximum diameter of triangles $K \in \mathcal{C}_h$. We introduce the finite element space

$$V_h = \{v \in L_2(\Omega) : v|_K \text{ is linear}, K \in \mathcal{C}_h\}$$

and define a discrete solution operator $T_\mu^h : L_2(\Omega) \to V_h$ approximating T_μ by the following discontinuous Galerkin finite element method for (2.4):

$$\sum_{K\in\mathcal{C}_h} \left[(\mu\cdot\nabla u^h + u^h, v)_K + \int_{\partial K^-} [u^h]v_+\,|\mu\cdot\hat{n}|d\sigma\right] = \int_\Omega gvdx \qquad \forall v \in L_2(\Omega), \tag{4.1}$$

with

$$(u,v)_K = \int_K uv\,dx, \qquad \partial K^- = \{x \in \partial K : \mu \cdot \hat{n}(x) < 0\},$$
$$[v] = v_+ - v_-, \qquad v_\pm(x) = \lim_{s\to 0\pm} v(x+s\mu), \quad x \in \partial K,$$

where $\hat{n} = \hat{n}(x)$ is the outward unit normal to ∂K at $x \in \partial K$ and $u^h_- = 0$ on Γ^-_μ.

Now, let us formulate the following fully discrete analogue of (2.6): given f, find $u^h_n(\cdot,\mu) \in V_h$ such that

$$u^h_n(\cdot,\mu) = T^h_\mu(\lambda U^h_n + f), \qquad \mu \in \Delta, \tag{4.2}$$

where U^h_n is the totally discretized scalar flux, $U^h_n = \sum_{\mu\in\Delta} u^h_n(\cdot,\mu)\omega_\mu$. Equation (4.2) is equivalent to the problem of finding $U^h_n \in V_h$ such that

$$(I - \lambda T^h_n)U^h_n = T^h_n f, \tag{4.3}$$

where $T^h_n : L_2(\Omega) \to V_h$ is defined by $T^h_n = \sum_{\mu\in\Delta} T^h_\mu \omega_\mu$.

For $\lambda^{-1} \notin \sigma(T)$ and $\max(h, 1/\bar{n})$ sufficiently small, (4.3) has a unique solution $U^h_n \in V_h$; see [1, section 5]. In this section our main concern will be the estimates of the L_2-error for the scalar flux for the fully discrete problem (4.3), i.e., $\|U - U^h_n\|$. This error, as a result of combining the semidiscrete L_2-error (3.5) with our estimates of this section, is of order $\mathcal{O}(h^{1-\varepsilon'} + n^{-1/2})$. The parameters h and n will be related according to the following *compatibility conditions*:

$$h^{-1}(n) \sim \sqrt{n} = \sqrt{MN}, \qquad \text{and} \quad M \sim N. \tag{4.4}$$

As we shall see in the proof of Lemma 4.1 below, without this condition the contribution of the "bad directions" (cf. Remark 2.1 and also splitting (4.9) below) to the spatial error will not be of the desired order of $\sim h$. Otherwise we could, instead of (3.5), use improved semidiscrete L_2-estimates similar to (3.4) of the L_1 case in [4] and, for more consistent estimates, make the contribution to the L_2-error from the spatial and velocity discretizations both of the same order of magnitude $\mathcal{O}(h^{1-\varepsilon})$.

Our main result is the following theorem.

THEOREM 4.1. *Assume that* $\lambda^{-1} \notin \sigma(T)$. *Let* U *and* U^h_n *satisfy* (2.7) *and* (4.3), *respectively. Then there is a constant* C *such that for sufficiently small* h *(large* $\bar{n}$*) satisfying* (4.4), *for any small* ε *and* ε' *satisfying* $0 < \varepsilon < \varepsilon'$, *and for* $g \in H^{3/2-\varepsilon}(\Omega)$,

$$\|U - U^h_n\| \le C|\log h| h^{1-\varepsilon'} \|g\|_{H^{3/2-\varepsilon}(\Omega)},$$

where $g = \lambda U + f$.

To derive the relevant estimates we shall use the following two results.

PROPOSITION 4.1 (cf. [7]). *Given* $g \in L_2(\Omega)$, *there is a unique* $u^h(\cdot,\mu) = T^h_\mu g(\cdot) \in V_h$ *satisfying* (4.1). *Moreover, there is a constant* C *independent of* $g, \mu, h,$ *and* Ω *such that*

$$|||(T_\mu - T^h_\mu)g|||_\mu \le C h^{s-1/2} |T_\mu g|_s, \qquad s = 1, 2, \tag{4.5}$$

$$|||T^h_\mu g|||_\mu \le C\|g\|, \tag{4.6}$$

where

$$|||v|||_\mu = \left[\|v\|^2 + h \sum_K \|\mu\cdot\nabla v\|^2_K + \sum_K \int_{\partial K} |[v]|^2 |\mu\cdot\hat{n}|\,ds \right]^{1/2}, \tag{4.7}$$

$$\|v\|_K = (v,v)_K^{1/2}.$$

PROPOSITION 4.2 (stability; cf. [1]). *For $g \in L_2(\Omega)$ we have*

$$\|\mu \cdot \nabla T_\mu g\| + \|T_\mu g\| + \left[\int_\Gamma (T_\mu g)^2 |\mu \cdot \hat{n}| d\sigma\right]^{1/2} \leq C\|g\|. \tag{4.8}$$

We shall also need the following splitting of Δ in two sets:

$$\begin{aligned} J'_\delta &= \left\{\mu \in \Delta : \gamma(\mu) = \min_k(|\sin(\mu, S_k)|) \geq \delta \sim \frac{1}{M},\, k = 1,2,\ldots,P_0\right\}, \\ J_\delta &= \{\mu \in \Delta : \mu \notin J'_\delta\}, \end{aligned} \tag{4.9}$$

where $\{S_k\}_{k=1}^{P_0}$ are the directions of the sides of Ω and P_0 is the number of sides of Ω. Note that, because of our special radial quadrature rule, we have $|\mu| < 1/N$ for all $\mu \in \Delta$. Now, we show by the following two lemmas that, since the number of elements in J_δ is very few when compared with those in J'_δ, the weighted sum over $\mu \in J_\delta$, the "bad directions," is not worse than the one over $\mu \in J'_\delta$, the "good directions."

Finally, the proof of Theorem 4.1 is based on these lemmas.

LEMMA 4.1. *There is a constant C such that for $g \in L_2(\Omega)$,*

$$\sum_{\mu \in J_\delta} \omega_\mu \|(T_\mu - T_\mu^h)g\| \leq Ch\|g\|_{L_2(\Omega)}.$$

Proof. By the L_2-stability estimate resulting from (4.6)–(4.8), we have that

$$\sum_{\mu \in J_\delta} \omega_\mu \|(T_\mu - T_\mu^h)g\| \leq C\left(\sum_{\mu \in J_\delta} \omega_\mu\right)\|g\|_{L_2(\Omega)} \leq \frac{C}{MN} P_0 N \|g\|_{L_2(\Omega)} \leq h\|g\|_{L_2(\Omega)},$$

where we use the compatibility condition (4.4) and the fact that for $\gamma(\mu) \leq \frac{\pi}{M}$, J_δ contains at most NP_0 elements, where P_0 is the number of sides of Ω. $\square$

LEMMA 4.2. *For any ε and ε' satisfying $0 < \varepsilon < \varepsilon' << 1$, there is a constant C such that for $g \in H^{3/2-\varepsilon}(\Omega)$,*

$$\sum_{\mu \in J'_\delta} \omega_\mu \|T_\mu g\|_{H^{3/2-\varepsilon'}(\Omega)} \leq C|\log\delta| \|g\|_{H^{3/2-\varepsilon}(\Omega)}.$$

Let us postpone the proof of this lemma and first show that Theorem 4.1 follows from Lemmas 4.1 and 4.2.

Proof of Theorem 4.1. We have, using (2.7) and (4.3), that

$$(I - \lambda T_n^h)(U - U_n^h) = (T - T_n)(\lambda U + f) + (T_n - T_n^h)(\lambda U + f) := e_n + e_n^h.$$

According to a stability estimate (see [1, Theorem 5.1]), if $\lambda^{-1} \notin \sigma(T)$, then for sufficiently large n, $(I - \lambda T_n^h)^{-1} : L_2(\Omega) \to L_2(\Omega)$ exists and is uniformly bounded. Thus we have

$$\|U - U_n^h\| \leq C_\lambda \left(\|e_n\| + \|e_n^h\|\right).$$

Now, we replace, on the right-hand side of (4.5), $|T_\mu g|_s$ by $\|T_\mu g\|_s$ and interpolate using $\theta = \frac{1}{2} - \varepsilon'$, $0 < \varepsilon < \bar{\varepsilon} < \varepsilon' << 1$, $\bar{\varepsilon} = (\varepsilon + \varepsilon')/2$, in a weighted form of (3.7), which also interpolates in powers of h, to obtain

$$\begin{aligned}\|e_n^h\| &= \left\|\sum_{\mu\in\Delta} \omega_\mu (T_\mu - T_\mu^h) g\right\| \le \sum_{\mu\in J_\delta} \omega_\mu \|(T_\mu - T_\mu^h) g\| + \sum_{\mu\in J'_\delta} \omega_\mu \|(T_\mu - T_\mu^h) g\| \\ &\le Ch\|g\| + Ch^{1-\varepsilon'} \sum_{\mu\in J'_\delta} \omega_\mu \|T_\mu g\|_{B^{3/2-\varepsilon'}_{2,1}} \\ &\le Ch\|g\| + Ch^{1-\varepsilon'} \sum_{\mu\in J'_\delta} \omega_\mu \|T_\mu g\|_{3/2-\bar{\varepsilon}} \le C|\log h| h^{1-\varepsilon'} \|g\|_{H^{3/2-\varepsilon}(\Omega)},\end{aligned}$$

where we have also used (3.8) and Lemmas 4.1 and 4.2. Thus (3.5), with the compatibility condition (4.4), gives the desired result. ∎

In the proof of Lemma 4.2 we use the following result.

LEMMA 4.3. *There is a constant C such that for $g \in H^s(\Omega)$, $s = 3/2 - \varepsilon$, and for any ε' satisfying $0 < \varepsilon < \varepsilon'$, we have*

$$\int_0^1 t^{1-s'} \omega_2 \left(\frac{\partial}{\partial x_i}(T_\mu g)\right)(t) \frac{dt}{t} \le \frac{C}{|\mu|} \cdot \frac{1}{\gamma(\mu)} \|g\|_{H^s(\Omega)}, \quad i = 1, 2, \tag{4.10}$$

where $s' = 3/2 - \varepsilon'$, $\gamma(\mu) = \min_j |\sin(\mu, S_j)|$, and S_j are the directions of the sides of Ω.

Proof of Lemma 4.2. Since we have at most $g = (\lambda U + f) \in H^{3/2-\varepsilon}(\Omega)$, therefore s is at most $3/2 - \varepsilon$ and it suffices to show that for any ε'' satisfying $0 < \varepsilon < \varepsilon'' < \varepsilon'$,

$$\sum_{\mu\in J'_\delta} \omega_\mu \|T_\mu g\|_{B^{3/2-\varepsilon''}_{2,q}} \le C|\log\delta| \|g\|_{B^{3/2-\varepsilon''}_{2,q}}, \tag{4.11}$$

since then, by the embedding relation (3.8),

$$\sum_{\mu\in J'_\delta} \omega_\mu \|T_\mu g\|_{3/2-\varepsilon'} \le \sum_{\mu\in J'_\delta} \omega_\mu \|T_\mu g\|_{B^{3/2-\varepsilon''}_{2,q}} \le C|\log\delta| \|g\|_{B^{3/2-\varepsilon''}_{2,q}} \le C|\log\delta| \|g\|_{3/2-\varepsilon}.$$

To prove (4.11) we use the definition (3.6) of Besov space norm with $q = 1$ and write

$$\begin{aligned}\|T_\mu g\|_{B^{3/2-\varepsilon'}_{2,1}} &= \sum_{|\alpha|\le 1} \|\mathcal{D}^\alpha (T_\mu g)\| + \sum_{|\alpha|=1} \left(\int_0^1 t^{1-s'} \omega_2 \left(\mathcal{D}^\alpha (T_\mu g)\right)(t) \frac{dt}{t}\right) \\ &= \|T_\mu g\| + \|\mathcal{D}(T_\mu g)\| + \int_0^1 t^{1-s'} \omega_2 \left(\mathcal{D}(T_\mu g)\right)(t) \frac{dt}{t} \\ &\le C\left(\|g\| + \frac{1}{|\mu|}\|g\|\right) + \frac{C}{|\mu|} \cdot \frac{1}{\gamma(\mu)} \|g\|_{H^s(\Omega)} \le \frac{C}{|\mu|} \cdot \frac{1}{\gamma(\mu)} \|g\|_{H^s(\Omega)},\end{aligned}$$

where in the first inequality above we use the stability estimate (4.8) and Lemma 4.3. Now multiplying both sides by the positive weights ω_μ and summing over all $\mu \in J'_\delta$, and using the fact that by our choice of the quadrature rule $|\mu_1| = \min_{\mu\in\Delta} |\mu| > 1/N$, $\sum_{i=1}^N \frac{A_i}{|\mu_i|} \le C$ (see [1, proof of Lemma 6.1]),

$$\sum_{\mu\in J'_\delta} \frac{\omega_\mu}{|\mu|} \cdot \frac{1}{\gamma(\mu)} \le C \left(\sum_{i=1}^N \frac{A_i}{|\mu_i|}\right) \left(\frac{2\pi}{M} \sum_{\gamma(\mu)>\delta} \frac{1}{\gamma(\mu)}\right) \sim C|\log\delta|,$$

and we obtain the desired result. ◻

Remark 4.1. i) Below we consider the first partial derivatives of $T_\mu g$. Recall that

$$T_\mu g(x) = \int_0^{d(x,\mu)/r} e^{-s} g(x - s\mu) ds, \quad \frac{\partial d}{\partial \mu} = r, \quad \text{and} \quad \frac{\partial d}{\partial \nu} = r \frac{\nu \cdot \hat{n}}{\mu \cdot \hat{n}}, \tag{4.12}$$

where $\nu \in S^1$ is any unit vector which is not parallel to the direction of μ; see [1]. By an orthogonal coordinate transformation we may assume that $\mu = (\mu_1, 0), |\mu| = r$, and $\nu \perp \mu$. This is for notational convenience in $\omega_2(\frac{\partial}{\partial x_i}(T_\mu g(x)))$, $i = 1, 2$, and singles out the most singular term, i.e., the one corresponding to $i = 2$; see below. Using (4.12), since μ is parallel to the direction of the x_1-axis, we have

$$\frac{\partial}{\partial x_1}(T_\mu g(x)) = \frac{1}{\mu_1} e^{-d/r} g(\bar{x}) + \int_0^{d/r} e^{-s} \frac{\partial}{\partial x_1} g(x - s\mu) ds, \tag{4.13}$$

where $\bar{x} = (x - d\frac{\mu}{r}) \in \Gamma$. Further, $\frac{\partial d}{\partial x_2} = \frac{\sin(\mu, \hat{n})}{\hat{\mu} \cdot \hat{n}}$, i.e., $\frac{\partial (d/r)}{\partial x_2} = \frac{\sin(\mu, \hat{n})}{\mu \cdot \hat{n}}$, hence for the partial derivative of $T_\mu g(x)$ with respect to x_2, we have that

$$\frac{\partial}{\partial x_2}(T_\mu g(x)) = e^{-d/r} g(\bar{x}) \frac{\sin(\hat{\mu}, \hat{n})}{\mu \cdot \hat{n}} + \int_0^{d/r} e^{-s} \frac{\partial}{\partial x_2} g(x - s\mu) ds. \tag{4.14}$$

The geometry of the domain imposes singularities in $\psi(\bar{x}) := \frac{\sin(\hat{\mu}, \hat{n})}{\mu \cdot \hat{n}}$ stopping further differentiations, ($\psi \in H^{1/2-\varepsilon}(\Gamma)$). Besov spaces are employed to perform fractional derivatives. The cost, as we saw in the proof of Lemma 4.2, is that in the embedding procedures between Sobolev and Besov spaces we lose a small power of h of order $\mathcal{O}(h^{\varepsilon' - \varepsilon})$. The same result is obtained using the K-method of interpolation based on a splitting of $\psi(\bar{x})$. Since the whole calculation can concisely be done by Besov space techniques, we skip the tedious task of finding the appropriate splits in using the K-method.

ii) Recalling (4.13) and (4.14), we observe that $\omega_2 \left(\frac{\partial}{\partial x_i} T_\mu g \right)(t)$, $i = 1, 2$, contains differences of translations in g, d, and ψ. Below, in the proof of Lemma 4.3, we estimate some of these terms gaining a factor of $|\eta|^p \leq t^p$, $p \sim 1$, in order to make the integration in (4.10) possible, whereas some other terms, integrated as in (4.10), will be in the form of a Besov norm for g.

Proof of Lemma 4.3. To begin with, we use the same coordinate transformation as in Remark 4.1. It suffices to estimate the contribution from the less regular term, i.e.,

$$\omega_2 \left(\frac{\partial}{\partial x_2} T_\mu g(\cdot) \right)(t) = \sup_{|\eta| \leq t} \left\| \frac{\partial}{\partial x_2} \left(T_\mu g \right)(\cdot + \eta) - \frac{\partial}{\partial x_2} \left(T_\mu g \right)(\cdot) \right\|_{L_2(\Omega_\eta)},$$

since the other term will be similar and dominated by this one. Now denoting $\frac{\sin(\hat{\mu}, \hat{n})}{\mu \cdot \hat{n}}$ by $\psi(\bar{x}) = \psi(x_2)$ in (4.14), and using the notation $\frac{\partial}{\partial x_2}[\varphi]_\eta := \frac{\partial}{\partial x_2}\varphi(x + \eta) - \frac{\partial}{\partial x_2}\varphi(x)$, we get

$$\begin{aligned} \frac{\partial}{\partial x_2} \left[(T_\mu g)(x) \right]_\eta &= \left(e^{-d(x+\eta\,,\mu)/r} g(\overline{x+\eta}) \psi(x_2 + \eta_2) - e^{-d(x\,,\mu)/r} g(\bar{x}) \psi(x_2) \right) \\ &+ \left(\int_0^{d(x+\eta\,,\mu)/r} e^{-s} \frac{\partial}{\partial x_2} g(x + \eta - s\mu)\, ds - \int_0^{d(x\,,\mu)/r} e^{-s} \frac{\partial}{\partial x_2} g(x - s\mu)\, ds \right) \\ &:= \mathcal{F} + \mathcal{J}. \end{aligned}$$

Further, we can write $\mathcal{F}$ and $\mathcal{J}$ as

$$\begin{aligned}\mathcal{F} =& e^{-d(x+\eta\,,\mu)/r} g(\overline{x+\eta})\,[\psi(x_2+\eta_2)-\psi(x_2)] + e^{-d(x+\eta\,,\mu)/r}\psi(x_2)\,[g(\overline{x+\eta})-g(\bar{x})] \\ &+ \left[e^{-d(x+\eta\,,\mu)/r} - e^{-d(x\,,\mu)/r}\right] g(\bar{x})\psi(x_2) := \mathcal{F}_1+\mathcal{F}_2+\mathcal{F}_3\end{aligned}$$

and

$$\begin{aligned}\mathcal{J} =& \int_0^{d(x+\eta\,,\mu)/r} e^{-s}\frac{\partial}{\partial x_2}\,[g(x+\eta-s\mu)-g(x-s\mu)]\;ds \\ &+ \int_{d(x\,,\mu)/r}^{d(x+\eta\,,\mu)/r} e^{-s}\frac{\partial}{\partial x_2} g(x-s\mu)\,ds := \mathcal{J}_1+\mathcal{J}_2.\end{aligned}$$

Below we shall estimate each of the terms $\mathcal{F}_i$, $i=1,2,3$, and $\mathcal{J}_l$, $l=1,2$, separately. Now we use a new transformation of coordinates

$$x = P_j + \xi_1\hat{\mu} + \xi_2\hat{\tau}_j, \quad 0\le \xi_1 \le B_j(\xi_2),\ 0\le \xi_2 \le L_j,$$

where P_j is an endpoint of S_j and $\hat{\tau}_j = (\cos\vartheta_j, \sin\vartheta_j)$ is tangent to S_j. Thus $d(x,\mu) = \xi_1$, $d(x+\eta,\mu) = \xi_1+\eta_1$, and since the area element is $dx = |\sin\vartheta_j|d\xi = \frac{|\mu\cdot\hat{n}|}{|\mu|}d\xi$, thus we have, using the Hölder inequality, with $\bar{p} = 1+\tau$, for τ being small and $\frac{1}{\bar{p}}+\frac{1}{\bar{q}}=1$, that

$$\begin{aligned}\|\mathcal{F}_\infty\|^2_{L_2(\Omega_\eta)} &\le \int_\Omega e^{-2d(x+\eta\,,\mu)/r} g^2(\overline{x+\eta})|[\psi]|^2\,dx \\ &\le C\int_0^{\mathrm{diam}\Omega} e^{-2(\xi_1+\eta_1)/r}\,d\xi_1 \int_\Gamma g^2(\overline{x+\eta})|[\psi]|^2\frac{|\mu\cdot\hat{n}|}{|\mu|}\,d\xi_2 \\ &\le \frac{C}{r}\int_0^{\mathrm{diam}\Omega} e^{-2(\xi_1+\eta_1)/r}\,d\xi_1 \left(\int_\Gamma [\psi]^{2\bar{p}}|\mu\cdot\hat{n}|\right)^{1/\bar{p}}\left(\int_\Gamma g^{2\bar{q}}|\mu\cdot\hat{n}|\right)^{1/\bar{q}},\end{aligned}$$

where $[\psi]$ denotes the jump $\psi(x_2+\eta_2)-\psi(x_2)$, which is zero except on a finite number of intervals of length $\le 2|\eta|$, and we have used $\overline{x+\eta}\in\Gamma$. Thus, using Sobolev embedding (3.9), the trace theorem (3.12), and the relation (3.8), we obtain

$$\begin{aligned}\|\mathcal{F}_1\|_{L_2(\Omega_\eta)} &\le \frac{C}{\sqrt{r}}\|[\psi]\|_{L_{2\bar{p}}(\Gamma)}\|g\|_{L_{2\bar{q}}(\Gamma)} \le \frac{C}{\sqrt{r}}\|[\psi]\|_{L_{2\bar{p}}(\Gamma)}\|g\|_{B^{\frac12-\frac{1}{2\bar{q}}}_{2,1}(\Gamma)} \\ &\le \frac{C}{\sqrt{r}}\|[\psi]\|_{L_{2\bar{p}}(\Gamma)}\|g\|_{B^{1-\frac{1}{2\bar{q}}}_{2,1}(\Omega)} \le \frac{C}{\sqrt{r}}\|[\psi]\|_{L_{2\bar{p}}(\Gamma)}\|g\|_{H^1(\Omega)}.\end{aligned}$$

For the corresponding estimate for $\mathcal{F}_2$, we have, using a similar argument as above, that

$$\begin{aligned}\|\mathcal{F}_2\|^2_{L_2(\Omega_\eta)} &\le \int_0^{\mathrm{diam}\Omega} e^{-2(\xi_1+\eta_1)/r}\,d\xi_1\int_\Gamma |g(\overline{x+\eta})-g(\bar{x})|^2\left|\frac{\sin(\mu,\hat{n})}{\mu\cdot\hat{n}}\right|^2\frac{|\mu\cdot\hat{n}|}{|\mu|}\,d\xi_2 \\ &\le \frac{C}{|\mu|\min_j|\mu\cdot\hat{n}_j|^2}\int_\Gamma |g(\overline{x+\eta})-g(\bar{x})|^2|\mu\cdot\hat{n}| \\ &\le \frac{C}{r^2\gamma(\mu)^2}\int_\Gamma |g(\overline{x+\eta})-g(\bar{x})|^2|\mu\cdot\hat{n}|\,d\xi_2.\end{aligned}$$

For the estimate of $\mathcal{F}_3$ we note that, by the geometry, for a convex polygonal region $\Omega\subset\mathbb{R}^2$, $|d(x+\eta\,,\mu)-d(x\,,\mu)|$ is larger when $\bar{x}$ and $\overline{x+\eta}$ belong to the same side

S_j of Ω and the maximum occurs for $x+\eta$ being the tangent point of one of the two lines parallel to S_j, with the circle of radius $|\eta|$ centered at the point x. This implies that, on each side S_j,

$$|d(x+\eta\,,\mu)-d(x\,,\mu)| \le \frac{|\eta|}{|\sin(\mu,S_j)|} = \frac{|\eta|}{|\hat{\mu}\cdot\hat{n}_j|}, \tag{4.15}$$

where $\hat{n}_j$ is the outward unit normal to the side S_j. Moreover, using the identity

$$|e^{-a}-e^{-b}| = |e^{-|a-b|}-1| \times e^{-\min(a,b)},$$

with $a=d(x+\eta\,,\mu)$ and $b=d(x\,,\mu)$, and letting $\rho_j := r|\sin(\mu\,,S_j)|$, since we have that $|e^{-|\eta|/\rho_j}-1| < C\frac{|\eta|}{\rho_j}$, then by (4.15) and a similar argument as in the estimate for $\mathcal{F}_1$, we may write

$$\begin{aligned}\|\mathcal{F}_3\|^2_{L_2(\Omega_\eta)} &\le C\sum_j \int_0^{\operatorname{diam}\Omega} \frac{|\eta|^2}{r^2|\hat{\mu}\cdot\hat{n}_j|^2} e^{-\frac{2}{r}\min(\xi_1+\eta_1,\xi_1)}\,d\xi_1 \int_{S_j} g^2\psi^2|\hat{\mu}\cdot\hat{n}_j|\,d\xi_2 \\ &\le C\frac{|\eta|^2}{r^2\gamma(\mu)^2}\|g\|^2_{L_4(\Gamma)}\|\psi\|^2_{L_4(\Gamma)}.\end{aligned}$$

Further, using (3.9), (3.12) with $p=2$ and (3.8),

$$\begin{aligned}\|g\|_{L_4(\Gamma)} &\le C\|g\|_{B^{1/4}_{2,1}(\Gamma)} \le C\|g\|_{B^{3/4}_{2,1}(\Omega)} \le C\|g\|_{H^1(\Omega)}, \\ \|\psi\|_{L_4(\Gamma)} &\le C\|\psi\|_{B^{1/4}_{2,1}(\Gamma)} \le C\|\psi\|_{H^{1/4+\varepsilon}(\Gamma)},\end{aligned}$$

and since $\psi\in H^{1/2-\varepsilon}(\Gamma)$, then

$$\|\mathcal{F}_3\|_{L_2(\Omega_\eta)} \le \frac{C|\eta|}{r\gamma(\mu)}\|g\|_{H^1(\Omega)}.$$

Now it remains to consider the contributions from the $\mathcal{J}$-terms. For $\mathcal{J}_1$ we use the definition of $\omega_p(\cdot)$ and thus have the estimate

$$\|\mathcal{J}_1\|_{L_2(\Omega_\eta)} \le C\left\|\frac{\partial}{\partial x_2}g(\cdot+\eta)-\frac{\partial}{\partial x_2}g(\cdot)\right\|_{L_2(\Omega_\eta)} \le C\omega_2\left(\frac{\partial}{\partial x_2}g(\cdot)\right)(t).$$

To estimate $\mathcal{J}_2$ we use a similar technique as in the estimate of $\mathcal{F}_1$ and write, using also the estimates (4.8) and (4.15),

$$\begin{aligned}\|\mathcal{J}_2\|^2_{L_2(\Omega_\eta)} &\le \int_\Omega\left[\int_{d(x,\mu)/r}^{d(x+\eta,\mu)/r} e^{-s}|\nabla g(x-s\mu)|\,ds\right]^2 dx \\ &\le \frac{C|\eta|^2}{r^2\gamma(\mu)^2}\|\nabla g\|^2_{L_2(\Omega)} \le \frac{C|\eta|^2}{r^2\gamma(\mu)^2}\|g\|^2_{H^1(\Omega)}.\end{aligned}$$

Thus, summing up, we have that, for $q = 1$,

$$
\begin{aligned}
\int_0^1 t^{1-s'} \omega_2(T_\mu g(\cdot))(t) \frac{dt}{t} &\le \frac{C}{r\gamma(\mu)} \left(\int_0^1 t^{1-s'} \sup_{|\eta|\le t} |\eta| \frac{dt}{t} \right) \|g\|_{H^1(\Omega)} \\
&\quad + \frac{C}{\sqrt{r}} \left(\int_0^1 t^{1-s'} \sup_{|\eta|\le t} \|[\psi]\|_{L_{2\bar p}(\Gamma)} \frac{dt}{t} \right) \|g\|_{H^1(\Omega)} \\
&\quad + \frac{C}{r\gamma(\mu)} \int_0^1 t^{1-s'} \sup_{|\eta|\le t} \left(\int_\Gamma |g(\overline{x+\eta}) - g(\bar x)|^2 \, d\Gamma \right)^{1/2} \frac{dt}{t} \\
&\quad + C \int_0^1 t^{1-s'} \omega_2 \left(\frac{\partial}{\partial x_2} g(\cdot) \right)(t) \frac{dt}{t}.
\end{aligned} \tag{4.16}
$$

Now let $\bar s = s' - 1$, i.e., $\bar s = 3/2 - \varepsilon' - 1 = 1/2 - \varepsilon'$; then by the definition of the Besov norm,

$$
\int_0^1 t^{1-s'} \sup_{|\eta|\le t} \|[\psi]\|_{L_{2\bar p}(\Gamma)} \frac{dt}{t} = \|\psi\|_{B^{\bar s}_{2\bar p,1}(\Gamma)} \le \|\psi\|_{B^{\bar s+\bar\varepsilon}_{2\bar p,2}(\Gamma)} \le \|\psi\|_{H^{1/2-\varepsilon}(\Gamma)},
$$

where the first inequality is based on the Besov embeddings (3.11) and the second one on the embedding (3.10), since here $s - \frac{n}{p} \ge t - \frac{n}{q}$ of (3.10) is equivalent to $s - \frac{1}{2} \ge \bar s + \bar\varepsilon - \frac{1}{2\bar p}$, which gives, with $\bar s = \frac{1}{2} - \varepsilon'$ and $1/\bar p = 1 - 2\tilde\varepsilon$, ($\bar p = 1 + \tau$), a regularity requirement for ψ of order $s \ge \frac{1}{2} - \varepsilon' + \bar\varepsilon + \tilde\varepsilon$. Thus taking, for instance, $\bar\varepsilon = \tilde\varepsilon = \frac{1}{4}\varepsilon'$, we need to have $\psi \in H^{\frac{1}{2}-\frac{1}{2}\varepsilon'}$, which is the case. Similarly,

$$
\begin{aligned}
&\int_0^1 t^{1-s'} \left(\int_\Gamma |g(\overline{x+\eta}) - g(\bar x)|^2 d\Gamma \right)^{1/2} \frac{dt}{t} \\
&\qquad = \int_0^1 t^{[\bar s]-\bar s} \left(\int_\Gamma |g(\overline{x+\eta}) - g(\bar x)|^2 d\Gamma \right)^{1/2} \frac{dt}{t} \\
&\qquad \le \|g\|_{B^{\bar s}_{2,1}(\Gamma)} \le \|g\|_{B^{\bar s+1/2}_{2,1}(\Omega)} \le \|g\|_{H^1(\Omega)}.
\end{aligned}
$$

Finally, by the definition of the Besov norms, the last integral in (4.16) is dominated by $\|g\|_{B^{3/2-\varepsilon'}_{2,1}(\Omega)}$. Thus

$$
\int_0^1 t^{1-s'} \omega_2(T_\mu g(\cdot))(t) \frac{dt}{t} \le \frac{C}{r\gamma(\mu)} \|g\|_{B^{3/2-\varepsilon'}_{2,1}(\Omega)},
$$

and the result is followed by using the embedding relation (3.8). $\square$

5. Duality and eigenvalue estimates. In studies of the transport equation, the criticality condition of a multiplying system is specified by the largest eigenvalue, λ^{-1}, that makes $(I - \lambda T)^{-1}$ singular. Therefore, in the solution of the transport equation, the eigenvalue, as a global quantity, is of interest. Below we shall see, by means of a *weak norm* estimate of the scalar flux, that the eigenvalue can be found more accurately than the pointwise scalar flux. Observe that the kernel of the integral operator T is symmetric and positive; see the representation of T in [1, relation (1.9)]. Hence T is self-adjoint (on $L_2(\Omega)$) and thus has only real eigenvalues. Furthermore, by the Krien–Rutman theory, its largest eigenvalue is positive and simple. Our main result in this section is the following theorem.

THEOREM 5.1. *Let κ and κ_n^h be the largest eigenvalues of the operators T and T_n^h, respectively. Then for any $\varepsilon > 0$ and $\varepsilon_1 > 0$ there are constants $C = C(\varepsilon_1, \kappa)$ and $C(Q) = C(\varepsilon, \kappa, Q)$ such that, for sufficiently large N and M and sufficiently small h, we have*

$$|\kappa - \kappa_n^h| \le C\left(\frac{1}{N^4} + \frac{1}{M^{2-\varepsilon_1}}\right) + C(Q)h^{3-\varepsilon}, \tag{5.1}$$

where Q is an arbitrary quadrature set.

The first term on the right-hand side of (5.1) follows from a semidiscrete result of [4].

PROPOSITION 5.1. *Assume that M is even and let κ and κ_n be the largest eigenvalues of T and T_n, respectively. Then, for any $\varepsilon_1 > 0$, there is $C = C(\varepsilon_1, \kappa)$ such that, for N and M sufficiently large, we have*

$$|\kappa - \kappa_n| \le C\left(\frac{1}{N^4} + \frac{1}{M^{2-\varepsilon_1}}\right). \tag{5.2}$$

The above assumption on the number of angular quadrature points M (even) makes the quadrature set Δ symmetric in the sense that $\mu \in \Delta$ implies that $-\mu \in \Delta$. Then it follows that T_n is self-adjoint (see, e.g., [2, Lemma 2.1]), and thus its eigenvalues are real, which is crucial in the proof of (5.2).

Proof of Theorem 5.1. It remains to estimate $|\kappa_n - \kappa_n^h|$. Let us consider the discrete ordinates method to be exact and the fully discrete method, i.e., the space discretization, to be approximate. We have discrete ordinates

$$U_n = \sum_{\mu \in Q} \omega_\mu u^\mu, \qquad u^\mu(x) = u_n^\mu(x, \mu),$$

and the fully discrete approximation

$$U_n^h = \sum_{\mu \in Q} \omega_\mu u_h^\mu, \qquad u_h^\mu(x) = u_{nh}^\mu(x, \mu),$$

where $Q \subset \mathbf{D}$ is an arbitrary n-point quadrature set. The fully discrete scalar flux is found by solution of the following bilinear finite element equation: find

$$U_n^h \in V_h = \{v : v|_K \in P_1(K) \quad \forall K \in \mathcal{C}_h\}, \qquad \Omega = \cup K,$$

such that

$$B_\mu(u_h^\mu, v) - \lambda(U_n^h, v) = (f, v) \qquad \forall v \in V_h,$$

where $B_\mu(u_h^\mu, v)$ is the usual bilinear form associated with the fully discrete transport equation

$$\mu \cdot \nabla u_h^\mu(x) + u_h^\mu(x) = \lambda \sum_{\mu \in Q} \omega_\mu u_h^\mu(x) + f(x).$$

Now for $\mu \in Q$, let $\psi^\mu(x)$ be the solution of the corresponding dual problem:

$$\begin{aligned} -\mu \cdot \nabla \psi^\mu(x) + \psi^\mu(x) = \lambda \Psi_n(x) + \sigma(x) \qquad &\text{in } \Omega \times \mathbf{D}, \\ \psi^\mu = 0 \qquad &\text{on } \Gamma_\mu^+ = \{x \in \Gamma : \mu \cdot \hat{n}(x) > 0\}, \end{aligned}$$

where σ is a given data and

$$\Psi_n(x) = \sum_{\mu \in Q} \omega_\mu \psi^\mu(x).$$

The solution ψ^μ may be found by solving the bilinear equation

$$B_\mu(w, \psi^\mu) - \lambda(w, \Psi_n) = (w, \sigma) \qquad \forall w \in V_h. \tag{5.3}$$

Substituting $u^\mu - u_h^\mu$ by w and using (5.3), we compute

$$\begin{aligned}(U_n - U_n^h, \sigma) &= \sum_{\mu \in Q} \omega_\mu (u^\mu - u_h^\mu, \sigma) = \sum_{\mu \in Q} \omega_\mu (w, \sigma) \\ &= \sum_{\mu \in Q} \omega_\mu \left[B_\mu(w, \psi^\mu) - \lambda(w, \Psi_n) \right].\end{aligned}$$

Replacing w with $u^\mu - u_h^\mu$ we obtain

$$\begin{aligned}(U_n - U_n^h, \sigma) &= \sum_{\mu \in Q} \omega_\mu \left[B_\mu(u^\mu - u_h^\mu, \psi^\mu) - \lambda \left(u^\mu - u_h^\mu, \sum_{\nu \in Q} \omega_\nu \psi^\nu \right) \right] \\ &= \sum_{\mu \in Q} \omega_\mu \left[B_\mu(u^\mu - u_h^\mu, \psi^\mu) - \lambda(U_n - U_n^h, \psi^\mu) \right].\end{aligned}$$

Now replace ψ^μ by $\psi^\mu - \tilde{\psi}^\mu$, where $\tilde{\psi}^\mu \in V_h$ is the interpolant of ψ^μ. Then we will have

$$(U_n - U_n^h, \sigma) = \sum_{\mu \in Q} \omega_\mu \left[B_\mu(u^\mu - u_h^\mu, \psi^\mu - \tilde{\psi}^\mu) - \lambda(U_n - U_n^h, \psi^\mu - \tilde{\psi}^\mu) \right].$$

Observe that we have not limited the quadrature set to just *"good"* directions. Thus the constant used below is a function of the quadrature set itself, so that, using Theorem 4.1,

$$(U_n - U_n^h, \sigma) \le C(Q)[h^{1-\varepsilon'} h^2 - \lambda h^{1-\varepsilon'} h^2] \le C(Q) h^{3-\varepsilon'}.$$

If, for example, $\sigma \equiv 1$, then

$$(U_n - U_n^h, \sigma) = \int (U_n - U_n^h)\, dx \le C(Q) h^{3-\varepsilon'},$$

i.e.,

$$\|U_n - U_n^h\|_{L_1(\Omega)} \le C(Q) h^{3-\varepsilon'}, \tag{5.4}$$

which is almost optimal, since $h^{1-\varepsilon'}$ is almost optimal for the scalar flux and h^2 is optimal for the interpolant. (5.4) can be compared with

$$\|U_n - U_n^h\|_{L_2(\Omega)} \le C h^{1-\varepsilon'}, \tag{5.5}$$

confirming our remark on suitability of the L_1-estimates for the scalar flux.

A similar estimate can be found for the eigenvalue κ_n using the same technique as in the proof of Theorem 6.1 in [4] or an estimate of the form

$$|\kappa_n - \kappa_n^h| \leq C\left[|((T_n - T_n^h)\Phi, \Phi)| + \|(T_n - T_n^h)\Phi\|^2_{L_1(\Omega)}\right]^{1/2} \leq C(Q)h^{3-\varepsilon'}, \tag{5.6}$$

where Φ is the (normalized) eigenfunction corresponding to the largest eigenvalue κ_n (see [10]), and (5.1) will follow from (5.2) and (5.6). ∎

Remark 5.1. We have, using Theorem 4.1, that

$$|\kappa_n - \kappa_n^h| \leq Ch^{1-\varepsilon'}, \tag{5.7}$$

where κ_n and κ_n^h are the largest eigenvalues of T_n and T_n^h, respectively, n is the number of discrete points on the unit disc, and the constant C is independent of the quadrature set. Combination of (5.2) and (5.7) gives

$$|\kappa - \kappa_n^h| \leq C\left(\frac{1}{N^4} + \frac{1}{M^{2-\varepsilon_1}} + h^{1-\varepsilon'}\right). \tag{5.8}$$

Comparing (5.1) with (5.8), we see that, in order to have the contributions from the spatial and angular errors to the global eigenvalue error be of the same order of magnitude, it is necessary to choose different compatibility relations $h = h(N)$ and $M = M(N)$ in these two approximations. Similar results hold for a two-dimensional problem with the cylindrical domain $\tilde{\Omega}$ replaced by $\Omega \subset \mathbb{R}^2$ and the velocity space $\mathbf{D}$ replaced by $S = \{\mu \in \mathbb{R}^2 : |\mu| = 1\}$.

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