**Applied Mathematics/ Partial Differential Equations, part A**

# Problems VI

## Vector Analysis

**1.** Compute $\nabla u$, $n \cdot \nabla u$, and $\triangle u$ for
(a) $u(x,\, y) = xy$; $\quad n = (1,\, 0)$,
(b) $u(x,\, y) = \sin(x)\cos(y)$; $\quad n = (1,\, 1)$,
(c) $u(x,\, y) = \log(r)$ where $r = \sqrt{x^2 + y^2}$ $\quad (r \neq 0)$; $\quad n = (x,\, y)$.

## Stiffness Matrix

**2.** Consider the triangulation of $\Omega = [0,\, 2] \times [0,\, 1]$ into 3 triangles drawn in Figure 1.

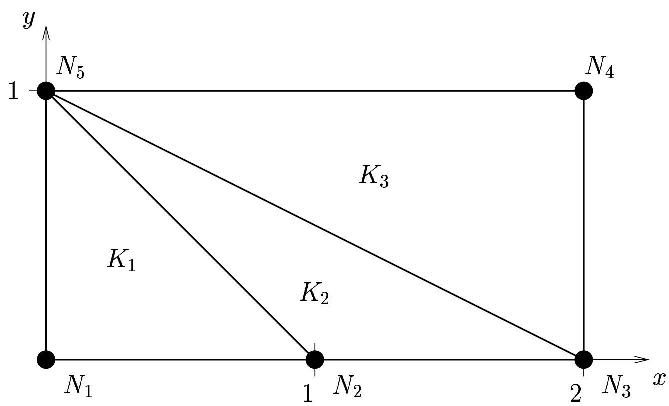


Figure 1: The triangulation in Problem 1 and Problem 4.

Compute by hand the stiffness matrix $A$ with elements $a_{ij} = \iint_\Omega \nabla\varphi_j \cdot \nabla\varphi_i \, dx\, dy$, $i, j = 1, \ldots, 5$.

*Hint:* Since $\varphi_i(x,\, y)$ is linear on each triangle, the gradient $\nabla\varphi_i$ will be a *constant* vector on each triangle. As an example, consider triangle $K_1$. On this triangle, it is easy to show that $\varphi_1(x,\, y) = 1 - (x + y)$, $\varphi_2(x,\, y) = x$, and $\varphi_5(x,\, y) = y$ (cf. how you did in Problem 2(a), Week 5). Therefore, on $K_1$: $\nabla\varphi_1 = (-1,\, -1)$, $\nabla\varphi_2 = (1,\, 0)$, and $\nabla\varphi_5 = (0,\, 1)$. Thus, $a_{11} = \iint_\Omega \nabla\varphi_1 \cdot \nabla\varphi_1 \, dx\, dy = \iint_{K_1} \nabla\varphi_1 \cdot \nabla\varphi_1 \, dx\, dy = \iint_{K_1} 2\, dx\, dy = 1$. Observe that some matrix elements will get contributions from more than one triangle.

**3.** Let $\mathcal{P}(K) = \{v(x) = c_0 + c_1x_1 + c_2x_2,\ c_i \in \mathbf{R},\ i = 1, 2, 3;\ x = (x_1, x_2) \in K\}$ be the space of linear polynomials defined on a triangle $K$ with corners $a^1$, $a^2$, and $a^3$. Derive explicit expressions (in terms of the corner coordinates $a^1 = (a^1_1,\, a^1_2)$, $a^2 = (a^2_1,\, a^2_2)$, and $a^3 = (a^3_1,\, a^3_2)$) for the gradients $\nabla\lambda_1$, $\nabla\lambda_2$, $\nabla\lambda_3$ of the basis functions $\lambda_1$, $\lambda_2$, $\lambda_3 \in \mathcal{P}(K)$

defined by

$$\lambda_i(a^j) = \begin{cases} 1 & i = j, \\ 0 & i \neq j, \end{cases} \tag{1}$$

with $i, j = 1, 2, 3$. Compare with the corresponding expressions in `MyFirst2DPoissonAssembler`.
*Hint:* Use the result from Problem 3, Week 6.

## Robin Boundary Conditions

**4.** Consider once more the triangulation of $\Omega = [0,\, 2] \times [0,\, 1]$ into 3 triangles drawn in Figure 1. Let $\Gamma = \partial\Omega$ denote the boundary of $\Omega$. Assuming that $\gamma(x,\, y) = 1$, $g_D(x,\, y) = 1 + x + y$, and $g_N(x,\, y) = 0$, compute by hand:
(a) The "boundary matrix" $R$ with elements $r_{ij} = \int_\Gamma \gamma\, \varphi_j\, \varphi_i\, ds$, $i, j = 1, \ldots, 5$.
(b) The "boundary vector" rv with elements $\text{rv}_i = \int_\Gamma (\gamma g_D - g_N)\, \varphi_i\, ds$, $i = 1, \ldots, 5$.
*Hint:* You can either compute the curve integrals analytically or use *Simpson's* formula which is exact in this case.

## The Finite Element Method: Stationary Problems (2D).

**5.** Show that the equation:

$$\iint_\Omega \nabla U \cdot \nabla v\, dx\, dy = \iint_\Omega f v\, dx\, dy \quad \text{for all } v \in V_{h0}, \tag{2}$$

is equivalent to

$$\iint_\Omega \nabla U \cdot \nabla \varphi_i\, dx\, dy = \iint_\Omega f \varphi_i\, dx\, dy \quad \text{for } i = 1, \ldots, N, \tag{3}$$

where $N$ is the number of internal nodes ("*nintnodes*") and $\{\varphi_i\}_{i=1}^N$ is the basis of "tent-functions" in $V_{h0}$.

**6*.** Show that the problem: find $U \in V_{h0}$ such that

$$\iint_\Omega \nabla U \cdot \nabla w dx\, dy = \iint_\Omega f w\, dx\, dy \quad \text{for all } w \in V_{h0}, \tag{4}$$

is equivalent to the minimization problem: find $U \in V_{h0}$ such that

$$\frac{1}{2}\iint_\Omega \nabla U \cdot \nabla U\, dx\, dy - \iint_\Omega f U\, dx\, dy = \min_{v \in V_{h0}} \frac{1}{2}\iint_\Omega \nabla v \cdot \nabla v\, dx\, dy - \iint_\Omega f v\, dx\, dy. \tag{5}$$

**7*.** (a) Consider the quadratic equation

$$at^2 + bt + c = 0, \tag{6}$$

Investigate under what condition on the coefficients $a, b, c$ equation (6) does *not* have two distinct real roots.
(b) Prove the Cauchy-Schwarz inequality:

$$|\iint_\Omega vw \, dx \, dy| \leq \|v\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)} \tag{7}$$

Hint: start from the fact that $\|v+tw\|^2_{L^2(\Omega)} \geq 0$. Expanding $\|v+tw\|^2_{L^2(\Omega)}$ gives a quadratic polynomial which can not have two distinct real roots (why?). Use (a) to prove the Cauchy-Schwarz inequality.

**8.** Calculate $\|\nabla f\|_{L^2(\Omega)}$ where $\Omega = [0,1] \times [0,1]$ and
(a) $f = x_1 x_2^2$.
(b) $f = \sin(nx_1)\sin(mx_2)$ with $n$ and $m$ arbitrary integers. What happens when $n, m$ tends to infinity?

**9.** Let $u = x_1 x_2^2$ and $a = 1 + x_2^2$. Calculate
(a) $\nabla u$.
(b) $\Delta u$.
(c) $\nabla \cdot a \nabla u$.

**10.** Consider the problem: find $u$ such that

$$-\Delta u + cu = f \quad \text{in } \Omega, \tag{8}$$
$$u = g_D \quad \text{on } \Gamma_D, \tag{9}$$
$$-n \cdot \nabla u = g_N \quad \text{on } \Gamma_N, \tag{10}$$

with the usual notation.
(a) Derive a finite element method for this problem using approximation of the Dirichlet boundary conditions.
(b) Prove that the finite element solution is unique when 1. $c > 0$ and 2. $\Gamma_D$ is nonempty.

**11.** Let $K$ be a triangle with corners $(0,0)$, $(0,1)$, and $(1,0)$, and let $f = x_1^2 + x_2$. Calculate

$$\iint_K f \, dx \, dy, \tag{11}$$

using
(a) one point ("center of gravity") quadrature,
(b) corner ("node") quadrature,
(c) mid-point (of the triangle sides) quadrature.
Also compute (11) analytically and compare with your results above.
**12.** Let $K$ be a triangle with corners $(0,0)$, $(0,1)$, and $(1,0)$.
(a) Calculate the three basis functions $\lambda_i$, $i = 1, 2, 3$, for the space $\mathcal{P}(K)$ of linear functions

defined on $K$.
(b) Calculate the $3 \times 3$ element mass matrix with elements $m_{ij} = \iint_K \lambda_j \lambda_i \, dx\, dy$ approximately using corner quadrature.
(c) Calculate the $3 \times 3$ element stiffness matrix with elements $a_{ij} = \iint_K \nabla\lambda_j \cdot \nabla\lambda_i \, dx\, dy$.