1 Laplace transforms

Exercises 1-6 are selected from Fourier Analysis, J. Petersson

Problem 1.1 *Find the Laplace transforms to the following functions*

a. $at + b$

b. $(1 + t)^2$

c. $(t + 3)\theta(t)$

d. $(t^2 - 1)e^{3t}\theta(t)$

e. $\sinh at$

f. $2te^t$

g. t^2e^{-4t}

h. $e^t \cos 2t$

Problem 1.2 *Find the Laplace transform of*

a. $\frac{1}{2}(e^t + e^{-t})$

b. $t \sinh t$

c. $te^{-2t}\theta(t - 1)$

d. $te^{-t} \cos^2 t$

Problem 1.3 *Find a function $f(t)$ with Laplace transform*

a. $\dfrac{1}{s + 1}$

b. $\dfrac{1}{s^2 + 4}$

c. $\dfrac{s + 1}{s^2 + 1}$

d. $\dfrac{s}{(s + 1)^2(s^2 + 1)}$

e. $\dfrac{s+1}{(s-3)^4}$

Problem 1.4 *Determine the functions, whos Laplace transform is*

a. $\dfrac{s}{s^2-2s-3}$

b. $\dfrac{s+2}{s^2+4s+5}$

c. $\dfrac{1}{(s-2)^2+9}$

d. $\dfrac{s+1}{s^3+s^2-6s}$

e. $\dfrac{3s}{s^2+2s-8}$

f. $\dfrac{1+e^{-s}}{s}$

Problem 1.5 *Solve the following differential equations for* $t>0$

a. $y'+2y=e^{-3t},\ y(0)=4$

b. $y'+ay=\theta(t),\ y(0)=b$

c. $\dot{y}-y=e^{2t},\ y(0)=-1$

d. $y''+2y'+y=e^{-t},\ y(0)=0,\ y'(0)=1$

e. $y''+\omega^2y=0,\ y(0)=A,\ y'(0)=B$

f. $y''+4y'+13y=2e^{-t},\ y(0)=0,\ y'(0)=-1$

g. $y''+4y=8e^{2t},\ y(0)=0,\ y'(0)=3$

h. $y''-2y'+2y=\cos t,\ y(0)=1,\ y'(0)=0$

i. $y''+4y'=3e^{t},\ y(0)=2,\ y'(0)=1$

j. $\ddot{y}+2\dot{y}+2y=2\theta(t),\ y(t)=0$ *for* $t<0$

Problem 1.6 *Solve the following integral- and integrodifferential equations*

a.

$$\int_0^t y(\tau)d\tau + 2y = 4$$

b.

$$\frac{dy}{dt} + 2y + \int_0^t y(\tau)d\tau = \cos t, \qquad y(0) = 1$$

c.

$$\frac{dy}{dt} + 4y + 5\int_0^t y(\tau)d\tau = e^{-t}, \qquad y(0) = 0$$

d.

$$\frac{d^2y}{dt^2} - 7y + 6\int_0^t y(\tau)d\tau = 1, \quad y(0) = 7 \quad y'(0) = -12$$

e.

$$\int_0^t y(\tau)\cos(t-\tau)d\tau = t, \quad t \geq 0$$

f.

$$y - 2\int_0^t y(t-\tau)\cos(\tau)d\tau = \sin t, \quad t \geq 0$$

Problem 1.7 *Use Laplace transforms to solve*

$$\begin{cases} y''(t) + 3y'(t) + 2y(t) = \cos(2t), \quad t > 0, \\ y(0) = y'(0) = 1. \end{cases}$$

Problem 1.8 *Use Laplace transforms to solve the differential equation*

$$\begin{cases} y''' - y'' - y' + y = te^t, \quad t > 0, \\ y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0. \end{cases}$$

Problem 1.9 *Solve the integral equation*

$$\begin{cases} u''(t) + u(t) = \int_0^t e^{(\tau - t)} u(\tau)\, d\tau - \cos(t), \\ u('0) = 0, \qquad u(0) = 1. \end{cases}$$

Problem 1.10 *Use Laplace transform to solve u in the following system of differential equations:*

$$\begin{cases} u' + 2u + v = te^{-t}, \\ v' - u = t + 1, \\ u(0) = 1, \quad v(0) = 0. \end{cases}$$

Problem 1.11 *Solve the integral equation*

$$\begin{cases} y'(t) + 2y(t) + 2\int_0^t y(\tau)\, d\tau = 1 + e^{-t}, \quad t > 0, \\ y(0) = 1. \end{cases}$$

Problem 1.12 *Solve, for $t > 0$, the integral equation below*

$$y'(t) + 2y(t) - 2\int_0^t y(t)\sin(t - \tau) d\tau = \cos t, \qquad y(0) = 1.$$

Problem 1.13 *Find the solution of the following integral equation*

$$y(t) + \int_0^t y(\tau)\sin(t - \tau) d\tau = 1 - \theta(t - 2\pi).$$

Problem 1.14 *Determine the function $y(x)$ satisfying the integral equation*

$$y'' - 2y' - 5y + 6\int_0^x y(t) dt = 1 - \theta(x - 1), \qquad y(0) = 1, \quad y'(0) = 0.$$

Problem 1.15 *Let $t > 0$ and solve the differential equation*

$$y'' + 2y + 10y = f(t), \qquad y(0) = y'(0) = 0,$$

where

$$f(t) = \begin{cases} \cos t, & då \quad 0 \le t \le \pi/2, \\ 0, & då \quad t > \pi/2. \end{cases}$$

Answer 1.1

a. $a/s^2 + b/s$

b. $2/s^3 + 2/s^2 + 1/s$

c. $s^{-2} + 3s^{-1}$

d. $2(s-3)^{-3} - (s-3)^{-1}$

e. $\dfrac{a}{s^2 - a^2}$

f. $2(s-1)^{-2}$

g. $2(s+4)^{-3}$

h.

$$\cos 2t \supset \frac{s}{s^2+4} \qquad e^t \cos 2t \supset \frac{s-1}{(s-1)^2+4} = \frac{s-1}{s^2-2s+5}$$

Answer 1.2

a. $s(s^2-1)^{-1}$

b. $2s(s^2-1)^{-2}$

c.

$$\theta(t-1) \supset \int_0^\infty e^{-st}\theta(t-1)\,dt = \int_1^\infty e^{-st}dt = e^{-s}s^{-1}.$$

$$t\theta(t-1) \supset -\frac{d}{ds}(e^{-s}s^{-1}) = e^{-s}(s^{-1}+s^{-2})$$

$$te^{-2t}\theta(t-1) \supset e^{-s-2}\left[(s+2)^{-1} + (s+2)^{-2}\right].$$

d. $\dfrac{1}{2(s+1)^2} + \dfrac{s^2+2s-3}{2(s^2+2s+5)^2}$

Answer 1.3

a. e^{-t}

b. $\frac{1}{2}\sin 2t$

c. $\cos t + \sin t$

d. $\frac{1}{2}(\sin t - te^{-t})$

e. $\frac{1}{6}(3t^2 + 4t^3)e^{3t}$

Answer 1.4

a. $e^t(\cosh 2t + \frac{1}{2}\sinh 2t) = \frac{3}{4}\,e^{3t} + \frac{1}{4}\,e^{-t}$

b. $e^{-2t}\cos t$

c. $\frac{1}{3}e^{2t}\sin 3t$

d. $-\frac{1}{6} + \frac{3}{10}e^{2t} - \frac{2}{15}e^{-3t}$

e. $2e^{-4t} + e^{2t}$

f. $1 + \theta(t-1)$

Answer 1.5

a. From the table we have that $\mathcal{L}y'(t) = sY(s) - 4$ and $\mathcal{L}e^{-3t} = (s+3)^{-1}$, i.e.

$$sY(s) - 4 + 2Y(s) = \frac{1}{s+3}.$$

We solve the preceding equation, which yields

$$Y(s) = \frac{1}{(s+3)(s+2)} + \frac{4}{s+2}.$$

To obtain $y(t) = \mathcal{L}^{-1}Y(s)$ we consult a table of inverse transforms and find that $y(t) = 5e^{-2t} - e^{-3t}$.

b. $\frac{1}{a}[1 + (ab-1)e^{-at}]$ if $a \neq 0$

c. $y = e^{2t} - 2e^t$

d. $y(t) = e^{-t}(t + \frac{1}{2}t^2)$

e. $y = A\cos\omega t + B/\omega \sin\omega t$

f. $y(t) = \frac{1}{5}e^{-t} - \frac{1}{5}e^{-2t}(\cos 3t + 2\sin 3t)$

g. $e^{2t} - \cos 2t + \frac{1}{2}\sin 2t$

h. $y = \frac{1}{5}(\cos t - 2\sin t + 4e^t\cos t - 2e^t\sin t)$

i. $y = \frac{3}{2} - \frac{1}{10}e^{-4t} + \frac{3}{5}e^t$

j. $\theta(t) - e^{-t}\theta(t)(\cos t + \sin t)$

Answer 1.6

a. The transform of the integral of $y(t)$ is $s^{-1}Y(s)$ which implies

$$\frac{1}{s}Y(s) + 2Y(s) = \frac{4}{s} \qquad Y(s) = \frac{4}{2s+1} \qquad y(t) = 2e^{-t/2}$$

b. $y(t) = \frac{1}{2}(e^{-t} - te^{-t} + \cos t)$

c. $y(t) = \frac{1}{2}(3e^{-2t}\sin t + e^{-2t}\cos t - e^{-t})$

d. $y(t) = e^t + e^{2t} + 5e^{-3t}$

e.

$$Y(s) \cdot \frac{s}{s^2+1} = \frac{1}{s^2} \qquad Y(s) = \frac{1}{s} + \frac{1}{s^3} \qquad y(t) = 1 + \tfrac{1}{2}t^2$$

f. $y = te^t$

Answer 1.7

$$y(t) = -\frac{1}{20}\cos(2t) + \frac{3}{10}\sin(2t) - \frac{7}{4}e^{-2t} + \frac{14}{5}e^{-t}.$$

Answer 1.8

$$y(t) = -\frac{7}{16}e^{-t} + \frac{7}{16}e^t + \frac{1}{8}te^t - \frac{1}{8}t^2e^t + \frac{1}{12}t^3e^t.$$

Answer 1.9

$$u(t) = \frac{2}{\sqrt{3}} e^{-t/2} \sin(\sqrt{3}t/2) + \cos(t) - \sin(t).$$

Answer 1.10

$$u = 1 - t - te^{-t} + \frac{t^2}{2} e^{-t} - \frac{t^3}{6} e^{-t}.$$

Answer 1.11

$$y(t) = (-1 + 2\cos t + \sin t)e^{-t}.$$

Answer 1.12

$$y(t) = 1 - te^{-t}.$$

Answer 1.13

$$y(t) = \frac{1}{2}\Big(1 + \cos(\sqrt{2}t)\Big) - \frac{1}{2}\Big(1 + \cos(\sqrt{2}(t - 2\pi))\Big)\theta(t - 2\pi).$$

Answer 1.14

$$y(x) = \frac{2}{5} e^{3x} + \frac{3}{5} e^{-2x} + \Big(\frac{1}{6} e^{x-1} - \frac{1}{10} e^{3(x-1)} - \frac{1}{15} e^{-2(x-1)}\Big)\theta(x - 1).$$

Answer 1.15

$$y(t) = \begin{cases} \frac{1}{85}\Big(2\sin t + 9\cos t\Big) - \frac{e^{-t}}{85}\Big(9\cos 3t + \frac{11}{3}\sin 3t\Big), & 0 \le t \le \pi/2 \\ -\frac{e^{-t}}{85}\Big(9\cos 3t + \frac{11}{3}\sin 3t\Big) - \frac{e^{\pi/2-t}}{85}\Big(2\sin 3t + \frac{7}{3}\cos 3t\Big), & t > \pi/2 \end{cases}$$